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# INSTITUTIONES ANALYTICÆ

A

VINCENTIO RICCATO

SOCIETATIS JESU

ET

HIERONYMO SALADINO

MONACHO CÆLESTINO

COLLECTÆ.

*TOMUS SECUNDUS.*



BONONIÆ MDCCLXVII.

EX TYPOGRAPHIA SANCTI THOMÆ AQUINATIS.

SUPERIORUM AUCTORITATE.

BIBLIOTECA NAZIONALE VITTORIO EMANUELE III NAPOLI

B. R. NAPOLI

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*Hæc qui ſpernit, ideſt has ſemitas ſapientiæ, ei denuncio non recte philoſophandum.* Boetius Arit. l. 1. c. 1.

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# TOMI SECUNDI

## *LIBER PRIMUS.*

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# LIBER PRIMUS

## DE QUANTITATIBUS INFINITESIMIS DE CURVARUM QUADRATURIS, ET RECTIFICATIONIBUS, ET DE INTEGRATIONE FORMULARUM, QUÆ UNAM SOLUM VARIABILEM CONTINENT.

### *CAPUT PRIMUM*

### De vera, ac germana quantitatum infiniteſimarum idea.

1 GEometræ veteres, quum comparationem inſtituere cæperunt inter rectilinea, & curvilinea, ut tutam methodum ſequerentur, ſemper confugerunt ad quantitates minores quacunque data. Nimirum ita augebant, & imminuebant rectilinea, ut eorum differentia a curvilineis minor fieret quacumque data; tum non ſine maxima induſtria, ac ingenii vi curvilineorum quantitatem, aut proportionem determinabant, eamdemque longo circuitu ſervata evidentia geometrica demonſtrabant. Hujuſce methodi luculenta exempla invenies in Archimede, & Pappo, qui ſcriptor illuſtriora Græcorum inventa collegit, & expoſuit. Qui autem ſequuti ſunt geometræ, longarum demonſtrationum pertæſi lemmata quædam præmiſerunt, quibus brevitatis cauſſa in peculiaribus perquiſitionibus uterentur. Lemmata autem omnia hoc unico continentur. Quantitates, & quantitatum rationes, quæ ita ad ſe invicem accedunt, ut earum differentia evadat minor quacumque data, ultimo fiunt æquales. Propoſitio hæc, quæ maxime fœcunda eſt, atque utilis, ita breviſſime per deductionem ad abſurdum oſtenditur. Si non æquales, fient ultimo inæquales; earum autem differentia ſit $D$; ergo nequeunt propius ad æqualitatem accedere, quam pro data differentia $D$, quod deſtruit hypotheſim. Hinc tamquam corollarium colligas velim, quantitates duas, quæ & ad alias ita accedunt, ut differentiæ fiant minores quibuſcumque datis, & dum augentur, aut minuuntur, ſemper eamdem rationem ſervant, ultimo eamdem rationem conſervaturas.

2. Ut ingenioſiſſimam methodum nobili aliquo exemplo declarem, placet eam, quam Archimedes tradidit, Parabolæ quadraturam paucis exponere. Hanc ob rem præmitto ſimpliciſſimum lemma, quod vix indiget demonſtratione. Sint duæ quantitates inæquales $A$, $B$, quarum differentia ſit $D$; minori $B$ addo dimidium differentiæ ideſt $\frac{1}{2}D$, tum reſiduæ diferentiæ $\frac{1}{2}D$ addo dimidium $\frac{1}{4}D$, atque ita deinceps; ajo, differentiam inter $A$, & $B$ ita auctam evadere minorem quacumque data. Si negas, ponamus minimam quantitatum differentiam eſſe $\frac{D}{m}$, exiſtente $m$ numero unitate majore; non enim differentia poteſt eſſe ma-

jor quam $D$; Manifeſtum eſt, in ſerie numerorum in dupla proportione creſcentium $2, 2^2, 2^3, 2^4$ &c. inveniri poſſe numerum, qui major ſit numero $m$; ſit hic $2^n$. Tot vicibus dimidia reſiduarum differentiarum addantur quantitati $D$, quot unitates inſunt in $n$. Differentia, quæ ſupereſt, erit $\frac{D}{2^n}$; ſed hæc minor eſt $\frac{D}{m}$, quia $2^n > m$; ergo $\frac{D}{m}$ non eſt minima; ergo tandem fit differentia minor quacumque data. Idem dicas velim, ſi dimidium differentiæ reliquæ ſucceſſive demas a quantitate $A$. Multo magis ſi quantitati $B$ ſucceſſive addas, vel demas ab $A$, plus quam dimidium reliquæ differentiæ, evadet differentia minor quacumque data.

3. His præmiſſis ducatur in parabola BAD (*Fig.* 1) chorda qualibet BD, quæ dividatur bifariam in C, & per C agatur diameter CA, cujus vertex A; junctis BA, DA efformatur triangulum BAD, quod differt a parabolico ſpatio per duo ſegmenta DGA, BKA. Diviſis AD, AB bifariam in E, H, ductiſque diametris GE, KH, formentur duo triangula ut ſupra AGD, AKB. Additis duobus hiſce triangulis triangulo BAD additur plus quam dimidium differentiæ inter hoc, & ſpatium parabolicum, quia hæc triangula ſunt dimidia parallelogrammorum circumſcriptorum, quæ ſine dubio ſegmentis parabolicis ſunt majora. Eodem modo, ſervatiſque iiſdem conditionibus efformata quatuor triangula addantur tribus ſuperioribus, probabitur addi plus quam dimidium differentiæ a ſpatio parabolico; atque ſucceſſive 8, 16, 32 &c. triangulis additis, ſumma triangulorum ad parabolicum ſpatium ita accedet, ut differentia quæcumque data minor evadat; igitur ſumma triangulorum ultimo fiet ſpatio parabolico æqualis.

4. Hæc quidem curvis aliis æque conveniunt; ſed in aliis curvis difficillime ſumma triangulorum determinatur, in parabola autem perelegant. Namque iiſdem poſitis ac antea, triangulum AGD, quod unum eſt ex ſecundis, quæ primo adduntur, ad triangulum BAD eſt ut 1:8. Ex punctis G, E agantur GM, EN parallelæ BD. Conſtat AN fore dimidium AC, & NE, ſeu MG, ſeu CF dimidium CD. Præterea MN erit quarta pars AC; nam eſt AC: AM:: $CD^2 : MG^2$ :: 4 : 1; ergo AM eſt quarta pars AC; ergo etiam MN, ſeu GE erit quarta pars AC, aut dimidium NC, aut EF. His poſitis Triangulum GED : EDF :: GE : EF :: 1 : 2, &

DEF : ADC :: $EF^2 : AC^2$ :: 1 : 4; ergo
GED : ADC :: 1 : 8. Igitur duplicatis antecedentibus
ADG : BAD :: 1 : 8. Q.E.D. Eadem methodo demonſtratur, triangulum item BKA eſſe ad BAD :: 1 : 8; quare triangula duo BKA, DGA erunt æqualia. Similiter diviſis rectis AG, GD, AK, BK bifariam, ductiſque congruis diametris, triangula, quæ efformantur erunt æqualia inter ſe, & ſingula erunt ad alterutrum ex triangulis AGD, AKB :: 1 : 8; ergo ad triangulum BAD :: 1 : $8^2$. Similiter octo lineis bifariam diviſis, peractiſque omnibus, ut antea, naſcentur octo triangula æqualia, quorum ſingula habebunt ad BAD rationem 1 : $8^3$; ſingula triangulorum ſexdecim, quæ ſequuntur, erunt ad idem BAD :: 1 : $8^4$; atque ita deinceps. Quapropter quodlibet ex ſuccel-

ceſſivis triangulis a ſequenti ſerie exprimitur

ABD. $1, \frac{1}{8}, \frac{1}{8^2}, \frac{1}{8^3}, \frac{1}{8^4}, \frac{1}{8^5}$ &c. Itaque primum triangulum æquat

$1$.ABD, unum ex ſecundis $= \frac{1}{8}$.ABD, unum ex tertiis $= \frac{1}{8^2}$ABD, & ſic deinceps.

5. Ad habendam omnium triangulorum ſummam, obſervandum eſt, primum triangulum unum eſſe, ſecunda duo, quæ ſequuntur quatuor, deinde octo, ſexdecim, triginta duo, atque ita deinceps. Ergo ſeries ſuperior erit multiplicanda per

$1, 2, 4, 8, 16, 32$ &c., ſive

$1, 2, 2^2, 2^3, 2^4, 2^5$ &c.; igitur ſumma omnium triangulorum erit

ABD. $1, \frac{2}{8}, \frac{2^2}{8^2}, \frac{2^3}{8^3}, \frac{2^4}{8^4}, \frac{2^5}{8^5}$ &c., aut

ABD. $1, \frac{1}{4}, \frac{1}{4^2}, \frac{1}{4^3}, \frac{1}{4^4}, \frac{1}{4^5}$, &c., cujus terminus generalis $= \frac{1}{4^{n-1}}$. ABD. Conſtat ex methodo tradita in Tomo primo, ſummam ſeriei, cujus terminus generalis $= \frac{1}{4^{n-1}}$ hujuſmodi formula contineri $A - \frac{B}{4^n}$. Pro $n$ ſcribe $n-1$, ut habeas $A - \frac{B}{4^{n-1}}$. Hanc a ſuperiore detrahens habebis ſeriei terminum generalem $= \frac{B}{4^{n-1}} - \frac{B}{4^n} = \overline{1 - \frac{1}{4}}.\frac{B}{4^{n-1}} = \frac{3}{4}.\frac{B}{4^{n-1}}$. Comparans hunc cum ſuperiori invenies $\frac{3}{4}B = 1$, vel $B = \frac{4}{3}$; ergo ſumma ſeriei erit $A - \frac{4}{3\cdot 4^n} = A - \frac{1}{3\cdot 4^{n-1}}$. Ad determinandam $A$, facta $n = 1$, inſtitue inter ſummam, & terminum generalem æqualitatem, ut fiat $A - \frac{1}{3} = 1$, aut $A = \frac{4}{3}$. Itaque ſeriei ſumma $= \frac{4}{3} - \frac{1}{3\cdot 4^{n-1}}$, & ſumma omnium triangulorum $\frac{4}{3}$. ABD $- \frac{ABD}{3\cdot 4^{n-1}}$. Quæ ſumma differt a $\frac{4}{3}$ ABD differentia $\frac{ABD}{3\cdot 4^{n-1}}$; ſed hæc aucto numero $n$ fit minor quacumque data; ergo ſumma omnium triangulorum ultimo fit æqualis $\frac{4}{3}$ABD; ſed eadem ſumma ultimo æquat ſpatium parabolicum; ergo ſpatium parabolicum $= \frac{4}{3}$ ABD. Q.E.I.

6. Forte fortuna accidit in parabola, ut triangula ſubſequentia habeant ad antecedentia rationem $1:8$, quæ nihil dependet a triangulorum lateribus, qua in proprietate innititur elegans, & ingenioſa demonſtratio Archimedis. Verum

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quoniam curvæ aliæ ſimili proprietate plerumque careant, iccirco ad aliam longe diverſam methodum, nempe ad rectangula inſcripta, & circumſcripta ſeſe contulerunt non minus veteres, quam recentiores Geometræ. Sit curva qualibet A D, (*Fig.* 2) cujus coordinatæ A B, B D angulum rectum faciant. Diviſa A B in partes æquales A e, e 2e, 2e 3e i &c., quarum numerus $= n$, ductiſque ordinatis e i, 2e 2i, 3e 3i &c., inſcribantur, & circumſcribantur curvæ rectangula, ut in figura. Perſpicuum eſt, differentiam inter rectangula circumſcripta, & inſcripta preciſe eſſe æqualem ultimo circumſcripto rectangulo $\overline{n-1}$. e B D; ergo inter circumſcripta & ſpatium curvilineum, inter hoc & inſcripta differentia erit minor ultimo circumſcripto rectangulo $\overline{n-1}$. e B D; ſed hoc, partium A e, e 2e &c. quantitate minuta, & aucto numero, fieri poteſt minus quocumque dato; ergo differentia inter rectangula circumſcripta & ſpatium, inter hoc & inſcripta evadit minor quacumque data; ergo ſumma rectangulorum, vel circumſcripta ſint, vel inſcripta, ultimo eſt æqualis ſpatio curvilineo. Quare ſi determinare liceat, licet autem ſæpiſſime, cuinam quantitati ultimo fiat æqualis ſumma rectangulorum, quæ aut inſcripta ſunt, aut circumſcripta; inveniemus quantitatem, cui æquale eſt ſpatium curvilineum. Methodum illuſtremus exemplo ejuſdem parabolæ apollonianæ.

7. Itaque curva A D ſit parabola, cujus axis A F, tangens autem normalis axi A B, parameter vero $= a$. Vocata A B $= x$, B D $= y$, erit æquatio $xx = ay$. Dividatur A B in partes æquales numero $n$, quarum ſingulæ fiant $= q$, patet, fore $nq = x$. Ex punctis diviſionis ordinentur
e i, 2e 2i, 3e 3i, 4e 4i &c., quæ analytice ita exprimentur
$\frac{q^2}{a}$, $\frac{4q^2}{a}$, $\frac{9q^2}{a}$, $\frac{16q^2}{a}$ &c. ..... $\frac{n^2q^2}{a}$; ergo rectangula circumſcripta, nam circumſcripta pro libito accipio, nempe
A e i, e 2e 2i, 2e 3e 3i, 3e 4e 4i &c. fierent
$\frac{q^3}{a}$, $\frac{4q^3}{a}$, $\frac{9q^3}{a}$, $\frac{16q^3}{a}$ &c. ..... $\frac{n^2q^3}{a}$. Horum autem ſumma æqualis eſt quantitati $\frac{q^3}{a}$ ductæ in ſummam ſeriei 1, 4, 9, 16 &c..... $n^2$. Series hæc, quæ coaleſcit ex quadratis numerorum naturalium, eſt algebraica ſecundi ordinis, quia ejus differentiæ ſecundæ conſtantes ſunt, & habet terminum generalem $= n^2$. Quare per methodum in Tomo primo traditam invenies ejus ſummam $= \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{2.3}$. Igitur omnium rectangulorum ſumma

$= \frac{q^3}{a} . \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{2.3}$; ſed $nq = x$; ergo ſumma eadem

$= \frac{x^3}{3a} + \frac{x^3}{2na} + \frac{x^3}{2.3.n^2a}$. Quapropter rectangulorum ſumma major eſt $\frac{x^3}{3a}$

per differentiam $\frac{x^3}{2na} + \frac{x^3}{2.3.n^2a}$; ſed hæc aucto numero $n$ evadit minor quacum-

cumque data; ergo ſumma rectangulorum ultimo fit $= \frac{x^3}{3a}$; ſed ſumma rectangulorum ultima æquat ſpatium parabolicum ABD; ergo ſpatium ABD $= \frac{x^3}{3a} = \frac{xy}{3}$. Quæ quadratura adamuſſim cum ſuperiore coincidit.

8. Poſſem oſtendere, methodum hanc patere latiſſime, neque deficere, tametſi ordinatæ eſſent abſciſſis obliquæ; quia in hoc caſu non rectangula, ſed parallelogramma cum circumſcripta, tum inſcripta ſimul ſumpta ultimo eſſent ſpatio curvilineo æqualia. Satis enim hæc ſunt ad veterum methodum indicandam, quod ſolum mihi in præſentia propoſitum eſt. Hoc unice omittendum non eſt, methodum hanc non eſſe temere uſurpandam in curvis aſymptoticis, qualis eſt BD. (*Fig.* 3) Etenim differentia inter ſpatium curvilineum, & rectangula inſcripta non eſt primum rectangulum Aei, ſed hoc ſimul cum ſpatio infinite longo HiD, de quo ignoratur, utrum minus evadat quocumque dato. Primum vero rectangulum circumſcriptum habetur nuſquam, quia nuſquam curva incidit in aſymptotum. Quamobrem induſtria, & cautione maxima opus eſt, quum quadranda veniunt ſpatia infinite longa incluſa inter curvam, & aſymptotum. Sed de his ſuo loco.

9. Nunc ſpectemus, quanam via Geometræ veteres curvarum tangentes determinabant, ut diſcamus, quomodo uterentur rationibus, quarum differentia evaderet minor quacumque data. Incipiamus a circulo curva omnium notiſſima. Ad punctum E (*Fig.* 4) ducenda ſit tangens circuli AEB. Sit hæc TEL. Agantur EM, EH altera normalis, altera parallela diametro AB. Sumpto quocumque arcu EI agatur ſecans EIK; demum accipe EH æqualem datæ, & parallelam ordinatis age GFN tranſeuntem per punctum I, & HL occurrentem tangenti in L, ſecanti in K. Evidens eſt, ita imminui poſſe arcum EI, ut minor fiat quocumque dato; igitur angulus GEI, ſeu LEK evadet minor quocumque dato; nam ducta diametro ECP, junctaque IP angulus GEI = P; ſed, exiſtente arcu EI minore quocumque dato, angulus P eſt minor quocumque dato; ergo etiam angulus GEI, ſeu LEK. Igitur lineola LK evadet minor quacumque data; ergo ratio LH : HE erit quidem major ratione KH : HE, ſed earum differentia eſt minor quacumque data; ergo duæ rationes iſtæ ultimo fiunt æquales; ſed ratio LH : HE eadem eſt cum ratione GF : FE, & ratio KH : HE eadem cum ratione IF : FE; ergo duæ rationes GF : FE, IF : FE, ultimo æquales fiunt; atqui ratio GF : FE eadem eſt cum ratione ordinatæ ad ſubtangentem, ſcilicet EM : MT; ergo ratio IF : FE ultimo fit æqualis rationi EM : MT; ſed FI eſt differentia ordinatarum EM, IN, & MN differentia abſciſſarum AM, AN; ergo differentia ordinatæ ad differentiam abſciſſæ ultimo fit, ut ordinata EM ad ſubtangentem MT. Quare qui determinaverit, cuinam rationi ultimo ſit æqualis differentia ordinatæ ad differentiam abſciſſæ, determinabit quoque ſubtangentem, adeoque tangentem ducet.

10. Hanc rationem inveniamus in circulo. Vocetur radius $= a$, abſciſſa AM $= x$, ejus differentia MN $= dx$, ordinata ME $= y$, ejuſque differentia $= dy$. Ex natura circuli duæ æquationes oriuntur

$$2ax - xx = yy$$

$$2ax + 2adx - x^2 - 2xdx - dx^2 = y^2 + 2ydy + dy^2$$

Prima ex ſecunda auferatur, ut fiat

$2adx$

$2adx - 2xdx - dx^2 = 2ydy + dy^2$; ergo

$dy:dx :: a - x - \frac{dx}{2} : y + \frac{dy}{2}$; ſed quoniam $dx$, $dy$ fiunt minores quibuſcumque datis, hæc ratio ultimo eadem eſt cum ratione $a - x : y$; Ergo $dy:dx$ ultimo eſt, ut $a - x : y$; ſed $dy:dx$ ultimo eſt ut EM:MT; ergo $a - x : y$, ſeu CM:ME ſe habet, ut EM:MT, ex qua analogia determinatus eſt valor ſubtangentis MT; quæ determinatio apprime cohaeret cum illis, quæ traduntur in elementis, ex quibus conſtat, angulum TEC rectum eſſe oportere.

11. Methodus, quæ cauſſa facilitatis in circulo uſurpata eſt, valet in omnibus curvis continuis. Nam facta eadem præparatione, evidens eſt accedenti puncto I ad E, evadere tam angulum LEK, quam interceptam LK minorem quacumque data; ergo æquales ultimo fient rationes LH:EH, KH:EH; imo & rationes GF:FE, IF:EF; ſed prima eadem eſt ac ratio ordinatæ EM ad ſubtangentem TM; ergo ratio IF:EF ultimo fit æqualis rationi EM:TM. Methodum applicemus parabolæ, cui ſit æquatio $ax = yy$; ergo aucta $x$ quantitate $dx$, & $y$ proportionali quantitate $dy$, nova orietur æquatio $ax + adx = yy + 2ydy + dy^2$, ex qua ſi prima dematur, fiet $adx = 2ydy + dy^2$, ſive $dy:dx :: a : 2y + dy$; atqui, quum $dy$ evadit minor quacumque data, ratio $a : 2y + dy$ ultimo fit æqualis rationi $a : 2y$; ergo $dy:dx$ ultimo eſt, ut $a : 2y$; atqui $dy:dx$ ultimo eſt ut $y$ ad ſutangentem MT; ergo $a : 2y :: y : MT$; ergo $MT = \frac{2yy}{a} = 2x$ ex æquatione parabolæ. Igitur ſubtangens MT dupla eſt abſciſſæ, quod a nobis quoque in tomo ſuperiore eſt demonſtratum.

12. Novi veteres geometras, poſtquam hac methodo curvæ tangentem determinaſſent, alia via conſueviſſe demonſtrationem proponere. Nempe ex puncto E, ubi curva tangenda erat, ducebant ordinatam ME, cui ex utraque parte ducebant parallelas ſecantes curvam, & eam, quæ probanda erat tangens. Si intercepta inter hanc lineam & abſciſſam foret utrobique major, aut utrobique minor ordinata, colligebant, ductam lineam eſſe tangentem. In primo caſu curva eſt concava abſciſſis, in ſecundo convexitatem obvertit. Methodus hæc genuina eſt, quotieſcumpue curva ad utramque plagam extendatur, & ſit ad eamdem partem cava; ſed ſi in eo puncto curva aut deſinat, aut flexum habeat contrarium, methodus deficit. Quare eam anteponendam cenſeo methodum, quæ utititur ratione, cui ultimo ſit æqualis ratio differentiæ abſciſſæ ad differentiam ordinatæ, quia nunquam eſt ulli exceptioni obnoxia. Sed methodus quantitatum aut rationum qualibet data minorum, tam ſæpe a veteribus adhibita, ſatis ſuperque eſt explicata. Nunc quomodo ad rem noſtram traducenda ſit, accurate eſt declarandum.

13. Methodus duabus conſtat veluti partibus. In prima parte oſtenditur, duas quantitates, aut rationes ita ad ſeſe invicem accedere, ut earum differentia evadat minor quacumque data. In illis quantitatibus, quarum major imminui, aut minor augeri poteſt, prout libet, res palam eſt, neque indiget probatione; ita accidit duabus abſciſſis AM, AN, (*Fig.* 4) quarum alteram minuere poſſum ultra quoſcumque limites. In aliis, quæ ad ſeſe certa quadam lege accedunt, res non eſt ſupponenda ſed demonſtranda; quod nos præſtavimus

in

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in quadratura parabolæ Archimedæ, ubi triangula, quæ ſucceſſive ſegmentis inſcribuntur, ita ad parabolam accedere oſtendimus, ut eorum ſumma differat a ſpatio parabolico quantitate minore quacumque data. In altera methodi parte revocato theoremate, quod initio per deductionem ad abſurdum probatum eſt, deducitur, eas quantitates, aut rationes ultimas fieri æquales. Adverte, non aſſumi tamquam æquales eas quantitates, aut rationes, quibus conveniat differentia aliqua, quæcumque ſit; ſed dumtaxat ex eo, quod differentia minor fiat quacumque data, ſine paralogiſmi periculo demonſtratur, differentiam ultimo evaneſcere, nullamque fieri, & quantitates, rationeſve ultimo vere eſſe æquales. Itaque æqualitas hujuſmodi obtinetur tantum in limite extremo, ad quem quantitates, aut rationes ultra quamcumque menſuram accedunt, licet illum fortaſſe numquam attingant. In demonſtrationibus igitur, quæ hac methodo peraguntur, nulla contemnitur quantitas neque magna, neque parva, ſed quantitates, rationeſque ibi ſolummodo dicuntur æquales, ubi nulla omnino ſit earum differentia.

14. Novi equidem, quantitates in limite ſæpiſſime nullas fieri. Nihilo tamen minus limitis proportionem utile eſt conſiderare, quia quantitates aliæ non nulleſcentes in limite, quæ eamdem proportionem ſervabant, in limite limitis proportionem acquirunt. Hoc clariſſime apparet in determinatione tangentis. Namque duæ rationes IF : FE, GF : FE (*Fig.* 4) differunt quidem inter ſe, ſed ita ut differentia fiat minor quacumque data; ergo in limite æquales erunt. At ubinam habetur limes? nimirum in puncto E, coeuntibus cum hoc punctis G, I, F. At in hac hypotheſi lineæ omnes GF, IF, EF nullæ omnino ſunt. Hoc quidem non inficior. Verumtamen LH, KH, EH prioribus proportionales, in limite non evaneſcunt, ſed finitæ eſſe perſeverant, atque in hoc rationes duæ LH : HE, KH : HE veriſſime æquales fiunt. Itaque qui ſpectat limitis proportionem etiam in quantitatibus prorſus evaneſcentibus, determinat proportionem limitis omologarum quantitatum, quæ in limite minime nulleſcunt, ſed finitæ eſſe perſeverant. Quod methodi artificium maxime utile eſt, quia elegantiam, & brevitatem demonſtrationibus addit, neque figuras implicat novis lineis, quæ cum evaneſcentibus eamdem proportionem conſervent. Hoc apparet in determinatione tangentis, quæ dependet a proportione, quam in limite habent rectæ IF, FE, ut antea declaravi. Quia non video, me demonſtrationem eodem modo perficere, ſi introducam novas lineas KH, HE prioribus proportionales, & in limite, in quo iſtæ non evaneſcunt, earum rationem inveniam; ſed brevius accidet, & elegantius, ſi utar ratione limitis, ad quem IF, FE magis magiſque accedunt, neque attingunt niſi evaneſcentes.

15. Jam vero explicandum eſt, quid ſint differentiæ, fluxiones, elementa, quantitates infiniteſimæ, quibuſcumque tandem vocibus appellentur. Per hæc nomina nihil aliud intelligas volo, niſi differentias quibuſcumque datis minores a veteribus uſurpatas, ex quibus æqualitates, & proportiones ultimæ, quæ valent in limite, tuta conſequtione deduci poſſunt. Verum ne ad ſatietatem, & faſtidium repetere cogamur; quantitates iſtæ, aut rationes non differunt inter ſe niſi differentia minore quacumque data; ergo ultimo, & in limite æquales fiunt; brevitati methodi conſulentes utimur compendio, & limitis æqualitatem aſſignamus quantitatibus, aut rationibus præditis differentia quacumque data minori, non quod judicemus quantitates, aut rationes æquales eſſe poſſe, donec conſervent differentiam aliquam quantumvis minimam; ſed quod certo

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ſciamus, differentiam ultimam, & in limite nullam fieri, & ibi quantitates, rationeſque veriſſime æquales eſſe. Quod artificium nihil detrahit evidentiæ, & perſpicuitati demonſtrationis, quæ quum unice reſpiciat caſum limitis, ultimis quantitatum æqualitatibus, & proportionibus indiget. Hæ autem obtinentur, ſi differentiæ quacumque data minores omittantur, & tamquam nullæ habeantur. Fac igitur caveas, ne methodo infiniteſimorum, aut fluxionum hanc injuriam inferas, quod ipſa contemnat differentias, quæ licet exiguæ, tamen nullæ non ſunt, & quod ea pro æqualibus habeat, quibus convenit aliqua differentia. Nihil, hac methodo utentes, nihil omnino contemnimus. Etenim dum dicimus æquales quantitates, & rationes, quarum differentia eſt infiniteſima, aut minor quacumque data, non intelligimus eas æquales eſſe, donec earum differentia eſt aliqua, ſed in limite, ubi differentia omnino nulla eſt, quæ ſolumodo contemnitur. Quapropter negligere differentias infiniteſimas, ſeu quibuſcumque datis minores, & pro nullis habere, nihil eſt aliud, quam limitis veriſſimas æqualitates, & proportiones determinare. Quæ quum ita ſint, jure optimo Cel. Nevvtonus ſpectavit fluxiones, aut quantitates infiniteſimas non ut determinatas, ſed ut evaneſcentes, aut naſcentes. Nam ſi ſpectentur aut poſtquam evanuerint, aut antequam natæ fuerint, quum ſint omnino nullæ, earum proportio prima aut ultima, quæ valet in limite, determinari nullo modo poteſt. Si vero ſpectentur ut naſcentes, aut evaneſcentes, hoc eſt dum ſunt quaibet data minores, theoremata ſuppetunt tutiſſimam methodum determinandi earum primam, aut ultimam proportionem, quæ nuſquam habetur niſi in limite.

16. His clariſſime, ut mihi videtur, explicatis, declarandum eſt, quomodo oriantur diverſi infiniteſimorum ordines, quos hæc methodus maximo cum fructu advocat, & uſurpat. Hoc non obſcure palam facit methodus antea tradita determinandi tangentes. Etenim iiſdem poſitis, quæ antea, conſtat fore HK : KL :: FI : IG; (*Fig.* 4) ergo quum KL, FI evadant infiniteſimæ, ſeu minores quibuſcumque datis, erit GI quarta proportionalis poſt HK, quæ inter datas numeratur, & duas infiniteſimas LK, GI, quas dicimus infiniteſimas primi ordinis. Vocanda igitur erit GI infiniteſima ordinis ſecundi. Jam vero ſicuti LH, HK, quæ non differunt niſi quantitate infiniteſima, æquales in limite cenſendæ ſunt; ita GF, IF infiniteſimæ, quæ non differunt niſi quantitate infiniteſima ſecundi ordinis, hoc eſt infiniteſima relate ad ipſas, eodem jure æquales erunt in limite.

17. Sed diverſos infiniteſimorum ordines clarius deducamus ex parabolarum familia. Sit primo parabola apolloniana, ALC (*Fig.* 5) cujus axis AB, tangens AE, & cujus æquatio, ſumptis abſciſſis $= x$ in tangente, & ordinatis parallelis axi, ſit $ay = xx$. Accipiatur infiniteſima AD, & agatur ordinata DL. Erit ex natura parabolæ $a : AD :: AD : DL$; ergo DL tertia proportionalis poſt finitam & infiniteſimam primi ordinis infiniteſima ſecundi ordinis eſt appellanda. Similiter ducta parabola cubica AMC, cujus æquatio eſt $a^2 y = x^3$, quæ ſecet DL in M, erit $a :: AD :: \frac{AD^2}{a} :: DM$; ſed AD eſt primi ordinis infiniteſima, $\frac{AD^2}{a}$ ſecundi; ergo DM ad tertium ordinem ſpectabit. Eodem modo DN ordinata parabolæ quadrato quadraticæ æquationis $a^3 y = x^4$, erit infini-

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finiteſima quarti; enim $a :: AD :: \frac{AD^2}{a} :: \frac{AD^3}{a^2} :: y$; & DO, quæ ordinatur parabolæ quintæ poteſtatis, erit infiniteſima quinti ordinis. Adverte, ducta recta AC ſecante biſariam angulnm rectum A, in quam incidat DM prodncta in P, fore continue proportionales DP, DL, DM, DN, DO &c. eſt enim $DP = AD$, $DL = \frac{AD^2}{a}$, $DM = \frac{AD^3}{aa}$, $DN = \frac{AD^4}{a^3}$ atque ita deinceps.

18. Ne tamen putes, hos ordines ita definitos eſſe, ut inter hos nulli exiſtant ordines medii; quin immo multi exiſtunt ultra quemlibet numerum, & limitem. Eornm autem gradns indicatur ab exponente $m$ parabolæ $a^{m-1}y = x^m$, exiſtente $m$ numero fracto, & majore qnam nnitas. Itaque ſi $m$ medius fuerit inter 1, & 2, ut $\frac{3}{2}$, $\frac{4}{3}$, ordinata infiniteſima erit media inter primum ordinem, & ſecnndum, & ordo eo erit major, quo major erit numerus $m$; ſi $m$ medius fuerit inter 2, 3, ordinata infiniteſima erit ordinis medii inter ſecnndum, & tertium, atque ita deinceps. Adde inter infiniteſimum primi, & finitnm medios quoque eſſe ordines tam multos, ut omnem numerum, & limitem excedant. Hoc palam fiet, ſi eaſdem par bolas non ad tangentem, ſed ad axem referas. Diviſo angulo recto BAE (*Fig. 6*) biſariam recta AC, ſumptaque $AE = EC = a$, deſcribantur parabolæ æquationum $ax = yy$, $a^2x = y^3$, $a^3x = y^4$ &c. ſumptis abſciſſis $= x$ in axe AE, & ſumpta AD infinit ſima, quæ dicitur primi ordinis, excitetur normalis DPO. Exiſtente AMC parabola conica æquationis $ax = yy$, erit DM media proportionalis inter finitam $= a$, & infiniteſimam primi ordinis AD; adeoque erit infiniteſima ordinis $\frac{1}{2}$. Similiter DN, quæ deſinit in parabolam æquationis $a^2 x = y^3$, erit ordinis $\frac{1}{3}$; DO ordinata parabolæ æquationis $a^3x = y^4$ erit ordinis $\frac{1}{4}$, atque ita deinceps. Inter hos alii pariter innumeri exiſtunt ordines, quos determinant parabolæ æquationum $a^{\frac{m-1}{m}} x^{\frac{1}{m}} = y$, exiſtente $m$ medio inter duos numeros naturales; ordo enim determinatur a fractione $\frac{1}{m}$; ut ſi $m = \frac{5}{3}$, ordo erit $\frac{3}{5}$, qui medius eſt inter 1 & $\frac{1}{2}$; ſic de aliis. Adverte, arbitrarium omnino eſſe, conſtitnere in primo ordine potius AD, quam DM, aut aliam quamcumque; ſed ſpectata una tamquam primi ordinis, ordo reliquarum eſt definitus.

19. Quæ dicta ſunt de infiniteſimis, opportune transferenda ſunt ad infinita, quo nomine intellige quantitates majores quibuſcumqve datis, per quas fit tranſitus ad eas, quæ valent in limite, proportiones ultimas inveniendas. Quam ob rem ſit hoc lemma. Quantitates, & quantitatum rationes, quarum data ſit differentia, ſi ita augeri poſſint, ut evadant quibuſcumque datis majores, ultimo fiunt æquales. Quantitates, aut rationes creſcentes ſint $M$, $N$, quarum diffe-

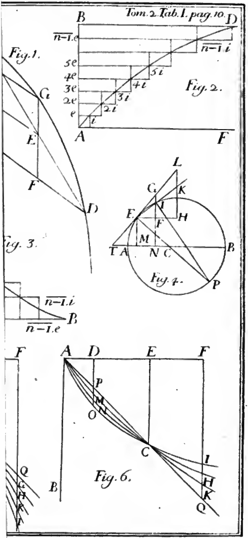
differentia data ſemper ſit $= D$. Accipe datam, & conſtantem $= m$, & fac ut $M : D :: m : d$. Manifeſtum eſt, decreſcere $d$ creſcente $M$, & quidem proportionaliter; ergo ita creſcente $M$, ut evadat major quacumque data, ita minuetur $d$, ut fiat quacumque data minor; igitur quum duæ quantitates $m$, $m \pm d$ ita ad ſeſe accedant, ut earum differentia minor fiat quacumque data, ultimo ſunt æquales; ergo etiam quantitates proportionales $M, M \pm D$ ultimo æquales fiunt; atqui $M \pm D = N$; ergo $M, N$ præditæ differentia data $D$, ſi fiant majores quibuſcumque datis, ultimo ad æqualitatem perveniunt. Adverte lemma hoc valere eodem modo, ſi creſcentibus $M, N$ non ſolum decreſcat $D$, (quod per ſe evidens eſt) ſed etiamſi creſcat, dummodo creſcendo datam aliquam quantitatem non poſſit ſuperare. Nam ſi $M, N$ diferrent differentia $C$, ultimo æquales evaderent; ergo potiori jure æquales fient, ſi habeant differentiam minorem, quæ ſucceſſive accedat ad $C$. Per hujuſmodi lemma fit, ut quemadmodum per quantitates, & rationes quibuſcumque datis minores determinantur proportiones ultimæ, quæ valent in limite continue decreſcentium, ita per quantitates quibuſcumque datis majores proportiones ultimæ determinentur, quæ valent in limite continue creſcentium. Verumtamen illud intercedit diſcrimen inter duos limites extremos, quod decreſcentium limitem aliquo modo licet aſſequi, non quod quantitates fiant individuæ, ſed quod evaneſcant prorſus, nullæque evadant; ſed limitem creſcentium numquam aſſequimur, quia tametſi quantitates ultra quemlibet datum terminum tranſire poſſint, tamen fieri nequit, ut fiant abſolute infinitæ. Attamen accidit non raro, ut hujuſce limitis proportiones in uſum veniant, ut apparet in duabus lineis rectis ſeſe interſecantibus, in quibus ſi punctum concurſus recedat in infinitum, eæ proportiones exurgunt, quæ reapſe locum habent in parallelis.

20. In infinitis diverſi ordines per parabolas eodem modo diſtinguentur, ac in infiniteſimis. Productis rectis A E, A C, (*Fig. 5*) producantur item parabolæ A L C, A M C &c.; tum acceptæ infinitæ A F excitetur perpendicularis ſecans A C in Q, parabolas vero in G, H &c. Quando eſt $a : AF = FQ :: FQ : FG$, & A F, ſeu F Q eſt infinita primi ordinis, erit F G infinita ſecundi. Item $a : FQ :: \frac{FQ^2}{a} : FH$, quæ erit infinita ſecundi ordinis, atque ita de reliquis. Ordines inter hos medios præbebunt parabolæ mediæ, perinde ac in infiniteſimis. Eodem modo determinantur ordines medii inter finitum, & infinitum. Nam productis rectis A E, A C, (*Fig. 6*) & parabolis A O C, A N C &c., infinitæ abſciſſæ A F ordinetur F I Q. Liquet F K ordinatam parabolæ apollonianæ eſſe infinitam ordinis $\frac{1}{2}$, F H ordinatam parabolæ cubicæ infinitam eſſe ordinis $\frac{1}{3}$; atque ita de aliis. Ordines inter hos medii a mediis parabolis definientur.

21. Illud curandum eſt ſedulo, ut ad determinandas limitum proportiones non negligantur niſi illæ quantitates, quæ præ illis quæ comparantur, minorem rationem habent quacumque data. Ut ſi comparanda veniant elementa ordinis ſecundi infiniteſimorum, negligi non debent niſi elementa ordinis tertii, aut ſaltem ordinis ſuperantis ſecundum; contra ſi comparentur duo infinita ſecundi ordinis, non ſunt omittenda niſi infinita ordinis aut primi, aut qui ſit infra ſecundum. Neque nos hoc adnotaviſſe ſuperfluum putes; namque non raro Auctores etiam

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non obſcuri nominis in hoc paralogiſmi genus lapſi ſunt, ut poſtquam probaverint, quantitatem aliquam infiniteſimam eſſe, ſtatim eam neglexerint, licet datam haberet rationem cum quantitatibus, quæ ad problematis ſolutionem inter ſe erant comparandæ. Quamobrem plurimum intereſt ad paralogiſmi periculum fugiendum tuta poſſidere criteria, quibus cognoſcamus, ad quem ordinem quantitates pertineant. Hujuſmodi criteria accurate propoſuit Hieronymus Saladinus edito libello, quem inſcripſit *Elementa Geometriæ Infiniteſimorum*. Quæ in hoc magis elementaria ſunt, ſequenti capite complectemur. Reliqua opportunis in locis diſponemus.

## CAPUT SECUNDUM.

### De elementis Geometriæ Infiniteſimorum.

Expoſita præcedenti capite genuina notione quantitatum infiniteſimarum, ordo, & neceſſitas poſtulat, ut proponamus prima, eaque precipua elementa Geometriæ earumdem quantitatum, ſcilicet illius ſcientiæ, quæ ſpectat relationes magnitudinum infiniteſimarum; hæc enim ſcientia, quum proprias habeat regulas, & præcepta, quæ a vulgari geometria aliquando non parum diſcrepant; valde neceſſaria videtur ad evitandos paralogiſmos, ubi per methodum aut ſyntheticam, aut analyticam infiniteſimorum proprietates expenduntur. Totam autem rem expendimus ſequentibus theorematibus.

1. Propoſitio prima. Si duæ quantitates AB, DE (*Fig.* 1) differant a duabus AC, DF infiniteſima quantitate, ratio AB ad DE non differt a ratione AC ad DF. Dem. AC:DF eſt in ratione compoſita AC:AB, AB:DE, & DE:DF; ſed rationes AC:AB, & DE:DF non differunt a ratione æqualitatis; ergo ratio AB:DE non differt a ratione AC:DF.

2. Scholium. Quantitas infiniteſima *B* ex. gr. poteſt referri ad aſſignabilem *A* duobus modis. Primum quatenus magnitudinem auget vel minuit, ſi ipſi addatur, vel dematur. Hic vero quantitas *B* pro nihilo habetur, quia ex ejus additione, vel detractione nihil mutatur proportio magnitudinis *A* ad aliam quamlibet aſſignabilem, quum mutatio minor ſit quacumque mutatione aſſignabili; atque hinc patet nihil referre utrum magnitudo *B* ſit major minorve modo ſit infiniteſima. Deinde poteſt referri quantitas *B* ad quantitatem *A*, quatenus in ipſam infinities ingreditur, & infinities repetita ipſam conſtituit. Quod nihil ſane aliud eſt, niſi conſiderare proportionem, quam habet *A* ad *B*; quæ proportio utique infinita eſt, nec ullus aſſignari poteſt tantus numerus, qui ipſam exprimat. Hic vero plurimum refert, utrum quantitas *B* ſit major, vel minor; etenim quamvis proportio finitæ *A* ad infiniteſimum *B* ſit infinita, tamen erit duplo major, ſi *B* fuerit duplo minor, & omnino tanto major erit proportio, quanto quantitas *B* minor erit.

3. Ex his illud ſequitur; ſi duabus quantitatibus aſſignabilibus AB, CD (*Fig.* 2) addantur infiniteſimæ duæ qualeſcumque ſint BX, DY, proportio aſſignabilium nihil mutabitur, ut demonſtratum eſt in Theoremate, & ipſarum magnitudines eædem æſtimabuntur quæ antea. Licebit ergo ponere AX:CY::AB:CD; nihilque impediet, quominus euclideis inſiſtens demonſtrationibus, & inverten-



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*do* argumenteris; ergo CY : AX :: CD : AB : & alternando; ergo AX : AB :: CY : CD : & componendo ; ergo AX + CY : CY :: AB + CD : CD ; & dividendo; ergo AX — CY : CY :: AB — CD : CD. Idque perpetuo licebit, donec quantitates tantum aſſignabiles AB, AX, CD, CY, AX — CY, AB — CD in proportionem venerint.

4. Tametſi neque invertendo, neque alternando, neque componendo ex illa proportione AX ; CY :: AB : CD nullæ aliæ quantitates præter quam aſſignabiles in proportionalitatem venire poſſint ; tamen dividendo prodire poterunt, & in proportionalitatem venire duæ infiniteſimæ BX, DY. Quod tum accidet, quum poſito AX : AB :: CY : CD dividendo argumenteris; ergo AX — AB : AB :: CY — CD : CD ; ſive BX : AB :: DY : CD ; vel quod eodem recidit AB : BX :: CD : DY. Quod ſi quantitates infiniteſimæ BX, DY in proportionalitatem venerint; argumentationem illam, quam dividendo inſtituiſti, & unde infiniteſimæ illæ prodierunt, pro falſa habebis; niſi forte minimæ quantitates BX, DY poſitæ ab initio fuerint proportionales quantitatibus AB, CD, vel a duabus, quæ ſint ijſdem AB, CD proportionales, differant quantitate infiniteſima ordinis ſuperioris reſpectu ejus ordinis, in quo ſunt ipſæ quantitates BX, DY. Etenim ſi BX, DY ſint qualeſcumque, erit etiam qualiſcunque proportio AB, ad BX, & qualiſcunque pariter erit proportio CD ad DY, & poterunt proportiones hæ duæ diverſæ eſſe; nec tuto poni poterit AB, BX :: CD, DY.

5. Quare ſi in triangulo ABC (*Fig.* 3) ducta ſit linea recta DE abſcindens partes infiniteſimas quaſcunque AD, AE, licebit ubique ponere AB : AC :: DB : EC, & AB : DB :: AC : EC ; neque hinc tamen licebit dividendo argumentari, ergo AD : DB :: AE : EC, niſi ab initio poſita fuerit linea recta DE parallela rectæ BC, vel diſcrepans a vero paralleliſmo quantitate infiniteſima, abſcindens lineolas DA, EA eandem habentes inter ſe proportionem, quam habent lineæ rectæ AB, AC.

6. Propoſitio ſecunda. Poſita ſerie finita quantitatum ejuſdem ordinis continue proportionalium AB, CD, MN &c., (*Fig.* 4) cujus duo proximi termini AB, CD differant quantitate infiniteſima SD ; dico, omnes proximos terminos finitæ ſeriei differre quantitate infiniteſima ejuſdem ordinis. Dem. Quoniam MN eſt ad CD, ut CD ad AB ; erit etiam dividendo MN — CD ad CD, ut CD — AB ad AB ; hoc eſt NL erit ad CD, ut DS ad AB, & permutando NL ad DS, ut CD ad AB ; ſed CD & AB ſunt ejuſdem ordinis ; ergo differentiæ NL, DS ſunt ſimiliter ejuſdem ordinis. Quumque eadem demonſtratio repeti poſſit per omnes terminos finitæ ſeriei, patet, differentiam duorum terminorum, qui proximi ſunt, eſſe infiniteſimam ejuſdem ordinis Q. E. D.

7. Propoſitio tertia. Si ſint plures quantitates *A*, *B*, *C*, &c. ita determinatæ, ut *B* ſit infiniteſima reſpectu *A* ; *C* ſit infiniteſima reſpectu *B*, & ſic deinceps ; dico, *C* eſſe infiniteſimam reſpectu *A* ejus ordinis, qui reſultat ex ſumma ordinis *C* reſpectu *B*, cum ordine *B* reſpectu *A*. Dem Quantitas *C* ut evadat ejuſdem ordinis, ac eſt quantitas *A*, tranſire debet per tot infinita, quot ſunt unitates in numero ordinis *C* reſpectu *B*, & in numero ordinis *B* reſpectu *A*. Ergo numerus ordinis *C* reſpectu *A* habetur, ſi addatur numerus ordinis *C* reſpectu *B* numero ordinis *B* reſpectu *A* ; adeoque ordo *C* reſpectu *A* reſultat ex ſumma ordinis *C* reſpectu *B*, & ordinis *B* reſpectu *A*. Q. E. D.

8. Propoſitio quarta. Si ſint plures quantitates *A*, *B*, *C* &c. infiniteſimæ cujuſcumque ordinis; erit factum omnium quantitatum prædictarum infiniteſimum

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mum ejus ordinis, qui reſultat ex ſumma ordinum quantitatum $A, B, C$ &c. Dem. Sumantur totidem quantitates aſſignabiles quæcunque $a, b, c$ &c. quot ſunt quantitates infiniteſimæ $A, B, C$ &c.; tum fiat $A$ ad $S$, ut $b$ ad $B$; $S$ ad $X$, ut $c$ ad $C$ &c. Factum omnium quantitatum aſſignabilium $a, b, c$ &c. eſt ad factum omnium quantitatum $A, B, C$ &c. in ratione compoſita $a$ ad $A$, $b$ ad $B$, $c$ ad $C$ &c. ſeu in ratione compoſita $a$ ad $A$, $A$ ad $S$, $S$ ad $X$ &c. ſeu in ratione $a$ ad $X$. Sed $X$ reſpectu ad aſſignabilem $a$ eſt infiniteſima ejus ordinis, qui reſultat ex ſumma ordinis $X$ reſpectu $S$, & $S$ reſpectu $A$, & $A$ reſpectu $a$, per 3. prop.; ergo factum omnium quantitatum infiniteſimarum $A$, $B$, $C$ &c. eſt ejus ordinis, qui reſultat ex ſumma ordinum omnium quantitatum $A, B, C$ &c. Q. E. D.

9. Propoſitio quinta. In circulo ſi anguli ad centrum vel ad peripheriam ſunt infiniteſimi cujuſcunque ordinis; erunt etiam arcus quibus inſiſtunt infiniteſimi ejuſdem ordinis, & viceverſa. Dem. Anguli ſive ad centrum ſive ad peripheriam proportionales ſunt arcubus quibus inſiſtunt, & viceverſa, ſed angulis finitis reſpondent arcus finiti, & viceverſa; ergo infiniteſimis infiniteſimi.

10. Propoſitio ſexta. In circulo arcus BC ( *Fig.* 5 ), & chorda BC ſunt ejuſdem ordinis. Dem. Sumatur arcus BC in peripheria circuli quoties fieri poteſt, & arcubus ducantur ſubtenſæ, quæ omnes æquales erunt chordæ BC. Erit arcus BC ad ſummam omnium arcuum in peripheria ſumptorum, ut ſubtenſa BC ad ſummam omnium ſubtenſarum reſpondentium; ſed ſumma omnium arcuum, & ſumma omnium ſubtenſarum ſunt ejuſdem ordinis; ergo etiam arcus, & chorda.

11. Corollarium primum. Ducta ex centro A recta AM ſecante biſariam arcum BC in M, hæc biſecabit quoque chordam in R ad angulos rectos; adeoque CR erit ſinus rectus anguli CAM; hinc ſinus rectus & arcus cujuſcunque anguli ſunt ejuſdem ordinis.

Corollarium ſecundum. Anguli ſive ad centrum ſive ad peripheriam ſunt ejuſdem ordinis ac ſunt arcus per hanc prop.; ſed arcus ſunt ejuſdem ordinis, ac ſunt ſinus recti per coroll. 1. hujus prop.; ergo anguli ſunt ejuſdem ordinis, ac ſunt ſinus recti.

Corollarium tertium. Hinc in triangulo, rectangulo ac proinde in triangulo quocumque, cujus anguli ſunt omnes finiti, latera ſunt ejuſdem ordinis.

12. Propoſitio ſeptima. In triangulo rectangulo, & acutangulo unus duntaxat angulus eſſe poteſt infiniteſimus; in triangulo vero obtuſangulo duo anguli eſſe poſſunt infiniteſimi, & quidem diverſorum ordinum. Dem. primæ par. Si duo anguli eſſent infiniteſimi in triangulo rectangulo, & acutangulo, quum tertius ex hypotheſi ſit rectus, aut acutus, hinc ſumma trium angulorum in triangulo non æquaret duos rectos, quod eſt abſurdum. Dem. 2. par. In triangulo obtuſangulo duo anguli infiniteſimi cum angulo obtuſo poſſunt conficere ſummam duorum rectorum.

13. Propoſitio octava. In triangulo acutangulo ſcaleno ABC (*Fig.6*) ſit angulus B infiniteſimus primi ordinis, duorum reliquorum angulorum A, & C minor ſit C. Dico primum, differentiam anguli C ab angulo recto eſſe infiniteſimam primi ordinis. Dico deinde, differentiam anguli A ab angulo recto poſſe eſſe infiniteſimam cujuſcumque ordinis. Dem. pri. par. Ex angulo B demittatur normalis BD in baſim AC, quæ normalis cadet intra triangulum ABC; erit igitur triangulum ABC diviſum in duo triangula rectangula ADB, CDB. Ergo angulus A cum angulo ABD æqualis eſt angulo C cum angulo CBD; ſed ex hypotheſi an-

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angulus C minor eſt angulo A; ergo angulus C B D major eſt angulo A B D; adeoque major eſt dimidio anguli A B C; hinc angulus C B D eſt infiniteſimus primi ordinis; ſed angulus C B D eſt differentia anguli C ab angulo recto; ergo angulus minor C differt ab angulo recto quantitate infiniteſima primi ordinis, Q. E. P. Pars ſecunda patet per ſe.

Corollarium. Hinc anguli A, & C poſſunt differre quantitate infiniteſima cujuſcunque ordinis incipiendo a primo.

14. Propoſitio nona. In triangulo rectangulo B A C (*Fig. 7.*) ſi unus ex angulis ex. gr. C ſit infiniteſimus primi ordinis, erit latus oppoſitum B A infiniteſimum ejuſdem ordinis; reliqua vero latera A C, B C infiniteſimum angulum comprehendentia differunt quantitate infiniteſima ſecundi ordinis. Et viceverſa ſi unum ex lateribus angulum rectum continentibus ex. gr. B A ſit infiniteſimum primi ordinis, angulus ipſi oppoſitus C erit infiniteſimus ejuſdem ordinis. Dem. pri. par. In Triangulo rectangulo B A C ſit angulus A rectus, latus oppoſitum ipſi erit B C, quod accipio pro ſinu toto; erit B A ſinus rectus anguli C; ergo latus B A eſt ejuſdem ordinis, cujus eſt angulus C per Cor. 2. prop. *6.* Q. E. D. Dem. ſec. par. Centro C intervallo C A abſcindatur C M æqualis C A; erit B M differentia laterum A C, B C; quum autem rectangulum B M in $\overline{BC + CA}$ æquetur quadrato B A, erit ut A C + C B ad B A, ita B A ad B M; ergo ordo rectæ B M reſpectu B A idem eſt ac ordo rectæ B A reſpectu B C; adeoque B M reſpectu B A eſt infiniteſima primi ordinis; ergo B M reſpectu B C eſt infiniteſima ſecundi ordinis Q. E. S. Pars tertia patet ex corollario 2. Prop. VI.

Scholium. In hoc capite deinceps quantitas infiniteſima, quæ eſt baſis, ut ita dicam, hypotheſis, cauſſa claritatis accipietur primi ordinis, licet demonſtratio ſeſe extendat ad quamcumque hypotheſim infiniteſimorum.

15. Propoſitio decima. In triangulo acutangulo ſcaleno B A C (*Fig. 8.*) poſito angulo C infiniteſimo primi ordinis, erit latus oppoſitum infiniteſimum ejuſdem ordinis; reliqua vero latera B C, A C differre poſſunt quantitate infiniteſima cujuſcunque ordinis, incipiendo a ſecundo; & viceverſa ſi latus unum B A ſit infiniteſimum primi ordinis, erit angulus oppoſitus C infiniteſimus ejuſdem ordinis. Dem. pri. par. Ponatur angulus A minor angulo B. Quoniam angulus A ex hypotheſi eſt finitus, hinc ducta ex angulo majori B normale B R in latus C A, erunt in triangulo rectangulo A B R ſinus totus A B & ſinus rectus B R ejuſdem ordinis per corol. 2. prop. *6.*; ſed ſinus rectus B R eſt infiniteſimus primi ordinis per prop. *9.*; ergo latus A B erit infiniteſimus primi ordinis, Q. E. P. Dem. ſec. par. Angulus acutus A differt a recto per angulum A B R infiniteſimum primi ordinis per prop. 8.; ergo in triangulo rectangulo A B R recta A R eſt infiniteſima reſpectu A B ejuſdem ordinis, cujus eſt A B reſpectu A C per pri. par. hujus prop., adeoque A R eſt reſpectu A C infiniteſima ſecundi ordinis; ſed C B excedit C R quantitate infiniteſima ſecundi ordinis per prop. *9.*; ergo latera A C, C B differre non poſſunt niſi quantitate infiniteſima ſecundi ordinis, vel alterius ordinis ſuperioris Q. E. S. Pars tertia patet ex demonſ. pri. par.

16. Propoſitio undecima. In triangulo obtuſangulo B A C (*Fig. 9.*) ſi ſolus angulus C ſit infiniteſimus, erit latus B A oppoſitum infiniteſimum ejuſdem ordinis; reliqua vero latera B C, C A differunt quantitate infiniteſima ordinis ipſius anguli C; & viceverſa ſi unus ex angulis acutis ex. gr. B ſit finitus, & latus tantum, quod continetur inter angulum obtuſum A & acutum finitum B ſit infiniteſimum, erit angulus C oppoſitus infiniteſimus ejuſdem ordinis. Dem. pri. par. Ex angulo obtuſo A demittatur normalis A R in latus B C; erunt anguli

B, et

B, et BRA, RAB finiti; hinc AB, BR, RA ſunt ejuſdem ordinis, per corol. 3. prop. *6.*; ſed AR eſt ejuſdem ordinis cujus eſt angulus C per *9.* prop. Ergo BA, BR ſunt ordinis ejuſdem cujus eſt angulus C; Q. E. P. Par. ſecun. BC excedit CR quantitate infiniteſima BR ordinis anguli C, ſed CR non differt a CA niſi quantitate infiniteſima ſecundi ordinis per *9.* prop. Hinc (cum hæc ſecunda differentia ſit infiniteſima reſpectu ad primam BR) BC excedit CA quantitatæ infiniteſima ordinis ejuſdem cujus eſt angulus C; Q. E. S. Pars tertia patet ex Dem. pri. par.

17. Propoſitio duodecima. In triangulo obtuſangulo ABC (*Fig.* 10.) in quo angulus unus C eſt infiniteſimus primi ordinis, ſi angulus obtuſus ABC major ſit angulo recto quantitate infiniteſima cujuſcunque ordinis incipiendo ab ordine primo; erit differentia laterum angulum infiniteſimum comprehendentium infiniteſima ſecundi ordinis; ſi vero angulus obtuſus ABC major ſit angulo recto quantitate infiniteſima ordinis inferioris relate ad ordinem anguli C ex. gr. ſi ſit hæc differentia primi ordinis; angulus vero C ſecundi, differentia laterum CA, CB erit infiniteſima ordinis, qui reſultat ex ſumma ordinis anguli C cum ordine differentiæ anguli ABC a recto, ſeu tertii ordinis. Dem. pri. par. Ex puncto B erigatur normalis BS ad BC; erit angulus ABS differentia anguli ABC a recto. Quoniam angulus ABS, vel eſt infiniteſimus primi ordinis vel ordinis ſuperioris per ſuppoſitionem, in triangulo obtuſangulo ASB erit AS reſpectu AB vel infiniteſima ordinis anguli C vel ordinis ſuperioris per prop. 11., adeoque AS reſpectu BC vel erit infiniteſima ſecundi ordinis, vel ordinis ſuperioris per 3. prop. Ergo quum SC excedat BC quantitate infiniteſima ſecundi ordinis, per *9.* prop. AC excedit BC quantitate infiniteſima ſecundi ordinis. Q. E. P. Dem. ſec. par. AS reſpectu AB eſt infiniteſima ordinis primi; AB reſpectu BC eſt infiniteſima ſecundi ordinis; ergo AS reſpectu BC eſt infiniteſima ordinis tertii per 3. prop. Sed SC excedit CB quantitate infiniteſima quarti ordinis; ergo quum hæc infiteſima ſit reſpectu AS, erit differentia inter latera AC, BC infiniteſima ordinis, qui reſultat ex ordine anguli ABS accepto cum ordine anguli C, ſeu tertii. Q. E. S.

18. Scholium. Si triangulum habens unum angulum infiniteſimum fuerit acutangulum ſcalenum, differentia laterum infiniteſimum angulum continentium eſſe poteſt cujuſcumque ordinis ſuperioris incipiendo ab ordine ſecundo. Si fuerit rectangulum, vel obtuſangulum, in quo angulus obtuſus major eſt recto quantitate infiniteſima cujuſcumque ordinis incipiendo ab ordine, in quo eſt angulus infiniteſimus trianguli, qui in hypotheſi eſt primi ordinis, erit differentia laterum ordinis ſecundi. Si fuerit obtuſangulum, in quo angulus obtuſus major eſt recto quantitate infiniteſima ordinis inferioris relate ad ordinem anguli infiniteſimi trianguli, erit differentia laterum ordinis, qui reſultat ex ordine anguli infiniteſimi trianguli accepto cum ordine differentiæ anguli obtuſi a recto. Si tandem angulus ſit finite obtuſus, differentia erit ordinis primi.

19. Propoſitio decimatertia. In triangulo obtuſangulo ABC (*Fig.* 11) ſi duo anguli A, & C ſint infiniteſimi ejuſdem ordinis ex. gr. primi, omnia latera ſunt ejuſdem ordinis; differentia vero lateris AC obtuſo angulo B oppoſiti a lateribus AB, BC obtuſum angulum continentibus eſt infiniteſima ſecundi ordinis; viceverſa ſi in triangulo obtuſangulo ABC latera AB, BC, CA ſunt ejuſdem ordinis, & angulus B differat a duobus rectis quantitate infiniteſima primi ordinis, erunt anguli A, & C infiniteſimi primi ordinis. Dem. pri. par. Ex angulo obtuſo B in baſim AC demittatur normalis BS; hæc reſpectu ad latus BC erit infi-

infiniteſima primi ordinis; ſimiliter BS reſpectu A B eſt infiniteſima primi ordinis per 9. prop.; ergo normalis BS eſt ejuſdem ordinis tum reſpectu BC, tum reſpectu BA; ergo latera BC, BA ſunt ejuſdem ordinis; quod tertium AC ſit ejuſdem ordinis ac ſunt BA, BC per ſe clarum eſt. Patet igitur Q.E.P. Dem. ſec. par. BC differt a CS quantitate infiniteſima ſecundi ordinis; ſimiliter AB differt ab AS quantitate infiniteſima ordinis ſecundi per 9. prop.; ergo BA+BC differt ab AC quantitate infiniteſima ordinis ſecundi Q.E.S. Pars Tertia patet ex prima.

20. **Propoſitio decimaquarta. In triangulo obtuſangulo ABC ſi duo Anguli A & C ſint infiniteſimi, ita ut A ſit infiniteſimus primi ordinis, & C ſecundi; erit latus AB oppoſitum angulo C, infiniteſimum primi ordinis, qui ordo habetur, ſi videlicet ſubtrahatur ordo anguli A ab ordine anguli C. Latera vero BC, CA angulum infiniteſimum ſecundi ordinis comprehendentia differunt quantitate AB. Viceverſa ſi latus BA ſit infiniteſimum primi ordinis, & angulus B differat quantitate infiniteſima primi ordinis a duobus** rectis; erit angulus oppoſitus C infiniteſimus ſecundi ordinis, qui ordo ſcilicet habetur, ſi ſummatur ordo lateris AB cum ordine anguli, quo differt angulus B a duobus rectis. Dem. Pri. Par. Ex angulo obtuſo B in latus AC demittatur normalis BS. Ordo rectæ BS reſpectu BC idem eſt, ac ordo anguli C, ſeu ſecundus per 9. prop.; ergo recta BS ut evadat ejuſdem ordinis, cujus eſt BC, tranſire debet per tot infinita quot ſunt unitates in numero ordinis anguli C, ſeu per duo infinita; recta vero BS reſpectu BA eſt infiniteſima primi ordinis per 9. prop; ergo recta AB ut ſit ejuſdem ordinis, cujus eſt BC, tranſire debet per tot infinita, quot ſunt unitates in differentia ordinis anguli A ab ordine anguli C, ſcilicet per unum infinitum, adeoque erit infiniteſima primi ordinis Q.E.P. Dem. ſec. par. Quoniam angulus A eſt infiniteſimus primi ordinis, rectæ AB, AS differunt quantitate infiniteſima ſecundi ordinis relate ad AB, quemadmodum rectæ BC, CS differunt quantitate infiniteſima quarti ordinis; ergo cum AS, ſeu AB ſit differentia laterum AC, SC, erit etiam differentia laterum AC, BC Q.E.S. Pars tertia patet ex prima.

21. Propoſitio decimaquinta. In triangulo obtuſangulo ABC, in quo ſunt duo anguli A & C infiniteſimi, A ordinis primi, C ſecundi; erit differentia, quæ intercedit inter latus AC obtuſo angulo oppoſitum, & latera AB, BC obtuſum angulum continentia infiniteſima tertii ordinis. Dem. Differentia, quæ intercedit inter AS, & AB, eſt infiniteſima ſecundi ordinis per 9. prop.. Relate ad AB; ſed AB relate ad AC eſt infiniteſima primi ordinis per 14. prop. ergo differentia, quæ intercedit inter latera AS, AB eſt infiniteſima tertii ordinis per 3. prop. Differentia quæ intercedit inter latera BC, CS eſt infiniteſima quarti ordinis per 9. prop. Igitur, quum differentia quarti ordinis ſit infiniteſima relate ad tertium ordinem, erit illa differentia, quæ intercedit inter latus AC, & latera AB, BC, infiniteſima tertii ordinis. Q.E.D.

22. Propoſitio decimaſexta. In triangulo rectangulo, & acutangulo, in quo angulus unus eſt infiniteſimus, reliqui anguli pro æqualibus accipi poſſunt; minime vero in triangulo obtuſangulo, in quo angulus unus eſt infiniteſimus. Pri. pars patet ex natura trianguli rectanguli & ex corollario prop. 8. Altera pars per ſe eſt manifeſta.

23. Scholium. Hæc propoſitio licet per ſe clara, attamen ſedulo perpendenda eſt; multum enim conducit ad evitandos paralogiſmos, dum agitur de quantitatibus infiniteſimis. Etenim innuit, quod licet aliqua figura poſitis aliquibus quantitatibus infini-

infiniteſimis poſſit habere nonnullas proprietates communes cum aliis figuris, nihilominus non ſequitur, illam habere omnes proprietates iſtarum figurarum. Sic ex. gr. licet ex Propoſitionibus 9., 10., 11., ſpretis quantitatibus infiniteſimis, quibus differunt latera angulum infiniteſimnm compræhendentia, ſequatur, latera illa eſſe æqualia, ac proinde triangula in hoc caſu habere præcipuam proprietatem trianguli iſoſcelis: male tamen ex hoc dedneeretur, prædicta triangula habere omnes proprietates trianguli æquicruris, ut videre eſt potiſſimum in triangulo obtuſangulo, in quo angulus unus eſt infiniteſimus. Sic etiam per propoſitionem præſentem in triangulo rectangulo, in quo angulus unus eſt infiniteſimus, reliqui anguli, ſpretis differenriis infiniteſimis, ſunt æquales. Ex hoc tamen colligendum non eſt, triangulum illud habere omnes proprietates trianguli æquicruris, quod ſic demonſtro. Proprietas una trianguli æquicruris eſt, qnod ducta ex vertice in baſim normali, baſis biſecetur; hæc proprietas non verificatur in triangulo rectangulo, in quo angulus unus eſt infiniteſimus. Quapropter omnes proprietates, quæ de aliqua figura, ſpretis infiniteſimis, adſtruuntur, geometrico ratiocinio ſunt demonſtrandæ.

24. Propoſitio decimaſeptima. Si latera BD, BH (*Fig.* 12) rectanguli BA augeantur vel minuantur infiniteſima quantitate BG, BF, dico in primo caſu rectangulum AB augeri rectangulis BD in BF, BH in BG, FB in BG; in altero caſu minui rectangulis BD in BF, BH in BG minus rectangulo FB in BG. Demonſtratio utriuſque patet ex ſola inſpectione figuræ.

25. Scholion primum. Ex ſola inſpectione etiam figuræ ſolidæ, ſi adeſſet, conſtaret differentiale paralIepipedi rectanguli facti a rectis $A$, $B$, $C$ auctis quantitatibus infiniteſimis $S$, $X$, $Z$ eſſe parallelepipeda $BAZ$, $BCS$, $ACX$, $SXC$, $XZA$, $SZB$, $SXZ$. Hinc facile erui poteſt, differentiale facti plurium quantitatum $A$, $B$, $C$, $D$, &c. contineri ſumma productorum ex ſingulis differentialibus quanritatum $A$, $B$, $C$, $D$, &c. in factum omnium quantitatum, excluſa illa, cujus eſt differentiale, una cum ſumma productorum ex omnibus binis differentialibus in factum omnium quantitatum, excluſis illis, quarum ſunt differentialia, & ſic deinceps ex omnibus ternis, quaternis, quinis &c. uſque ad factum omnium differentialium.

26. Scholion ſecundum. Quum autem ſummæ prodnctorumex omnibus binis, ternis, quaternis &c. differentialibus in facta reſpectiva quantitatum aſſignabilium, juxta ſcholion præcedens, ſint infiniteſimæ relate ad ſummam productorum ex ſingulis differenrialibus in factum omnium quantitatum aſſignabilium, excluſa illa, cujus eſt differentiale (poſitis tamen differentiis ejuſdem ordinis); hinc ſummæ illæ relate ad hanc ſpernuntur. Verum aliquando evenit ut ſumma hæc ſit nulla, & in hoc caſu non erit ſpernenda ſumma productorum ex omnibus binis &c.; & ſi hæc pariter ſit nulla, tunc non erit negligenda ſumma productorum ex omnibus ternis &c., & ſic deinceps uſque ad ultimum factum ex omnibus differentialibus. En exemplum, quo declaratur quomodo fieri poſſit, ut factum ex ſingulis differentialibus &c. ſit nullum: latus BD (*Fig.* 13.) quadrati AB minuatur infiniteſima quantitate BG; erit quadratum AB imminutum rectangnlo infiniteſimo HG ordinis GB. Latus LG augeatur infiniteſima quantitate GF æquali BG; rectangulum AG auctum erit rectangulo DF infiniteſimo ordinis GF; ſed rectangulum HG excedit rectangulum GC quadrato GB infiniteſimo relate ad ipſa rectangula. Ergo differentia ordinis GB evadit nulla, & remanet differentia ordinis duplo majoris GB. Verum hæc clarius per analyſim infiniteſimorum cognoſcentur.

27. Propoſitio decimaoctava. Potentiæ eædem rectarum *A*, *B* quæ differunt quantitate infiteſima *L*, & ipſæ differunt quantitate infiniteſima ejuſdem ordinis; & viceverſa. Dem. Pri. Par. Potentia rectæ *A* eſt ad potentiam rectæ *B* ut *A* ad unam *M* ex continue proportionalibus poſt *A*, *B* in ſerie finita; ergo dividendo potentia *A* minus potentia *B* eſt ad potentiam *B*, ut *A* minus *M* ad *M*; ſed *A* minus *M* eſt infiniteſima ejuſdem ordinis *L*; ergo differentia potentiarum eſt infiniteſima ejuſdem ordinis. Q. E. D. Pars ſecunda fere eodem modo demonſtratur.

Corollarium. Hinc differentiæ quadratorum, circulorum, cuborum, ſphærarum habentium radices vel radios, qui differunt quantitate infiniteſima, erunt & ipſæ infiniteſimæ ejuſdem ordinis.

28. Scholion. Hæc propoſitio applicari etiam poteſt poligonis ſimilibus, quorum latera relative differunt quantitate infiniteſima tum ejuſdem ordinis, tum diverſorum ordinum; ſed in hac ultima ſuppoſitione aliqua cautio eſt adhibenda; etenim ad evitandos paralogiſmos juvat recurrere ad quadrata totius peripheriæ poligonorum, vel ad quadrata laterum, quæ differunt quantitate infiniteſima ordinis inferioris. Ratio hujus rei invenitur, ſi poligona in triangula ſimilia dividantur, & ſi revocentur propoſitiones 19., & 20. libri ſexti Euclidis. Hic quoque rurſus adverto, non eſſe præcipiti ratiocinio concludendum, figuras, quæ ſpretis infiniteſimis habent aliquam proporietatem cum aliis communem, habere etiam omnes; ſed prius videndum eſt, num differentiæ illæ infiniteſimæ turbent demonſtrationes, quibus probatur unam ab alia profluere.

29. Appendix. Quoniam rectæ a vero parallelismo declinantes quantitate infiniteſima acceptæ a minus cautis pro parallelis facilem viam ſternunt ad paralogiſmos; hinc propoſueram ſupprimere phraſim illam: *accipi poſſunt ut parallelæ:* contentus dumtaxat monuiſſe, confugiendum eſſe ad doctrinam triangulorum expoſitam, quando hujuſmodi linearum proprietates examinandæ occurrunt. Verum ut clarius paralogiſmorum fontes aperiantur, operæ pretium duxi, in hac appendice de rectis parallellis ea, quæ ſequuntur, adnotare. Parallelismus rectarum A X, D Y (*Fig.* 14) duas conditiones includit. Prima eſt, quod in utramlibet partem quantumvis productæ nunquam concurrant. Altera eſt, quod diſtantiæ duæ ubivis ſumptæ A D, Z E æquales plane ſint. Hinc omnes parallelarum proprietates ducuntur.

30. Declinabunt lineæ rectæ A X, D Y a parallelismo infinite parum, ſi inclinetur A X ſic, ut tranſeat in A T, ſitque angulus X A T infiniteſimus. Quo poſito concurrent A T, D T ad diſtantiam infinitam E T. At intervalla A D, C E different differentia infiniteſima C Z. Cenſebuntur autem A T, D T habere utramque parallelismi conditionem: primam, quia ad nullam aſſignabilem diſtantiam concurrunt: alteram, quia diſtantiæ A D, C E pro æqualibus haberi poſſunt, quum differentia ſit infiniteſima. In hoc autem illud excipe: ſi ſumpta D R infinita, minori tamen quam D T, notaretur intervallum V R; differentia profecto inter A D, & V R eſſet aſſignabilis; neque diſtantiæ A D, V R poſſent haberi pro æqualibus. Quare donec intercepta D E inter intervalla A D, C E erit aſſignabilis, intervalla ipſa A D, C E haberi poterunt pro æqualibus, & tractus A C, D E ſervabunt alteram parallelismi conditionem; ſed ſi intercepta D R ſit infinita, quoniam intervalla A D, V R nequaquam pro æqualibus haberi poterunt, tractus A V, D R carebunt conditione ſecunda parallelismi; primam conditionem ſervabunt, quia eſt angulus T infiniteſimus, ideoque diſtantia R T, ad quam occurrunt A V, D R infinita ſit oportet, ſi cum

R V

RV compararetur. Igitur ſi lineas rectas parallelas ex prima conditione definias, erunt AV, DR parallelæ; ſi ex ſecunda, non erunt. Si quis autem de parallelismo rectarum AV, DR contendere velit, videat ne quæſtionem de nomine instituat.

31. Quod vero ad rem maxime pertinet, ſcire oportet, rectas AV, DR (ſive parallelæ dicantur, ſive non) hoc certe habere, quod ducta ubivis GL, quæ ſecet AV in G, DR in L, anguli alterni DLG, LGT ſunt æquales; nam quum angulus DLG, ut externus, ſit æqualis duobus internis, & oppoſitis LGT, LTG, ſitque LTG infiniteſimus, poteſt DLG æqualis cenſeri angulo LGT. Habent ergo lineæ rectæ AV, DR hanc utique proprietatem, aliaſque, quæ maxime in parallelis celebrari ſolent, quæque non ex æqualitate intervallorum AD, VR pendent, ſed ex ſola infinita parvitate anguli T. Hæc hactenus dixi ponens CZ quidem infiniteſimam, AZ vero, & CE aſſignabiles. Videamus jam quid de parallelismo cenſendum ſit, ſi tres rectæ, CZ, AZ, CE ponantur vel omnes infiniteſimæ, vel tantum aliquæ: idque quatuor, quæ ſequuntur, theoremata expedient.

32. Theorema primum. Si ſit CZ infiniteſima, tum reſpectu AZ, tum reſpectu CE, tractus AC, DE habere cenſebuntur utramque parallelismi conditionem. Primam, quia quum ſit CZ infiniteſima reſpectu AZ, erit infiniteſimus angulus CAZ; ergo etiam angulus T; ergo erit ET infinita reſpectu EC; ergo AC, DE concurrent ad diſtantiam infinitam. Secundam, quia quum CZ infiniteſima ſit reſpectu CE, poterunt AD, CE haberi pro æqualibus.

Scholion. Si ſumpſeris ER infinitam reſpectu EC, ſed minorem quam TE, ac notaveris intervallum RV, tractus AV, DR habebunt primam conditionem parallelismi propter angulum infiniteſimum T; non habebunt ſecundam, quia, ut patet, non poterunt AD, VR ſumi pro æqualibus.

Scholion alterum. Quamvis AV, DR careant ſecunda conditione parallelismi, tamen ſi GL ſecet AV in G, DR in L, erunt anguli alterni GLD, LGT æquales, ſcilicet propter angulum infiniteſimum T.

33. Theorema ſecundum. Si ſit CZ infiniteſima reſpectu AZ, non vero reſpectu CE, tractus AC, DE habebunt primam conditionem parallelismi, non vero ſecundam. Habebunt primam, quia quum ſit CZ infiniteſima reſpectu AZ, erit angulus CAZ infiniteſimus; ergo etiam angulus T erit infiniteſimus; ergo erit ET infinita reſpectu EC; ergo lineæ AC, DE concurrent ad diſtantiam infinitam. Non habebunt ſecundam, quia quum CZ non ſit infiniteſima reſpectu CE, non poterunt AD, CE ſumi pro æqualibus.

Scholion. Ubicumque tamen ducatur GL ſecans AT in G, DT in L, anguli alterni GLD, LGT erunt æquales propter angulum infiniteſimum T.

34. Theorema tertium. Si ſit CZ infiniteſima reſpectu CE, non vero reſpectu AZ, tractus AC, DE carebunt prima conditione parallelismi; habebunt ſecundam. Carebunt prima, quia quum CZ non ſit infiniteſima reſpectu AZ, angulus CAZ erit aſſignabilis; ergo etiam angulus T; ergo ET non erit infinita reſpectu EC; ergo AC, DE non concurrent ad diſtantiam infinitam. Habebunt ſecundam, quia, quum ſit CZ infiniteſima reſpectu CE, poterunt AD, CE haberi pro æqualibus.

Scholion. Ubicumque ducatur GL, ut ſupra, anguli alterni minime pro æqualibus haberi poterunt, cum ſit angulus T aſſignabilis.

35. Theorema quartum. Si CZ non ſit infiniteſima neque reſpectu AZ, neque reſpectu CE, tractus AC, DE neutram habebunt parallelismi conditio-

nem; ac ducta GL, ut ſupra, anguli alterni minime ſumi poterunt pro æqualibus. Hæc ſatis patent ex dictis.

36. Propoſitio decimanona. In circulo, poſito arcu CM (*Fig.* 15) infiniteſimo primi ordinis, erunt tangens CS, chorda CM, ſinus rectus LM infiniteſimi ejuſdem ordinis; pars SM ſecantis BS, & ſinus verſus CL ernnt infiniteſimi ordinis ſecundi. Dem. pri. par. Qunm arcus CM ſit infiniteſimus primi ordinis, erit angnlus CBM infiniteſimus ejuſdem ordinis; ſed anguli BCM, BMC, BCS, CSB ſunt finiti; ergo SC, CM, ML ſunt infiniteſimæ primi ordinis. Q. E. P. Dem. ſec. par. In triangnlo rectangulo BCS, quia angulus CBS eſt infiniteſimus primi ordinis, erit MS ordinis ſecnndi per prop. 9. ergo quia LM parallela eſt CS, erit CL & ipſa infiniteſima ordinis ſecundi. Q. E. S.

37. Corollarium. BM eſt ad BL, ut MS eſt ad CL; ergo dividendo BM — BL = CL eſt ad LB, ut MS — CL eſt ad CL, ſed CL eſt infiniteſima ordinis ſecnndi, ergo differentia inter MS & CL eſt ordinis quarti.

38. Propoſitio vigeſima. In Circulo, poſito arcu MN (*Fig.* 16) infiniteſimo primi ordinis, ſegmentum MXN terminatum ab arcu MXN & ejus chorda MFN, & trilineum MXNC terminatum ab arcu MXN, tangente MC & intercepta CN, erunt infiniteſima tertii ordinis. Dem. prim. par. Ex centro S ducatur SFX normalis ad MN, conjungantur MX, XN; erit FX ſecnndi ordinis infiniteſimorum per præcedentem prop.; ergo rectangulum MN in FX erit infiniteſimum tertii ordinis per 4. prop.; ſed ſegmentum MXNF mediat inter rectangulum MN in FX & triangulnm MXN, ſeu dimidinm prædicti rectanguli; igitur quum rectangulum MN in FX, & ejus dimidium MXN ſint infiniteſima tertii ordinis, erit etiam ſegmentum MXNF & ipſum infiniteſimum tertii ordinis. Q. E. P. Dem. ſec. Par. Ex puncto N ducatur tangens LN; erunt ML, LN æquales; ſimiliter LN, & LC non differnnt niſi quantitate infiniteſima reſpective; nam in triangulo rectangulo LNC angulus NLC eſt infiniteſimus; ergo etiam LC non differet ab LM niſi quantitate infiniteſima reſpective; ac proinde triangula CMN, LNC ſunt ejuſdem ordinis; ſed triangulum MNC eſt infiniteſimum ſecundi ordinis relate ad triangulum CMS ſeu infiniteſimum tertii relate ad quantitatem aſſignabilem; ergo triangnlum NLC eſt infiniteſimum tertii ordinis; quum autem trilineum MXNC mediet inter triangulum MCN, & triangulum LCN, erit & ipſum infiniteſimum tertii ordinis.

39. Propoſitio vigeſimaprima. In Circulo, poſito arcu NM infiniteſimo primi ordinis, tangens MC, arcus MN, chorda MN, ſinus rectus GN differunt quantitate infiniteſima tertii ordinis. Dem. Tangens MC, arcus MN, chorda MN ſunt ut triangulum MSC, ſector circnli MSNX, & quadrilaterum SMXN; ſed iſta differnnt qnantitate infiniteſima tertii ordinis; quum trilineum MXNC, & ſegmenta MX, XN ſint infiniteſima tertii ordinis per propoſ. 20; ergo & illa differunt quantitate infiniteſima tertii ordinis, quum omnes termini tam in prima quam in ſecunda ſerie ſint infiniteſimi primi. Quod chorda MN excedat ſinum rectum NG infiniteſima quantitate tertii ordinis patet. Q. E. D.

40. Propoſitio vigeſimaſecunda. In Triangnlo ABC obtnſangulo, in quo duo anguli A & C ſunt infiniteſimi, erunt latera AB, BC (*Fig.* 11.) in ratione angulorum oppoſitorum. Dem. Per punctum C ducatnr CL parallela ad AB, quæ concurrat cum BS (normali ex obtuſo B ad rectam AC) producta in

L, &

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L, & centro C intervallo CS deſcribantur arcus MS, NS. Recta AB eſt ad rectam BC, ut AS eſt ad SC per prop. 9., ſeu ut BS eſt ad SL, ſeu ut arcus SM ad arcum SN per prop. 21. ſeu ut angulus BCA eſt ad angulum ACL, ſeu ut angulus BCA eſt ad angulum A. Q. E. D.

41. Corollarium primum. Hinc duo triangula rectangula ASB, CSB, quæ habent eandem altitudinem, & angulum iſti oppoſitum infiniteſimum, habent etiam baſes AS, SC, ac proinde hypothenuſas AB, BC in ratione reciproca angulorum infiniteſimorum.

Corollarium ſecundum. Quinimmo ſi duo triangula ALM, OBC (*Fig.* 17) habent duo latera AL, BO æqualia, angulos ALM, OBC jacentes ad latera AL, BO habeant ſimiliter æquales, & angulos M, & C prædictis lateribus oppoſitos habeant infiniteſimos; habent etiam latera LM, CB, & AM, OC in ratione reciproca angulorum infiniteſimorum. Etenim ex punctis A, & O ducantur normales AQ, OP ad rectas LM, CB productas, ſi opus eſt; quum anguli MLA, CBO ſint æquales, & anguli Q, & P ſint recti, erunt etiam æquales anguli LAQ, BOP; adeoque in triangulis QAL, BOP, quum ſint etiam rectæ AL, BO æquales, erunt etiam æquales normales AQ, OP; hinc latera AM, OC, & QM, PC ſunt in ratione reciproca angulorum infiniteſimorum M, & C; ſed QM non differt ab LM, PC non differt a BC; ergo etiam LM, BC erunt in ratione reciproca angulorum infiniteſimorum M, & C.

42. Propoſitio vigeſimatertia. Iiſdem poſitis quæ in propoſitione 21. differentia ſinus recti GN (*Fig.* 16) a chorda MN non differt a quarta parte differentiæ ſinus recti GN a tangente MC. Dem. Ex puncto N ducatur NR normalis ad tangentem MC, erit RC differentia ſinus recti a tangente: tum centro M intervallo MN deſcribatur arcus NQ, erit RQ differentia ſinus recti a chorda. Quum SM ſit ad MC, ut NR ad RC erit rectangulum SM in RC æquale rectangulo MC in RN. Similiter quum MQ+MR ſit ad RN ut RN ad RQ, & MQ+MR non differat a 2MR, ſeu a recta ON; ratio ON ad NR non differt a ratione NR ad RQ; ſed ratio ON ad NR ſeu ad GM eadem eſt ac ratio 2GA ad GN, ſeu MR; ergo ratio 2GA ad MR non differt a ratione NR ad RQ; ſed ratio 2GA ad MR non differt a ratione 2MA ad MR, & ratio 2MA ad MR non differt a ratione 2MA ad MC; ergo ratio 2MA ad MC non differt a ratione RN ad RQ, adeoque rectangulum 2MA in RQ non differt a rectangulo MC in RN, ſeu a rectangulo SM in RC, & per conſequens 2MA eſt ad MS, ut RC ad RQ; ſed MS eſt quarta pars 2MA; ergo RQ non differt a quarta parte RC. Q. E. D.

Corollarium. Hinc differentia ſinus recti GN a chorda MN non differt a tertia parte differentiæ chordæ MN a tangente MC.

43. Propoſitio vigeſimaquarta. Sit AOSM (*Fig.* 18) quadrans circuli CML, in radio AO ſumantur duo puncta P, Q, per quæ ducantur duæ ordinatæ normales ad AO intercipientes arcum MN infiniteſimum primi ordinis diſſitum ab extremitatibus quadrantis AS arcubus finitis MA, NS; dico differentiam abſciſſarum AP, AQ, & ordinatarum PM, QN eſſe quantitatem infiniteſimam primi ordinis. Dem. Ordinata QN producatur uſque dum concurrat cum circulo in L, & ex puncto M ducatur MG parallela ad radium AO, quæ producatur uſquedum concurrat cum circulo in C; erit PQ ſeu GM differentia abſciſſarum AP, AQ; & GN erit differentia ordinatarum PM, QN.

Quum

Quum ex hypotheſi arcus NS, MA ſint finiti, erunt etiam arcus CN, LM finiti; hinc anguli CMN, LNM ſunt finiti; adeoque in triangulo rectangulo MGN omnes anguli ſunt finiti, & per conſequens latera MN, MG, GN ſunt ejuſdem ordinis per corol. 3. prop. 6.; ſed arcus MN, ac proinde chorda MN eſt infiniteſima primi ordinis per ſuppoſitionem; ergo etiam latera MG, GN, ſeu differentia abſciſſarum, & ordinatarum erit infiniteſima primi ordinis. Q. E. D.

44. Corollarium. Si arcus NS ſit infiniteſimus primi ordinis, erit arcus MS infiniteſimus primi ordinis; ac proinde arcus CS cum arcu NS, ſeu arcus CN erit infiniteſimus primi ordinis; hinc angulus CMN erit infiniteſimus primi ordinis; adeoque GN reſpectu NM erit infiniteſima primi ordinis per 9. prop.; ſed NM eſt infiniteſima primi ordinis; ergo GN erit infinitaſima ſecundi ordinis. Si vero arcus MA ſit infiniteſimus, erit arcus AN, ac proinde arcus LA infiniteſimus primi otdinis; adeoqua etiam arcus LM & per conſeguens angulus LNM erit infiniteſimus primi ordinis; ergo GM reſpectu MN erit infiniteſima primi ordinis per 9. prop.; ſed MN aſt infiniteſima primi ordinis; ergo GM erit infiniteſima ſecundi ordinis.

45. Prop. vigeſimaquinta. Sit MS arcus circuli AMN infiniteſimus primi ordinis, cui ſubtendatur chorda MS (*Fig. 19*); per punctum M ducatur MR tangens circulum, & per punctum S ducatur utcumque recta NS ſecans circulum extra arcum infiniteſimum MS, & concurrens cum tangente MR; dico angulum MSR factum a chorda MS cum ſecante NS producta vel eſſe finitum, vel infiniteſimum primi ordinis; angulum NRM factum a ſecante NR, & tangente MR, vel eſſe finitum vel infiniteſimum cujuſcumque ordinis; ac tandem angulum SMR factum a chorda SM & tangente MR eſſe infiniteſimum primi ordinis. Dem. pri. par. Tota peripheria circuli aſt menſura duorum rectorum ad peripheriam ipſam; quum igitur arcus NAM ſit menſura anguli NSM, ſeu duorum SMR, SRM, erit arcus NSM menſura anguli MSR; ſed arcus NSM, vel eſt finitus, vel infiniteſimus primi ordinis; quia ad minimum ſemper remanet arcus MS infiniteſimus primi ordinis, ergo angulus MSR vel erit finitus vel infiniteſimus primi ordinis. Q. E. P. Dem. ſec. Par. Ex arcu NM abſcindatur arcus NX æqualis MS, & conjungantur puncta X, S, erit angulus NSX æqualis angulo SMR; quum igitur arcus NM ſit menſura duorum angulorum SMR, SRM, & arcus NX ſit menſura anguli NSX, ſeu SMR; erit arcus XM menſura anguli NRM; ſed arcus XM poteſt eſſe finitus & infiniteſimus cujuſcunque ordinis; argo angulus NRM poteſt eſſe finitus & infiniteſimus cujuſcunque ordinis. Q. E. S. Pars. ter. patet, quia arcus MS infiniteſimus primi ordinis eſt menſura anguli SMR.

46 Corollarium. Quum arcus XAM, NSM ſint menſura angulorum MRS, MSR; hinc ſi arcus XAM, & NSM ſint finiti, & æquales, erunt etiam anguli MRS, MSR æquales; ſi vero ſunt finiti, ſed differunt quantitate infiniteſima; & tunc anguli MSR, & MRS ſunt pariter finiti, & differunt quantitate infiniteſima, vel tandem arcus ſunt finiti, & differunt quantitate finita, tunc anguli MRS, & MSR ſunt finiti, & eorum differentia erit finita, hoc eſt unus erit obtuſus. In prima ſuppoſitione SR intercepta inter circulum & tangentem erit ſecundi ordinis, & differentia inter MS, MR erit abſolute nulla; in ſecunda ſuppoſitione intercepta SR erit ſecundi ordinis, & differentia inter MS, MR poteſt eſſe cujuſcunque ordinis incipiendo a tertio; in tertia tandem ſuppoſitione intercepta SR erit ſecundi ordinis, & differentia

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inter M S, M R erit ſimiliter ſecundi ordinis; quod totum patet ex ſcholio prop. 12. Si vero arcus X A M vel N S M ſunt infiniteſimi primi ordinis, erit angulus M R S, vel M S R infiniteſimus primi ordinis; hinc S R, S M, M R in hac ſuppoſitione erunt infiniteſimæ primi ordinis, & differentia inter M S & M R erit ejuſdem ordinis per prop. 13. Si tandem arcus X A M ſit infiniteſimus ordinis ſuperioris, erit angulus R infiniteſimus ordinis ſuperioris; adeoque S R reſpectu S M poteſt eſſe infiniteſima cujuſcunque ordinis per prop. 14.

47. Propoſitio vigeſimaſexta. Sit quadrans circuli A B C (*Fig.* 20), ducanturque ordinatæ P M, Q N productæ uſquedum concurrant cum circulo in T, & S, quæ intercipiant arcum M N infiniteſimum primi ordinis; tum per punctum M ducatur tangens M O, quæ concurrat cum Q N producta in O. Si arcus A M, N C ſunt finiti, vel ſi arcus A M eſt infiniteſimus cujuſcumque ordinis ſuperioris, incipiendo a ſecundo, ducta ex M normali M R ad Q O; ratio Q O ad O R non differt a ratione Q N ad N R; ſi vero arcus N C ſit infiniteſimus cujuſcunque ordinis, vel A M ſit primi, tunc ratio Q O ad O R differt a ratione Q N ad N R. Dem. pri. par. Si arcus A M, N C ſunt finiti, erunt etiam finiti arcus T A M, S N M; hinc, quum arcus T A M ſit menſura anguli S O M, & arcus S N M ſit menſura anguli M N O; igitur quum angulus N M O ſit infiniteſimus primi ordinis, erit intercepta N O infiniteſima ſecundi ordinis per doctrinam triangulorum, ſed R N eſt infiniteſima primi ordinis per prop. 24., ergo Q N non differt a Q O, neque R N differt ab R O; & per conſequens ratio Q O ad R O non differt a ratione Q N ad N R per prop. 1. Q. E. P. Dem. ſec. par. Quoniam arcus A M eſt infiniteſimus cujuſcunque ordinis ſuperioris incipiendo a ſecundo, erit recta P M infiniteſima ejuſdem ordinis per prop. 19. ſed Q N eſt infiniteſima primi ordinis per ſuppoſitionem; ergo P M, ſeu Q R eſt infiniteſima reſpectu R N; adeoque ratio Q N ad N R non differt a ratione æqualitatis; & multo magis non differt a ratione æqualitatis ratio Q O ad O R; adeoque ratio Q O ad O R non differt a ratione Q N ad N R. Q. E. S. Dem. terti. Par. Si arcus N C eſt infiniteſimus cujuſcunque ordinis, erunt arcus T A M, S N M finiti, ac proinde erunt etiam finiti anguli M O N, O N M; quare N O eſt infiniteſima ſecundi ordinis per doctrinam triangulorum, ſed R N eſt pariter infiniteſima ſecundi ordinis per coroll. prop. 24.; ergo ratio Q O ad O R differt a ratione Q N ad N R. Q. E. T. Dem. quar. par. Si tandem arcus A M eſt infiniteſimus primi ordinis, erit N O infiniteſima primi ordinis per coroll. prop. 25.; ſed R N eſt infiniteſima primi ordinis per 24. prop.; ergo R N, N O ſunt infiniteſimæ ejuſdem ordinis; & per conſequens ratio Q O ad O R differt a ratione Q N ad N R. Q. E. Q.

Corollarium. Si arcus N C ſit infiniteſimus cujuſcunque ordinis, erunt rectæ R N, R O infiniteſimæ ſecundi ordinis per coroll. prop. 23, 25, & R M infiniteſima primi ordinis, tandem Q N, Q O erunt finitæ. Ergo quarta proportionalis poſt N R, N Q, R M; vel poſt O R, O Q, R M erit infinita. Si arcus A M ſit infiniteſimus primi ordinis, erit R M ſecundi ordinis; Q N, N R, & R O, Q O ſunt infiniteſimæ primi ordinis per coroll. prop. 24. Ergo quarta proportionalis poſt Q N, N R, R M, vel poſt Q O, O R, R M erit infiniteſima ſecundi ordinis.

Propoſitio vigeſimaſeptima. Poſitis quæ ſupra poſuimus propoſitione præcedenti, dico triangulum M R N ſumi poſſe pro triangulo M R O ad determinandam ſubtangentem circuli. Dem. Si arcus A M, & N C ſunt finiti, vel ſi arcus A M ſit infiniteſimus cujuſcunque ordinis ſuperioris incipiendo a ſecundo,

tunc

tunc ratio QN ad NR non differt a ratione QO ad OR per 16. prop.; adeoque quarta proportionalis poſt RN, QN, RM non differt a quarta proportionali poſt QO, OR, QM, hoc eſt non differt a ſubtangente. Si vero arcus NC ſit infiniteſimus cujuſcunque ordinis, vel AM ſit primi, quum quarta proportionalis in primo caſu ſit infinita, & in ſecundo caſu infiniteſima ſecundi ordinis, quodcunque ex illis duobus triangulis ſumatur, patet utrumlibet accipi poſſe. Q. E. D.

48. Scholion. In ultima ſuppoſitione licet quarta proportionalis inventa per triangulum NRM ſit infinita, aut infiniteſima, quemadmodum etiam ſubtangens, attamen in ſe multum diſcrepant, quod ſequenti exemplo demonſtro. Sit quadrans circuli VQNM (*Fig.* 21), in radio VQ ſumatur QP infiniteſima primi ordinis, tum per punctum P ducatur PM parallela ad QN, per punctum M ducatur tangens XMT, quæ cum QV producta concurrat in X, deinde ducatur chorda NM, quæ concurrat cum QV producta in S; tandem per punctum M ducatur MR parallela ad QV. Eſt proprietas circuli notiſſima, quod angulus RMN æqualis eſt angulo NMT, ac proinde ratio TM ad RM eſt eadem ac ratio TN ad NR; ſed ratio TM ad MR non differt a ratione æqualitatis per 21. prop. ergo ratio TN ad NR non differt a ratione æqualitatis; ac proinde recta RN non differt a dimidio rectæ RT. Præterea in triangulo QNS erit QS ad RM, ut QN ad NR; ac proinde rectangulum QS in RN eſt æquale rectangulo QN in RM. Similiter QX in RT eſt æquale rectangulo QT in RM; ſed rectangulum QN in RM non differt a rectangulo QT in RM: ergo etiam rectangulum QS in RN non differt a rectangulo QX in RT, & per conſequens ratio RN ad RT non differt a ratione QX ad QS; ſed RN non differt a dimidia parte rectæ RT; ergo etiam QX non differt a dimidia parte QS; ſed QX eſt vera ſubtangens; ergo in hac ſuppoſitione vera ſubtangens, licet infinita, tamen eſt dimidium ſubtangentis ſuppoſitæ QS. Quod animadverſione dignum eſt.

49. Propoſitio vigeſimaoctava. Rectæ AB, BC (*Fig.* 22, 23) ad quemcumque angulum finitum ABC poſitæ ſint lineæ coordinatarum cujuſcumque curvæ MN, ducanturque ordinatæ QN, PM intercipientes arcum MN infiniteſimum primi ordinis, cujus chorda ſit MN; dico latera trianguli MRN eſſe infiniteſima primi ordinis, ſi caſum excipias, ubi MN producta faciat angulum infiniteſimum cum una ex diametris, & ad ejus paralleliſmum accedat. Dem. Chorda MN producatur uſque dum concurrat cum diametris coordinatarum, in punctis X, & Z; quum hæc non faciat angulos infiniteſimos cum diametris XB, BZ, erunt anguli BXZ, XZB finiti; adeoque in triangulo XBZ, & ob paralleliſmum rectarum MR, XB, & RN, BZ, in triangulo etiam RMN anguli omnes ſunt finiti; ac proinde latera NM, MR, NR ſunt ejuſdem ordinis; ſed NM ex hypotheſi eſt infiniteſima primi ordinis, quia arcus MN eſt infiniteſimus primi ordinis; ergo NR, RM ſunt infiniteſimæ primi ordinis. Q. E. D.

50. Coroll. 1. Si per punctum N ducatur tangens NS, efformatum erit triangulum MNS, in quo anguli NSM, NMS ſunt finiti per ſuppoſitionem, angulus vero SNM eſt infiniteſimus, quod curvarum natura per ſeſe manifeſto indicat; ergo SM eſt infiniteſima reſpectu NM, ac proinde reſpectu RM; adeoque triangulum MNR non differt a triangulo SNR.

Coroll. 2. Cum NS+SM major ſit arcu MN, qui major eſt chorda MN; & cum MN differat ab MS+SN quantitate relate ad ipſas infiniteſima; hinc

hinc arcus MN non differt a chorda MN, vel a tangente NS.

Scholion. In punctis in quibus MN eſt parallela diametro coordinatarum obſervanda veniunt, quæ diximus in ſcholio propoſitionis 27.

Propoſitio vigeſimanona. Ex puncto A (*Fig.* 24. 25.) ad quamcumque curvam ducantur rectæ AQ, AB intercipientes arcum QB infiniteſimum primi ordinis, cui ſubtendatur QB, & ex puncto Q in AB demittatur normalis QR: dico, tria latera trianguli QRB eſſe infiniteſima primi ordinis, exceptis illis punctis, in quibus QB vel eſt normalis ad AB, vel in directum cum AQ, vel ſaltem a vera normali, & a vera directione AQ ſolum differt quantitate infiniteſima. Dem. patet. Nam per ſuppoſitionem angulus RBQ neque eſt rectus, neque infiniteſimus; igitur quum angulus BRQ ſit rectus, in triangulo RBQ tres anguli ſunt finiti, ac proinde tria latera infiniteſima primi ordinis, quum BQ ſit infiniteſimum primi ordinis per ſuppoſitionem Q. E. D.

Corollarium. Si ex puncto Q ducatur tangens QS, quæ concurrat cum RB producta, ſi opus eſt, in S; quum anguli QBS, QSB ſint finiti per ſuppoſitionem, & angulus BQS ſit infiniteſimus per naturam curvæ, erit BS intercepta inter tangentem & curvam infiniteſima reſpectu BQ, ac proinde reſpectu RB; adeoque triangulum QRB ultimo æquat triangulum QRS; & triangula QAR, QAB, QAS ultimo æquantur, & promiſcue accipi poſſunt.

Scholion 1. Quod dictum eſt aliquam exceptionem patitur, ubi chorda BQ vel eſt normalis ad BA, vel in directum cum AQ; & generaliter loquendo quando anguli fluentes vel evadunt recti, vel æquales duobus rectis, vel ſaltem differentia horum angulorum aut a recto, aut a duobus rectis tranſit ex uno ordine infiniteſimorum ad alium, tunc nonnulli anguli fluentes, ac proinde nonnullæ rectæ fluentes tranſeunt ex uno ordine ad alium, atque adeo omnium fluentium interſeſe proportio mutatur. Attamen per expoſita principia, ſi probe teneantur, omnia poterunt non difficulter determinari.

Scholion ſecundum. Animadvertendum eſt etiam, quantitates fluentes poſſe fluere ex omni parte; adeoque ſedulo perpendendum eſt ex quibuſnam partibus data quantitas fluat, & utrum creſcat, vel decreſcat ex omni parte, vel ex aliquibus tantum creſcat, decreſcat ex aliis; ſic radius AQ creſcit vel decreſcit tantum ex parte Q, non vero ex puncto fixo A; tangens autem PQ deſinens in lineam poſitione datam AP creſcit ex parte P, & ex parte Q, vel decreſcit, prout radius AQ propius accedit, vel recedit a data AP.

Scholion tertium. Quæ duobus capitibus de quantitatibus infiniteſimis ſunt expoſita, ideo exactiſſima lance expendimus, eorumque adæquatam cognitionem iteratis præceptis Tyronibus inculcavimus, quia reputari debent veluti baſis & fundamentum totius doctrinæ difficillimæ de rebus ſublimioribus mathematicis; incredibile profecto eſt, quibuſnam difficultatibus, contradictionibus, erroribus, & paralogiſmis obnoxii ſint ſublimiorum mathematicarum ſtudioſi, qui germanam ideam, affectiones, proprietates, relationeſque magnitudinum infiniteſimarum probe non teneant.



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## CAPUT TERTIUM.

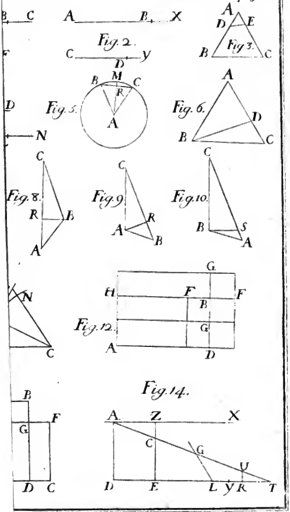
### De differentiatione, & integratione formularum, quæ unica variabili conſtant.

1. Differentiare nihil aliud eſt, quam dividere quantitates in ſua elementa. Integrare, ſive ſummare nihil eſt aliud, quam elementa eadem in exactam ſummam colligere. Sit quantitas quælibet $y$ data unice per $x$, & conſtantes. Exprimantur $x$ per abſciſſas, $y$ vero per ordinatas curvæ A C. (*Fig.* 1) Accipiatur quælibet abſciſſa A B, cui reſpondeat ordinata B C. Dividatur abſciſſa A B in ſua elementa B i, 1 2 i, 2 i 3 i &c., quæ vel æqualia ſint, vel inæqualia nihil refert, dummodo ſint infiniteſima. Ordinentur i e, 2 i 2 e, 3 i 3 e &c., & parallelæ abſciſſis agantur e u, 2 e 2 u, 3 e 3 u &c. Erunt C u, e 2 u, 2 e 3 u &c. differentiæ ordinatarum. Itaque ille differentiabit quantitatem $y$, qui invenerit ſingula elementa C u expreſſa per ſingulas abſciſſas A B, earumque elementa B i. Contra ille integrabit, qui ex datis elementis C u per A B, & B i, omnium elementorum ſummam accipiet, quæ ordinatæ B C erit æqualis. Dixi ſummam omnium elementorum C u æquare ordinatam C B, quod veriſſimum eſt, ſi abſciſſæ, & ordinatæ idem habeant initium, ut una nulleſcente, altera quoque nulleſcat. Verum ſi abſciſſa eſſet F G, vel H K, ut hac evaneſcente, ordinata ſit vel A F poſitiva, vel A H negativa; tunc ſumma omnium elementorum C u æquaret differentiam inter ultimam ordinatam C G, & primam A F, vel ſummam ultimæ ordinatæ C K, & primæ A H. Quare in ſummatoriis accipiendis addenda vel detrahenda eſt quantitas conſtans, quia quantitates omnes, quæ non differunt niſi per conſtantem, eodem præditæ ſunt differentiali. In problematum ſolutione quantitas addenda vel detrahenda ſæpe opportunis adhibitis criteriis determinatur. Eſt in more poſitum, ut quum quæritur differentia abſciſſæ A B $= x$, & ordinatæ B C $= y$, augeatur A B elemento infiniteſimo B I, & ordinata I E, ductaque C V parallela abſciſſis, ſpectetur E V tamquam elementum ordinatæ. Qui ergo invenerit E V datam per $x$ & ejus elementum, ille differentiam accipiet quantitatis $y$ datæ per $x$. Qui autem dato elemento E V omnia elementa ſimul collegerit, inveniet ſummam, adeoque ordinatam E I, addita, demptave conſtante, ſi opus fuerit. Nos quoque morem hunc ſequemur.

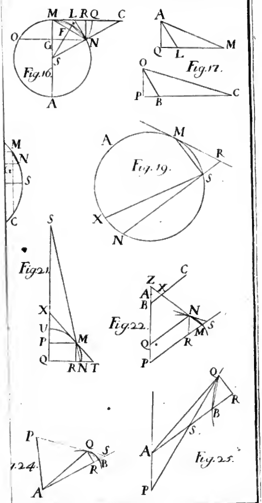
2. Elementa, ſeu differentiæ variabilium $x$, $y$ ſolent deſignari præfixa littera $d$. Ita $dx$, $dy$ exprimunt minimas differentias variabilium $x$, $y$. Quum vero indicanda eſt differentia quantitatis complexæ, utar littera D. Ita D $\frac{a^3x - x^4}{a\sqrt{a^2 + x^2}}$ exprimet differentiam quantitatis complexæ $\frac{a^3x - x^4}{a\sqrt{a^2 + x^2}}$. Si quantitas differentianda ſit $x$, aut $A + x$, ejus differentia erit $dx$; viceverſa integrale quantitatis differentialis $dx$ erit $x$, vel $A + x$. Eodem modo ſi quantitas differentianda coaleſcat ex ſumma, aut differentia plurium variabilium, ut ſi fuerit $A + x - y + z$, ejus differentia erit $dx - dy + dz$; vice verſa hujus integrale erit $x - y + z$, vel $A + x - y + z$. Hæc facillima exempla

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empla demonſtrant, in ſummatione addendam ſemper eſſe conſtantem $A$, quod memoria retinendum eſt, nam nos deinceps in methodo exponenda brevitati ſtudentes conſtantem hanc omittemus. Tutiſſima ratio differentiale inveniendi eſt addere ſingulis variabilibus elementum cuique ſuum, tum ex nova formula priorem deducere, quæ reliqua eſt, differentiam præbebit. Ita ſi $x - y + z$ addantur differentialia, fiet $x + dx - y - dy + z + dz$, ex hac ſi propoſitam detrahas, habebis $dx - dy + dz$, quæ erit differentialis quæſita. Multiplicatores, & diviſores conſtantes nihil turbant differentiam aut ſummatoriam, quæ eodem multiplicatore, ac diviſore erit affecta. Ita differentia quantitatis $\frac{ax}{b}$ erit $\frac{adx}{b}$, & hujus integrale erit $\frac{ax}{b}$.

3. Nunc quæramus differentiam quadrati $x^2$. Addamus $x$ ejus differentiam $dx$, ut fiat $x + dx$, quam ad quadratum eveheamus, ut fit $xx + 2xdx + dx^2$. Ablato quadrato $x^2$ remanet differentia $2xdx + dx^2$; atqui $dx^2$ habet ad $2xdx$ minorem rationem quacumque data, eſt enim ut $dx : 2x$; ergo differentia quæſita eſt $2xdx$. Animadvertas volo, incipere modo omiſſionem eorum infiniteſimorum, quæ excedunt primum ordinem; namque in ſumenda differentiali $x^2$ omiſſum eſt quadratum $dx^2$. Quæ omiſſio omnino facienda eſt, quotieſcumque $x$ finita ſit; quia, ut demonſtravimus $dx^2$ in ultimam proportionem limitis non ingreditur. Verum ſi $x$ nulleſceret, tunc proportio limitis a ſolo quadrato $dx^2$ unice conſtitueretur, ac proinde $dx^2$ nulla ratione deberet omitti. Harum omiſſionum, quæ in ſumendis reliquis differentiis ſemper contingunt, oportet non oblviſci, ut certo cognoſcatur, quouſque calculus differentialis ſeſe extendat. Quando $2xdx$ eſt differentiale $x^2$, erit $x^2$ integrale $2xdx$; ergo $\frac{x^2}{2}$ erit integrale $xdx$. Eadem methodus applicatur rectangulo $xy$ coaleſcente ex duabus variabilibus: nam augeantur ambæ ſuis elementis, ut fiant, $x + dx$, $y + dy$, quæ in ſeſe ducantur, ut fit $xy + xdy + ydx + dxdy$; demantur $xy$, & proveniet differentia $xdy + ydx + dxdy$. Negligatur terminus $dxdy$ infiniteſimus præ reliquis duobus, & provenit differentia quæſita $xdy + ydx$. Hæc coaleſcit ex ſumma productorum variabilis $x$ in differentiam $y$, & variabilis $y$ in differentiam $x$. Quare viciſſim ſi habeantur duo producta hujuſmodi, eorum ſummatoriam dabit productum ex duabus variabilibus nempe $xy$.

4. Nunc gradatim differentiam ſumamus $x^m$ exiſtente $m$ numero integro, & poſitivo. Poteſt $x^3$ ſpectari ut productum ex $x$ in $x^2$; ergo ejus differentiale $= x\,\mathrm{D}.\,x^2 + x^2dx$; atqui $\mathrm{D}.\,x^2 = 2xdx$; ergo
$\mathrm{D}.\,x^3 = 2x^2dx + x^2dx = 3x^2dx$. Similiter $x^4 = x.\,x^3$; ergo
$\mathrm{D}.\,x^4 = x.\,\mathrm{D}x^3 + x^3dx$; ſed $\mathrm{D}\,x^3 = 3x^2dx$; igitur
$\mathrm{D}.\,x^4 = 3x^3dx + x^3dx = 4x^3dx$. Simili modo invenies



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$D . x^5 = x D x^4 + x^4 dx = 4x^4 dx + x^4 dx = 5x^4 dx$; & ſic deinceps. Quæ quum ita ſint, apparet generatim formulam $x^m$ habere pro differentiali $m x^{m-1} dx$; & formulam hanc habere pro integrali $x^m$; ergo ſummatoria differentiæ $x^m dx$ erit $\frac{x^{m+1}}{m+1}$, exiſtente $m$ numero integro, & poſitivo.

5. Antequam inquiro differentiale $\frac{1}{x}$, præmitto axioma, quod per ſeſe manifeſtum eſt, nimirum: quantitates variabiles, quæ aut æquales ſint, aut non differant niſi per conſtantem, eodem præditæ ſunt differentiali; contra ſummatoriæ differentialium æqualium aut æquales ſunt, aut conſtanti differentia donatæ. Jam vero pono $\frac{1}{x} = y$; ergo $1 = xy$, & differentiis acceptis $o = x dy + y dx$, ſeu $-\frac{y dx}{x} = dy$, & ſubſtituto valore $y$ fit $-\frac{dx}{x^2} = dy$, quæ eſt differentia quæſita. Differentiale itaque $\frac{1}{x}$ eſt $\frac{-dx}{xx}$, & integrale $\frac{-dx}{xx}$ eſt $\frac{1}{x}$, adeoque $\frac{dx}{xx}$ habebit pro integrali $\frac{-1}{x}$. Quod ſi ad differentiandum propoſita eſſet fractio $\frac{x}{y}$, hac methodo licebit uti $\frac{x}{y} = x . \frac{1}{y}$; ergo $D . \frac{x}{y} = x D . \frac{1}{y} + \frac{dx}{y}$; atqui $D . \frac{1}{y} = \frac{-dy}{y^2}$; ergo $D . \frac{x}{y} = \frac{-x dy}{y^2} + \frac{dx}{y} = \frac{y dx - x dy}{yy}$: ſcilicet ſi diviſor ductus in differentiam numeratoris, dempto numeratore ducto in differentiam denominatoris, dividatur per hujus quadratum dabit differentiale quæſitum. Contra $\frac{y dx - x dy}{y^2}$ habebit pro integrali $\frac{x}{y}$.

6. Nunc gradatim progredientes determinemus differentiale formulæ $\frac{1}{x^m}$. Primum $D . \frac{1}{xx} = D \frac{1}{x} . \frac{1}{x} = \frac{1}{x} . \frac{-dx}{x^2} + \frac{1}{x} . \frac{-dx}{x^2} = \frac{-2dx}{x^3}$. Similiter $D \frac{1}{x^3} = D \frac{1}{x} . \frac{1}{x^2} = \frac{1}{x} . \frac{-2dx}{x^3} + \frac{1}{x^2} . \frac{-dx}{x^2} = \frac{-3dx}{x^4}$. Eodem modo $D \frac{1}{x^4} = D \frac{1}{x} . \frac{1}{x^3} = \frac{-4dx}{x^5}$ atque ita deinceps. Quapropter generatim $D . \frac{1}{x^m} = \frac{-m dx}{x^{m+1}}$, & hujus formulæ integrale erit $\frac{1}{x^m}$, adeoque indicata ſummatoria per litteram S, quem morem ſequemur deinceps, erit $S \frac{dx}{x^m} = \frac{-1}{m-1 . x^{m-1}}$.

Itaque rem paucis contrahentes, exiſtente $m$ numero integro vel poſitivo, vel ne-

negativo $D.x^m = m x^{m-1} dx$, & $S x^m dx = \frac{x^{m+1}}{m+1}$. Hoc obſervandum eſt diligenter, ex differentiatione formulæ $x^m$ nunquam prodire $\frac{dx}{x}$; quia ſi $m = 1$, habetur $dx$; ſi $m = -1$, oritur $\frac{-dx}{xx}$; ſi $m = o$, formula exhibet quantitatem conſtantem, quæ caret differentia. Quapropter licet $x^m dx$ ſit algebraice integrabilis; tamen in unico caſu, in quo $m = -1$, & habeatur $\frac{dx}{x}$, de ejus integratione nihil licet pronunciare. De hac autem formula ſuo loco agemus pro dignitate.

7. Demonſtrandum eſt, eaſdem formulas æque valere, licet $m$ ſit numerus fractus vel poſitivus, vel negativus. Pone $m = \frac{p}{q}$, ut $p$, $q$ ſint numeri integri; tum fac $x^{\frac{p}{q}} = y$; ergo $x^p = y^q$; & acceptis differentiis $p x^{p-1} dx = q y^{q-1} dy$; ergo $\frac{p}{q} . \frac{x^{p-1} dx}{y^{q-1}} = dy$; atqui $y^{q-1} = x^{\frac{p . \overline{q-1}}{q}}$; ergo

$$\frac{p}{q} . \frac{x^{p-1} dx}{x^{\frac{p . \overline{q-1}}{q}}} = \frac{p x^{p-1\frac{-pq+p}{q}}}{q} dx = \frac{p}{q} . x^{\frac{p}{q}-1} dx = dy;$$

ſubſtitutoque valore $m$ pro $\frac{p}{q}$, proveniet differentia quæſita eadem ac antea, $m x^{m-1} dx$. Quare ex converſo $S x^m dx = \frac{x^{m+1}}{m+1}$, quicumque ſit numerus $m$, dummodo rationalis.

8. Quæ explicata ſunt hactenus, ſufficiunt tuta principia cujuslibet differentiationis, & integrationis. Differentia cujuslibet formulæ licet valde complexæ ſemper obtineri poteſt, & praxim faciliorem efficient ſubſtitutiones, in quibus peragendis induſtria, & uſus plurimum valent. Nonnulla exempla proponam. Sit differentianda $\sqrt{2ax - xx}$. Potes ad poteſtates eam reducere ſcribendo $\overline{2ax - xx}^{\frac{1}{2}}$, cujus differentia $= \frac{1}{2} . \overline{2ax - xx}^{\frac{1}{2}-1} . D. \overline{2ax - xx} = \frac{D . \overline{2ax - xx}}{2 . \overline{2ax - xx}^{\frac{1}{2}}}$; atqui $D. \overline{2ax - xx} = 2adx - 2xdx$; ergo, facta ſubſtitutione, differentia erit $\frac{adx - xdx}{\sqrt{2ax - xx}}$. Sed, ſi mavis uti ſubſtitutione, fac $\sqrt{2ax - xx} = y$; ergo $2ax - xx = yy$, & facta differentiatione, & diviſione per 2, erit $adx - xdx = ydy$; igitur $\frac{adx - xdx}{y} = \frac{adx - xdx}{\sqrt{2ax - xx}} = dy$,

ut

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ut per primam methodum. Simili modo differentiam invenies formulæ $\frac{1}{\sqrt{2ax-xx}}$. Nam ſi optas ad poteſtatem perducere, ita erit ſcribenda $\overline{2ax-xx}^{-\frac{1}{2}}$, cujus differentia $=-\frac{1}{2}.\overline{2ax-xx}^{-\frac{3}{2}}.\overline{2adx-2xdx}$ $=\frac{-(adx-xdx}{\overline{2ax-xx}^{\frac{3}{2}}}$. Si mavis uti ſubſtitutionibus, pone $\frac{1}{\sqrt{2ax-xx}}=y$; ergo $\frac{1}{y}=\sqrt{2ax-xx}$, differentiiſque acceptis $\frac{-dy}{y^2}=\frac{adx-xdx}{\sqrt{2ax-xx}}$, vel $dy$ $=-y^2.\frac{adx-xdx}{\sqrt{2ax-xx}}=\frac{-(adx-xdx}{2ax-xx.\sqrt{2ax-xx}}$, ut antea.

9. Proponatur differentianda formula $\overline{\sqrt[2]{a^4x^5}+\sqrt[3]{a^5x^4}}^{\frac{3}{2}}$. Eam ita diſpone $a^2x^2.\overline{x^{\frac{1}{3}}+a^{\frac{1}{3}}}^{\frac{3}{2}}$. Fac $\overline{x^{\frac{1}{3}}+a^{\frac{1}{3}}}^{\frac{3}{2}}=y$ ut ſit $a^2x^2y$. Hujus differentia eſt $2a^2yxdx+a^2x^2dy$, in qua pro $y$ ejus valorem ſubſtitue, ut habeas $2a^2xdx.\overline{x^{\frac{1}{3}}+a^{\frac{1}{3}}}^{\frac{3}{2}}+a^2x^2dy$. Ad arcendam $dy$, ex formula ſubſtitutionis habes $x^{\frac{1}{3}}+a^{\frac{1}{3}}=y^{\frac{2}{3}}$, & acceptis differentiis $\frac{dx}{3.x^{\frac{2}{3}}}=\frac{2}{3}.\frac{dy}{y^{\frac{1}{3}}}$, ſive $\frac{y^{\frac{1}{3}}dx}{2x^{\frac{2}{3}}}=dy$, ſubſtitutoque valore $y$ erit $\frac{dx.\overline{x^{\frac{1}{3}}+a^{\frac{1}{3}}}^{\frac{1}{2}}}{2.x^{\frac{2}{3}}}=dy$. Igitur formula, quæ quæritur, ita ſeſe habebit $2a^2xdx.\overline{x^{\frac{1}{3}}+a^{\frac{1}{3}}}^{\frac{3}{2}}+\frac{a^2x^{\frac{4}{3}}dx.\overline{x^{\frac{1}{3}}+a^{\frac{1}{3}}}^{\frac{1}{2}}}{2}$, quæ ad ſequentem reducetur $\overline{\frac{5}{2}a^2x^{\frac{4}{3}}dx+2a^{\frac{7}{3}}xdx}.\overline{x^{\frac{1}{3}}+a^{\frac{1}{3}}}^{\frac{1}{2}}$. Sit differentianda

da formula $\frac{1}{\overline{a^{\frac{1}{5}}-x^{\frac{1}{5}}}^{\,2}}$. Fac $a^{\frac{1}{5}}-x^{\frac{1}{5}}=y$, ut fit $\frac{1}{y^{2}}$, cujus differentia

$$=\frac{-2\,dy}{y^{3}}=\frac{-2\,dy}{\overline{a^{\frac{1}{5}}-x^{\frac{1}{5}}}^{\,3}}.$$ Ut invenias $dy$, differentias ſume in formula, & invenies $\frac{-dx}{5x^{\frac{4}{5}}}=dy$, quo valore ſubſtituto prodit $\frac{2\,dx}{5x^{\frac{4}{5}}\cdot\overline{a^{\frac{1}{5}}-x^{\frac{1}{5}}}^{\,3}}$ differentia quæſita.

10 Neque vero methodus ſolumodo utilis eſt, ubi formula unicam dumtaxat indeterminatam continet, ſed æque valeret, ſi duas, aut plures contineret. Primum exemplum ſufficiat formula $\overline{\frac{x}{y}+\frac{y}{x}}^{\,p}$, quæ differentiata eſt hujuſmodi

$$p\cdot\overline{\frac{x}{y}+\frac{y}{x}}^{\,p-1}\cdot D.\overline{\frac{x}{y}+\frac{y}{x}};$$ atqui

$$D\overline{\frac{x}{y}+\frac{y}{x}}=\frac{y\,dx-x\,dy}{y^{2}}+\frac{x\,dy-y\,dx}{x^{2}}=\frac{yx^{2}dx-x^{3}dy-y^{3}dx+xy^{2}dy}{x^{2}y^{2}}=$$

$$\frac{\overline{y\,dx-x\,dy}\cdot\overline{x^{2}-y^{2}}}{x^{2}y^{2}}=\overline{y\,dx-x\,dy}\cdot\overline{\frac{1}{y^{2}}-\frac{1}{x^{2}}};$$ ergo facta ſubſtitutione differentia formulæ propoſitæ erit hujuſmodi

$$p\cdot\overline{\frac{x}{y}+\frac{y}{x}}^{\,p-1}\cdot\overline{\frac{1}{y^{2}}-\frac{1}{x^{2}}}\cdot\overline{y\,dx-x\,dy}.$$ Secundum exemplum habebis in formula $\overline{xy+a^{\frac{4}{3}}\,\overline{xy}^{\frac{1}{3}}}^{\,\frac{1}{4}}$, quam pone $=z$, & ad quartam poteſtatem elevando $xy+a^{\frac{4}{3}}\cdot\overline{xy}^{\frac{1}{3}}=z^{4}$. Accipe differentias

$$y\,dx+x\,dy+\frac{\frac{1}{3}\,a^{\frac{4}{3}}\cdot\overline{y\,dx+x\,dy}}{\overline{xy}^{\frac{2}{3}}}=4z^{3}dz;$$

ſive

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ſive $\dfrac{\overline{xy}^{\frac{2}{3}} + \frac{1}{3} a^{\frac{4}{3}} . \overline{ydx + xdy}}{\overline{xy}^{\frac{2}{3}} . 4 z^3} = \dfrac{\overline{xy}^{\frac{2}{3}} + \frac{1}{3} a^{\frac{4}{3}} . \overline{ydx + xdy}}{4 . \overline{xy}^{\frac{2}{3}} . \overline{xy + a^{\frac{4}{3}} \overline{xy}^{\frac{1}{3}}}^{\frac{3}{4}}} = dz$, quæ erat invenienda. Opportunum judicavi, præmittere methodum differentiandi formulas etiam coaleſcentes ex pluribus variabilibus, tum quia brevius hic factum eſt, quod eſſet in ſecundo libro omnino faciendum, tum præſertim quia eas identidem utemur etiam in primo libro ad elegantiam calculis conciliandam.

11. Si quemadmodum cujuſcumque algebraicæ formulæ differentia invenitur; ita cujuſcumque differentialis poſſet integralis algebraica determinari, nihil in hoc calculo eſſet amplius deſiderandum. Verum formularum pars maxima algebraica ſummatoria caret, & perdifficile eſt, eam, qua formulæ prædiiæ ſunt, algebraicam integrationem determinare. Quare duo potiſſimum in hoc libro nobis præſtanda ſunt, primum ut tradamus criteria cognoſcendi, quænam formulæ algebraicam integrationem recipiunt, deinde ut, non neglecta elegantia, doceamus rationem conſtruendi eas, quæ inter tranſcendentes enumerantur. Interim ſpectemus eas formulas, quæ ex modo traditis ſtatim integrationem recipiunt. Si quantitas differentiale multiplicans, non curato exponente, & ipſa differentiata exhibeat differentiale, in quod multiplicatur, ductum, & diviſum per conſtantes quaslibet, formula eſt ſemper integrabilis algebraice. Hoc conſtat, quia ſi illa quantitas vocetur $= z$, formula mutabitur in hujuſmodi $a z^m dz$, quæ ſemper integrabilis eſt. Excipe tamen caſum, in quo exponens $m = -1$. Ita $dy\sqrt{a+y}$, quia $a+y$ habet pro differentiali $dy$, integratur, ejuſque integrale $= \frac{2}{3} . \overline{a+y}^{\frac{3}{2}}$. Similiter $\dfrac{dy}{\sqrt{a+y}}$ habebit pro integrali $\frac{1}{2}\sqrt{a+y}$. Et generatim ſummatoria $dy . \overline{a+y}^{p}$ erit $\dfrac{\overline{a+y}^{p+1}}{p+1}$. Simili modo $\dfrac{xdx}{\sqrt[3]{aa-xx}}$ erit algebraice ſummabilis, quia differentiale $aa - xx$ æquat differentiale, quod dividit, ductum in $-2$. Quare ita diſponatur $\frac{-1}{2} . -2xdx . \overline{aa-xx}^{\frac{-1}{3}}$, cujus integrale $= \frac{-1}{2} . \frac{3}{2} . \overline{aa-xx}^{\frac{2}{3}}$; & generatim $xdx . \overline{aa-xx}^{p}$ habebit pro integrali $-\frac{1}{2} . \dfrac{\overline{aa+xx}^{p+1}}{p+1}$. Generalius etiam $x^{q-1} dx . \overline{fa^q + gx^q}^{p}$, quicumque ſint numeri $p$, $q$, & coefficientes $f$, $g$, vel poſitivi, vel negativi integrationem recipit, dummodo $p$ non $= -1$,

ejus

ejus vero integratio eſt hujuſmodi $\frac{\overline{fa^q+gx^q}^{\,p+1}}{gq\cdot\overline{p+1}}$;

12. Occurrunt interdum formulæ, quæ quomodo integrentur, primo intuitu non apparet, ſed facta congrua præparatione ad præſentem regulam perducuntur. In formula $dx\sqrt{a^2x^2+x^4}$ non invenitur requiſita conditio, ſed ſi extrahatur radix $x^2$, & ita ſcribatur $xdx\sqrt{a^2+x^2}$, conditio ſeſe manifeſtat, & integrale formulæ ita ſe habet $\frac{\overline{a^2+x^2}^{\frac{3}{2}}}{3}$. Formulam $dx\sqrt{a^3+3a^2x+3ax^2+x^3}$ præparabis, ſi extrahas radicem quadrati $aa+2ax+xx$, & hoc modo ſcribas $\overline{adx+xdx}\cdot\sqrt{a+x}=dx\cdot\overline{a+x}^{\frac{3}{2}}$, cujus ſumma $=\frac{2}{5}\cdot\overline{a+x}^{\frac{5}{2}}$. Contra in formula $3ax^3dx+4x^4dx\cdot\sqrt{ax+xx}$, ut præparetur, intra ſignum radicale $x$ transferenda eſt ſic

$\overline{3ax^2dx+4x^3dx}\cdot\sqrt{ax^3+x^4}$, cujus integrale invenitur $\frac{2}{3}\cdot\overline{ax^3+x^4}^{\frac{3}{2}}$.

Formula fracta præparatur interdum multiplicato tum numeratore, tum denominatore per eamdem quantitatem. In exemplum affero $\frac{adx+xdx}{\sqrt{3a+2x}}$, quæ multiplicato numeratore, & denominatore per $x$, in hanc convertitur $\frac{axdx+x^2dx}{\sqrt{3ax^2+2x^3}}$ cujus integrale $=\frac{1}{3}\sqrt{3ax^2+2x^3}$. Similiter formulam

$\frac{dx\cdot\overline{a+x}^{\frac{3}{2}}}{\sqrt{2a^2x+3ax^2+x^3}}$ ut præpares, divide tum numeratorem, tum denominatorem per $\sqrt{a+x}$, & invenies $\frac{adx+xdx}{\sqrt{2ax+xx}}$, cujus integrale eſt $\sqrt{2ax+xx}$. Sæpe quantitati, de cujus integratione non conſtat, juvat addere, & demere eamdem quantitatem integrabilem, nam ita quantitas propoſita, ſimul cum addita aut dempta, requiſita conditione prædita eſt. Hoc accidit formulæ $xdx.\overline{a+x}^p$, quicumque ſit exponens $p$, quæ profecto per preſentem regulam non integratur, ſed adde, & deme $adx.\overline{a+x}^p$ cujus integrationem cognoſcimus, ut fiat

 $dx$.

$dx . \overline{a+x}^{p+1} - a\,dx . \overline{a+x}^{p}$. Formulæ ita præparatæ liquet ſummatoriam eſſe $\frac{\overline{a+x}^{p+2}}{p+2} - a . \frac{\overline{a+x}^{p+1}}{p+1} = \frac{a+x}{p+2} - \frac{a}{p+1} . \overline{a+x}^{p+1} = \frac{\overline{p+1} . x - a}{\overline{p+2} . \overline{p+1}} . \overline{a+x}^{p+1}$.

Ita accidet $xx\,dx . \overline{a+x}^{p}$, ſi eidem addas & demas $\overline{aa+2ax} . \overline{a+x}^{p} . dx$ quæ partim in regula continetur, partim ex caſu ſuperiore integratur. Fit namque $\overline{a+x}^{2} . dx . \overline{a+x}^{p} - \overline{aa\,dx - 2ax\,dx} . \overline{a+x}^{p}$, de cujus integratione conſtat. Idem dicas de formula $x^3 dx . \overline{a+x}^{p}$ addita, demptaque $\overline{a^3 + 3a^2 x + 3ax^2} . dx . \overline{a+x}^{p}$, quam methodum extende ad $x^4 dx . \overline{a+x}^{p}$, $x^5 dx . \overline{a+x}^{p}$; atque ita deinceps. Ex hoc itaque progreſſu palam eſt, $x^m dx . \overline{a+x}^{p}$, ſemper integrari algebraice poſſe, ſi $m$ ſit numerus integer, & poſitivus, dummodo non occurrat differentiale per ſuum integrale diviſum.

13. Conducit interdum formulam dividere in duas, & producere ad canonicam $x\,dy + y\,dx$, cujus ſummatoria eſt $xy$. Proponatur integranda formula $\frac{-a^2 dx}{x^2 \sqrt{a^2 - x^2}}$. Hæc ita diſponatur $\frac{\overline{-aa + xx - xx} . dx}{x^2 \sqrt{a^2 - x^2}}$ quæ in duas dividatur hoc modo $\frac{-dx\sqrt{aa-xx}}{x^2} - \frac{x\,dx}{x\sqrt{a^2-x^2}}$, ſeu $\sqrt{aa-xx} . \mathrm{D}\frac{1}{x} + \frac{1}{x}\mathrm{D}\sqrt{aa-xx}$; ergo ejus ſummatoria $= \frac{\sqrt{aa-xx}}{x}$. Eadem methodo tractatur formula $\frac{-a^2 dx}{\overline{xx-aa}^{\frac{3}{2}}}$, ſi ita diſponatur $\frac{x^2 dx - a^2 dx - x^2 dx}{\overline{xx-aa}^{\frac{3}{2}}}$, & dividatur in duas ſic $\frac{dx}{\sqrt{xx-aa}} - \frac{x^2 dx}{\overline{xx-aa}^{\frac{3}{2}}}$, ſive $\frac{1}{\sqrt{xx-aa}} . dx + x\,\mathrm{D} . \frac{1}{\sqrt{xx-aa}}$.

Hujus autem ſummatoria $= \frac{x}{\sqrt{xx-aa}}$.

14. Ad integrale formulæ cognoſcendum neceſſe eſt identidem, actu elevare binomia, aut polynomia ad integras poteſtates, per quam operationem formula in plures terminos dividetur omnes integrabiles, niſi occurrat fractio $\frac{dx}{x}$. Exemplum præbeat $x^{\frac{1}{3}} dx . \overline{a^2 - x^2}^{2}$, quæ, elevato binomio ad ſecundam poteſtatem, in tres terminos diſtribuitur hoc modo $a^4 x^{\frac{1}{3}} dx - 2a^2 x^{\frac{7}{3}} dx + x^{\frac{13}{3}} dx$,

qui

qui omnes integrantur ita $\frac{3a^4x^{\frac{4}{3}}}{4} - \frac{3a^2x^{\frac{10}{3}}}{5} + \frac{3x^{\frac{16}{3}}}{16}$. Univerſalius autem $x^m dx . \overline{a^n \pm x^n}^p$, quicumque ſint numeri $m$,$n$, dummodo $p$ ſit integer, & poſitivus, ſemper integrabilis eſt algebraice, excepto caſu, ubi prodit $\frac{dx}{x}$. Nam ſi elevetur binomium ad poteſtatem integram $p$, mutabitur formula in ſummam plurium terminorum ſimplicium, quorum ſinguli ad algebraicam integrationem perducuntur. Neque vero neceſſe eſt, ut quantitas ad poteſtatem integram elevanda ſit binomium; quia etiamſi fuerit trinomium, aut polynomium quodcumque, calculus eadem prorſus methodo formulam ad integrationem perducet.

15. Reſtat, ut de artificio maxime uſitato verba faciamus, ſcilicet de ſubſtitutionibus, per quas formula, de cujus integratione non conſtat, in aliam convertitur, quæ cognoſcitur integrabilis. Proponatur integranda formula $\frac{adx + xdx}{\sqrt{2ax+xx} . \sqrt{a+\sqrt{2ax+xx}}}$. Ut eam ad ſimpliciorem formam perducamus, utemur ſubſtitutione $\sqrt{2ax+xx} = z$, & quadrando $2ax+xx = zz$, & differentiis acceptis, factaque diviſione per 2, erit $adx+xdx = zdz$. Itaque peractis ſubſtitutionibus fit $\frac{zdz}{z\sqrt{a+z}} = \frac{dz}{\sqrt{a+z}}$, cujus ſummatoria $= 2\sqrt{a+z}$; ergo integralis propoſitæ erit $= 2\sqrt{a+\sqrt{2ax+xx}}$. Spectemus formulam, quam integravimus num. 13., $\frac{-a^2dx}{\overline{xx-aa}^{\frac{3}{2}}}$, quam ita ſcribamus $\frac{a^2}{x}$. $\frac{-xdx}{\overline{xx-aa}^{\frac{3}{2}}}$. Quoniam $\frac{-xdx}{\overline{xx-aa}^{\frac{3}{2}}}$ algebraice integrabilis eſt, ponamus ejus ſummatoriam $\frac{1}{\sqrt{xx-aa}} = \frac{z}{aa}$, ut formula propoſita in hanc mutetur $\frac{dz}{x}$. Ut inveniamus $x$, formulam ſubſtitutionis elevemus ad quadratum $\frac{1}{xx-aa} = \frac{zz}{a^4}$, ex qua $x = \frac{a}{z}\sqrt{aa+zz}$; ergo formula fit $\frac{zdz}{a\sqrt{aa+zz}}$, cujus ſummatoria $= \frac{\sqrt{aa+zz}}{a}$, ſubſtitutoque valore $z$ dato per $x$, tandem invenies $\frac{x}{\sqrt{xx-aa}}$.

Nulla certa regula tradi poteſt, per quam cognoſcamus, quænam ſubſtitutio facienda ſit ad transformandam ſic formulam, ut appareat integrabilis algebraice. Uſus, induſtria, & identidem fortuna analyſtæ afferent adjumentum. Sæpe conducet per ſubſtitutiones tollere aſſymetriam, advocantes eas formulas, quibus utitur Diophantus ad ſua problemata reſolvenda. Ad exemplum aſſumo formulam

mulam $\frac{dx}{x\sqrt{ax-xx}}$. Ut radicalium formula evadat expers, oportet, ut $ax-xx$ ſit quadratum; itaque fiat $ax-xx=\frac{a^2x^2}{zz}$; ergo $x=\frac{az^2}{a^2+z^2}$, & proinde $\sqrt{ax-xx}=\frac{ax}{z}=\frac{a^2z}{a^2+z^2}$, & $dx=\frac{2a^3zdz}{\overline{aa+zz}^2}$. Factis porro ſubſtitutionibus, formula in hanc convertitur $\frac{2dz}{z^2}$, cujus integrale $=\frac{-2}{z}$; ſed $z=\frac{ax}{\sqrt{ax-xx}}=\frac{a\sqrt{x}}{\sqrt{a-x}}$; ergo ſummatoria propoſitæ $=\frac{-2\sqrt{a-x}}{a\sqrt{x}}$. Eodem artificio ſummatoriam invenies formulæ $xdx\sqrt[3]{xx+2ax+aa}$. Quoniam oportet, ut $xx+2ax+aa$ ſit cubus, fiat $x+a=\frac{y^3}{aa}$; ergo $x=\frac{y^3-a^3}{aa}$, & $dx=\frac{3y^2dy}{aa}$, demum $\sqrt[3]{xx+2ax+aa}=\frac{y^2}{a^{\frac{4}{3}}}$. Itaque factis ſubſtitutionibus provenit $\frac{3y^2dy}{aa}\cdot\frac{y^3-a^3}{aa}\cdot\frac{y^2}{a^{\frac{4}{3}}}=\frac{3y^7dy-3a^3y^4dy}{a^{5+\frac{1}{3}}}$, cujus integrale eſt

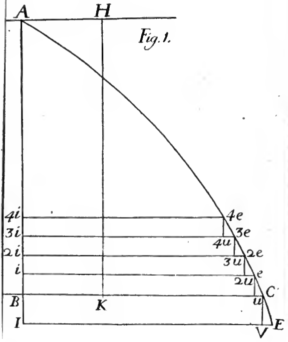
$$\frac{\frac{3}{8}y^8-\frac{3}{5}a^3y^5}{a^{5+\frac{1}{3}}}=\frac{y^5}{a^{5+\frac{1}{3}}}\cdot\overline{\frac{3}{8}y^3-\frac{3}{5}a^3}.$$

Reſtituamus formulæ ſpeciem $x$, & fiet $\frac{\overline{x+a}^{\frac{5}{3}}}{a^2}\cdot\overline{\frac{3}{8}a^2x+\frac{3}{8}a^3-\frac{3}{5}a^3}=\overline{x+a}^{\frac{5}{3}}\cdot\overline{\frac{3}{8}x-\frac{9}{40}a}$.

16. Pergo illuſtrare uſum ſubſtitutionum, ut formulam late patentem ad integrationem perducam. Ut claritati ſerviam, incipio a formula $x^2dx.\overline{a+x}^{\frac{3}{2}}$. Fac $a+x=z$, ergo $dx=dz$, $x=z-a$; igitur peractis ſubſtitutionibus oritur $z^{\frac{3}{2}}dz.\overline{z-a}^2$. Elevato binomio ad quadratum, fit $z^{\frac{7}{2}}dz-2az^{\frac{5}{2}}dz+a^2z^{\frac{3}{2}}dz$, quæ integrata eſt

$$\frac{2z^{\frac{9}{2}}}{9}-\frac{4az^{\frac{7}{2}}}{7}+\frac{2a^2z^{\frac{5}{2}}}{5}=2z^{\frac{5}{2}}.\overline{\frac{zz}{9}-\frac{2az}{7}+\frac{aa}{5}};$$

reſtitutaque ſpecie $x$ pro-


A
H
Fig. 1.
4i
3i
2i
i
4e
3e
4u
2e
3u
e
2u
C
B
K
u
I
V
E

provenit $2.\overline{a+x}^{\frac{5}{2}}.\overline{\frac{xx}{9}-4\frac{ax}{7\cdot9}-\frac{8aa}{5\cdot7\cdot9}}$. Luce clarius eſt, methodum hanc ſeſe extendere ad formulam generalem $x^m dx.\overline{a+x}^p$; quicumque ſit numerus $p$, dummodo numerus $m$ ſit integer, & poſitivus. Nam facta, ut antea ſubſtitutione, mutatur in hanc $z^p dz.\overline{z-a}^m$. Elato enim binomio ad poteſtatem integram, & poſitivam $m$, plures termini oriuntur numero $m+1$, qui omnes algebraice integrabiles ſunt, excepto ſemper caſu, ubi occurreret $\frac{dz}{z}$. Progrediens ajo, formulam $x^m dx.\overline{a^2+x^2}^p$ integrabilem fore, ſi $m$ ſit numerus impar, & affirmativus, quo in caſu $m-1$ erit par; ergo $\frac{m-1}{2}$ integer. Nam facta $a^2+x^2=az$, & $x^2=a.\overline{z-a}$, & $xdx=\frac{adz}{2}$ in hanc mutatur:

$$\frac{1}{2}a^{\frac{2+m+1}{2}}z^p dz.\overline{z-a}^{\frac{m-1}{2}},$$

in qua, quum numerus $\frac{m-1}{2}$ ſit integer, & affirmativus, binomium ad poteſtatem integram elevandum eſt. Quare formula in plures terminos omnes integrabiles, ut antea, diſtribuitur. Addo formulam $x^m dx.\overline{a^n+x^n}^p$, ſi $\frac{m-n+1}{n}$ ſit numerus integer, & affirmativus. Ita diſponatur $x^{m-n+1}.x^{n-1}dx.\overline{a^n+x^n}^p$, tum fiat ſubſtitutio $a^n+x^n=a^{n-1}z$; tum peractis neceſſariis operationibus deveniemus ad formulam

$$\frac{1}{n}a^{m+1+pn-p-\frac{m-1}{n}}z^p dz.\overline{z-a}^{\frac{m-n+1}{n}},$$

quæ ſi $\frac{m-n+1}{n}$ ſit poſitivus, & integer, æque ac ſuperiores, integrabilis eſt. Cuique obvium eſt, methodum hanc æque utilem eſſe, tametſi alteruter ex binomii terminis ſigno $-$ afficeretur. Hæc, & ſimilia artificia, auctor ſum, ut analyſta plurimum ſibi reddat familiaria. Nam quamquam generales aliquot canones trademus deinceps, per quos cognoſcamus formulas algebraice integrabiles; tamen niſi opportunum artificium advocemus, calculi ſæpiſſime moleſti ſunt, & inelegantes.

## *CAPUT QUARTUM.*

### De integratione formularum per ſeries infinitas.

1. Quotieſcumque occurrit formula, quæ aut algebraice nullo modo eſt integrabilis, aut de ejus integratione nihil ſcimus, licebit eam ad integrationem perducere ope ſerierum infinitarum, per quas ejus valorem ſi minus exactum, ſaltem vero proximum determinamus. Methodi duæ, de quibus in

in ſuperiore tomo verba fecimus, reducendi in ſeries fractiones, & poteſtates imperfectas, formulam differentialem in terminos numero infinitos divident, omnes integrabiles. Curandum tamen ut ſeries ſint convergentes, quia divergentes, & parallelæ inutiles ſunt. Sed præſtat prius oſtendere methodum magis analyticam reducendi in ſeries fractiones non omiſſis aliquot neceſſariis animadverſionibus; hoc enim perſpicuitatem iis, quæ dicenda nobis ſunt, conciliabit. Methodus in hoc ſita eſt, ut fractio æqualis ſupponatur ſeriei præditæ coefficientibus indeterminatis. Hæc deinde multiplicetur per denominatorem fractionis, ut exurgat ſeries æqualis numeratori. Termini ſeriei comparentur cum terminis numeratoris, & per hanc collationem coefficientes determinentur, quibus in ſuppoſita ſerie ſubſtitutis, obtinemus ſeriem fractioni æqualem. Methodus cæteroquin non difficilis exemplis illuſtranda eſt.

2. Sit fractio $\frac{a}{b+mx}$, quæ ponatur æqualis ſeriei, eujus coefficientes ſint indeterminati hoc modo $\frac{a}{b+mx} = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5$ &c. Multiplicetur æquatio per $b+mx$, ut fiat

$$a = bA + bBx + bCx^2 + bDx^3 + bEx^4 + bFx^5 \text{ \&c.}$$
$$+ mAx + mBx^2 + mCx^3 + mDx^4 + mEx^5 \text{ \&c.}$$

Facta terminorum comparatione, hoc eſt poſito primo termino $= a$, reliquis omnibus $= o$, provenient hæ coefficientium determinationes $A = \frac{a}{b}$, $B = \frac{-mA}{b}$, $C = \frac{-mB}{b}$, $D = \frac{-mC}{b}$, $E = \frac{-mD}{b}$, $F = \frac{-mE}{b}$ &c. Si hos coefficientes diligenter inſpicias, videbis, eos conſtituere ſeriem recurrentem primi ordinis, in qua terminus quilibet datur per antecedentem ductum in $\frac{-m}{b}$. Quapropter fractio $\frac{a}{b+mx} = \frac{a}{b} - \frac{mA}{b}x - \frac{mB}{b}x^2 - \frac{mC}{b}x^3 - \frac{mD}{b}x^4 + \frac{mE}{b}x^5$ &c. Hæc ſeries uſurpanda eſt, donec ſit $x < \frac{b}{m}$. Quod ſi $x$ fiat major $\frac{b}{m}$, præſtabit ſeriem ita formare $\frac{a}{mx+b} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x^4} + \frac{E}{x^5} + \frac{F}{x^6}$ &c., qua methodo eadem tractata, determinabuntur coefficientes, & orietur

$$\frac{a}{mx+b} = \frac{a}{mx} - \frac{bA}{mx^2} - \frac{bB}{mx^3} - \frac{bC}{mx^4} - \frac{bD}{mx^5} - \frac{bE}{mx^6} \text{ \&c.},$$

in qua litteræ $A$, $B$, $C$ &c. denotant coefficientem termini antecedentis.

3. Sit fractio $\frac{a+mx}{b+cx+nx^2}$, quam ſuppone æqualem ſeriei, ut infra

$$\frac{a+mx}{b+cx+nx^2} = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 \text{ \&c.}$$

Facta multiplicatione per denominatorem, obtinebis

$a + mx$

$$a+mx=bA+bBx+bCx^2+bDx^3+bEx^4+bFx^5 \text{ \&c.}$$
$$+cAx+cBx^2+cCx^3+cDx^4+cEx^5 \text{ \&c.}$$
$$+nAx^2+nBx^3+nCx^4+nDx^5 \text{ \&c.}$$ Si primum terminum facias $=a$, invenies $A=\frac{a}{b}$, ſi facias ſecundum $=mx$, invenies $B=\frac{m-cA}{b}$. Reliqui facti $=o$, præbent $C=\frac{-cB-nA}{b}$, $D=\frac{-cC-nB}{b}$, $E=\frac{-cD-nC}{b}$, atque ita deinceps, qui coefficientes conſtituunt ſeriem recurrentem ſecundi ordinis, in qua quilibet terminus datur per duos antecedentes, quorum qui propior eſt termino inveniendo, ducendus in $\frac{-c}{b}$, qui remotior in $\frac{-n}{b}$. Quare invenietur $\frac{a+mx}{b+cx+nx^2}=\frac{a}{b}-\frac{cA+m-cA}{b}.x$ $\frac{-cB-nA}{b}x^2\ \frac{-cC-nB}{b}x^3$ &c. Primæ litteræ $A$, $B$, $C$ &c. denotant coefficientem termini antecedentis, ſecundæ $A$, $B$ &c. denotant eum qui antecedentem præcedit. Quod ſi ſeries invenienda ſit, in cujus diviſore ſit $x$, hoc modo erit formanda $\frac{mx+a}{nx^2+cx+b}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x^3}+\frac{D}{x^4}$ &c., qua eodem modo tractata, invenientur coefficientes $A$, $B$, $C$, $D$ conſtituere ſeriem recurrentem ſecundi ordinis, in qua quilibet terminus datur per duos antecedentes. Eodem progreſſu, ſi denominator fractionis ſit formula tertii gradus, coefficientes ſeriei conſtituent ſeriem recurrentem tertii ordinis, atque ita deinceps. Quæ obſervatio facillimam reddit harum ſerierum inventionem.

4. Aliqua tamen ſunt neceſſario advertenda. Ut ſeries recurrens ſtatim oriatur, neceſſe eſt, ut exponens $x$ in numeratore minor ſit, quam in denominatore: nam ſi fuerit æqualis, aut major, ſeries reccurrens non erit continua, niſi poſt terminum præditum eo exponente, qui in numeratore invenitur. Exemplum det fractio

$$\frac{a^3+x^3}{a-x}=A+Bx+Cx^2+Dx^3+Ex^4+Fx^5 \text{ \&c.},$$ quæ, facta multiplicatione per $a-x$, in hanc tranſit

$$a^3+x^3=aA+aBx+aCx^2+aDx^3+aEx^4+aFx^5 \text{ \&c.}$$
$$-Ax-Bx^2-Cx^3-Dx^4-Ex^5 \text{ \&c.}$$

Fiat comparatio, & provenit $A=a^2$, $B=\frac{A}{a}$, $C=\frac{B}{a}$. Huc uſque res ordinatim procedit, ſed in ſequenti termino, ubi $aD-C$ æquanda eſt 1, ſit $D=\frac{1+C}{a}$, atque propterea ordo abrumpitur. Poſt hunc terminum ſeries non intercepta procedet. Quod ſi velles, exponentem ſpeciei $x$ continue decreſcere, ita

ita conſtituenda eſſet $\frac{x^3+a^3}{-x+a} = Ax^2+Bx+C+\frac{D}{x}+\frac{E}{x^2}+\frac{F}{x^3}$ &c., & facta multiplicatione

$$x^3+a^3 = -Ax^3 - Bx^2 - Cx - D - \frac{E}{x} - \frac{F}{xx} \text{ \&c.}$$
$$+aAx^2 + aBx + aC + \frac{aD}{x} + \frac{aE}{x^2} \text{ \&c.}$$

Facta comparatione proveniet $A=-1$, $B=aA$, $C=aB$; ordo qui hactenus ſervatus eſt, abrumpitur; nam $-D+aC$ æquanda $a^3$; ergo $D=aC-a^3$; ordo autem deinceps non interruptus procedet. Quare, ut defectus ordinis in ſerie vitetur, numerator toties per denominatorem dividatur, donec fractio ſeſe offerat, in qua exponens numeratoris minor ſit exponente denominatoris. Quare fractio in exemplum adducta in formulam partim integram, partim fractam ſic erit reſolvenda $x^2-ax-aa+\frac{2a^3}{a-x}$, tum fractio $\frac{2a^3}{a-x}$ in ſeriem infinitam erit convertenda.

5. Methodus hæc non poſtulat, ut quantitas $x$, cujus exponentes debent in ſerie creſcere, vel decreſcere ordinatim, in fractione affecta ſit exponentibus integris. Verum ſi exponentes ſint fracti, ne temere progrediamur, opus eſt determinare ſeriem arithmeticam, ſecundum quam exponentes debent in ſerie creſcere, aut decreſcere. Hinc ob rem ſic invenias ſeriem arithmeticam maxime ſimplicem, quæ complectatur exponentes omnes tum numeratoris tum denominatoris fractionis. Hæc erit progreſſio exponentium ſeriei, quæ debet incipere a $0$, ſi exponentes creſcant, & a maximo exponente denominatoris negative ſumpto, ſi decreſcant exponentes, ſive, quod idem eſt ſi creſcant exponentes $x$ poſiti in denominatore. Res neceſſario declaranda eſt exemplo. Sit propoſita fractio $\frac{a^{\frac{1}{3}}+x^{\frac{1}{3}}}{a-a^{\frac{1}{2}}x^{\frac{1}{2}}+x}$.

In hac exponentes diviſoris conſtituunt ſeriem aritmheticam $0, \frac{1}{2}, 1$, ſed in hac ſerie locum non tenet exponens numeratoris $\frac{1}{3}$. Ut inveniatur ſeries omnes complectens, reducendi ſunt exponentes ad eumdem denominatorem, qui invenietur $6$, & fractio ita exponenda erit $\frac{a^{\frac{2}{6}}+x^{\frac{2}{6}}}{a^{\frac{6}{6}}-a^{\frac{3}{6}}x^{\frac{3}{6}}+x^{\frac{6}{6}}}$. Apparet exponentes contineri in ſerie arithmetica $0, \frac{1}{6}, \frac{2}{6}, \frac{3}{6}, \frac{4}{6}, \frac{5}{6}, \frac{6}{6}$ &c. Itaque ſi velis ſeriem, in qua creſcat exponens $x$, ita æquatio ſtatuenda erit, ſuppletis in fractione terminis intermediis, qui deſunt in denominatore, facta mul-

multiplicatione per o, $\dfrac{a^{\frac{2}{6}}+x^{\frac{2}{6}}}{a^{\frac{6}{6}}+0.x^{\frac{1}{6}}+0.x^{\frac{2}{6}}-a^{\frac{3}{6}}x^{\frac{3}{6}}+0.x^{\frac{4}{6}}+0.x^{\frac{5}{6}}+x^{\frac{6}{6}}} =$ $A+Bx^{\frac{1}{6}}+Cx^{\frac{2}{6}}+Dx^{\frac{3}{6}}+Ex^{\frac{4}{6}}+Fx^{\frac{5}{6}}+Gx^{\frac{6}{6}}$ &c. atque factis neceſſariis operationibus oritur ſeries reccurrens ordinis ſexti. Quod ſi velis ſeriem, in qua decreſcant exponentes $x$, ſive in qua $x$ poſitus ſit in diviſore, eadem fractio æquanda eſt $\dfrac{A}{x^{\frac{6}{6}}}+\dfrac{B}{x^{\frac{7}{6}}}+\dfrac{C}{x^{\frac{8}{6}}}+\dfrac{D}{x^{\frac{9}{6}}}+\dfrac{E}{x^{\frac{10}{6}}}+\dfrac{F}{x^{\frac{11}{6}}}+\dfrac{G}{x^{\frac{12}{6}}}$ &c. ex qua orietur pariter ſexti ordinis ſeries recurrens.

6. Ut vites ſeries recurrentes altiorum ordinum, præſtabit non raro, fractionem in duas, aut plures diſtribuere, & ſingulis ſuas ſeries æquales invenire. Hoc in fractione num. ſuperioris palam fiet. Divide enim in duas, nempe $\dfrac{a^{\frac{1}{3}}}{a-a^{\frac{1}{2}}x^{\frac{1}{2}}+x}$, $\dfrac{x^{\frac{1}{3}}}{a-a^{\frac{1}{2}}x^{\frac{1}{2}}+x}$. Primam æqua ſeriei $A+Bx^{\frac{1}{2}}+Cx+Dx^{\frac{3}{2}}+Ex^{2}$ &c., & per methodum traditam determinabis coefficientes $A$, $B$, $C$ &c. per ſeriem recurrentem ſecundi ordinis. Ut alteram fractionem in ſeriem redigas ſepone multiplicatorem $x^{\frac{1}{3}}$, & fractioni $\dfrac{1}{a-a^{\frac{1}{2}}x^{\frac{1}{2}}+x}$ ſeriem æqualem inveni, quam deinde multiplica per $x^{\frac{1}{3}}$. Methodus hæc, quæ in ſeriebus coefficientes adhibet indeterminatos, elegantiam maxime adjuvat & ſimplicitatem. Sed niſi aut de induſtria determines, aut fortunate incidas in eam exponentium progreſſionem, quæ ad rem facit, fruſtra ſupputationes inſtitues; namque comparationes aut in abſurdum deſinent, aut ſeriem deſtruent. Quod ſi ex collatione inveniamus coefficientium valores congruos, qui ſeriem præbeant, certi fiemus, nos veram exponentium progreſſionem elegiſſe.

7. Ut methodum hanc in difficilioribus caſibus permultis uſurpare liceat, oportet facilem, & expeditam methodum cognoſcere, qua aliquot operationes in ſeriebus infinitis peraguntur. Ut facilitati conſulam in hiſce inſtituendis, ſpectabo ſemper ſeries, in quibus primus terminus eſt conſtans, ſecundus continens $x$ linearem. Quia, quum ſeries debeant habere exponentes $x$ in progreſſione arithmetica, ad eam formam omnes reduci poſſunt. Nam ſi in primo termino exiſtit $x$, hic ſeponatur tamquam multiplicator communis omnium terminorum, & in ſerie, quæ reliqua eſt, operationes inſtituantur. Si in ſecundo ter-

termino $x$ habeat exponentem fractum, aut negativum, fiat $= z$, & ſeries ad exponentes integros perducetur.

8. His præmiſſis quomodo ſeries ſit multiplicanda per ſeriem primum docebo. Sint ſeries duæ $M$, $N$

$$\begin{array}{l|l} M & A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 \;\&c. \\ N & a + bx + cx^2 + dx^3 + ex^4 + fx^5 \;\&c. \end{array}$$

Fiat multiplicatio more uſitato, & oritur ſeries $P$

$$\begin{array}{l|r} P & aA + aBx + aCx^2 + aDx^3 + aEx^4 + aFx^5 \;\&c. \\ & + bAx + bBx^2 + bCx^3 + bDx^4 + bEx^5 \;\&c. \\ & + cAx^2 + cBx^3 + cCx^4 + cDx^5 \;\&c. \\ & + dAx^3 + dBx^4 + dCx^5 \;\&c. \\ & + eAx^4 + eBx^5 \;\&c. \\ & + fAx^5 \;\&c. \end{array}$$

Diligenter inſpiciendum eſt, quomodo ptogrediuntur coefficientes ſingulorum terminorum ſeriei $P$. Primus eſt productum ex duobus primis terminis ſerierum $M$, $N$. Secundus naſcitur, ſi primus ſeriei $N$ multiplicetur per ſecundum $M$, & ſecundus $N$ per primum $M$. Tertius coefficiens obtinetur, ſi primus cum tertio, ſecundus cum ſecundo, tertius cum primo multiplicatur. Atque ita deinceps ſingulos coefficientes ſeriei $P$ determinabis, ſi tot coefficientes ordinatim ſumas in ſeriebus $M$, $N$, quotus eſt terminus inveniendus, tum primum unius ducas in ultimum alterius, ſecundus in penultimum, tertius in antepenultimum, atque ita deinceps, donec exhauſti fuerint omnes: atque hac facili methodo multiplicatio perficietur. In exemplum profero ſequentes ſeries

$$\begin{array}{l} a + x + \frac{x^2}{a} + \frac{x^3}{a^2} + \frac{x^4}{a^3} + \frac{x^5}{a^4} \;\&c. \\ a - 2x + \frac{3x^2}{a} - \frac{4x^3}{a^2} + \frac{5x^4}{a^3} - \frac{6x^5}{a^4} \;\&c. \end{array}$$

quæ ſi invicem multiplicentur, dant

$$aa - ax + 2x^2 - \frac{2x^3}{a} + \frac{3x^4}{a^2} - \frac{3x^5}{a^3} \;\&c.$$

9. Dividenda jam ſit unitas per ſeriem infinitam $a + bx + cx^2 + dx^3 + ex^4 + fx^5$ &c. Fractio fiat æqualis ſeriei infinitæ $A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5$ &c., cujus coefficienies $A$. $B$, $C$ &c. determinandi ſunt. Multiplicetur æquatio per ſeriem, quæ poſita eſt in denominatore, ut ſit

$$\begin{array}{r} 1 = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 \;\&c. \\ \text{ductæ in} \\ a + bx + cx^2 + dx^3 + ex^4 + fx^5 \;\&c. \end{array}$$

Fiat multiplicatio, ut præcipitur num. ſuperiore,

1 =

$$
1 = \begin{array}{r}
aA + aBx + aCx^2 + aDx^3 + aEx^4 + aFx^5 \text{ \&c.} \\
+ bAx + bBx^2 + bCx^3 + bDx^4 + bEx^5 \text{ \&c.} \\
+ cAx^2 + cBx^3 + cCx^4 + cDx^5 \text{ \&c.} \\
+ dAx^3 + dBx^4 + dCx^5 \text{ \&c.} \\
+ eAx^4 + eBx^5 \text{ \&c.} \\
+ fAx^5 \text{ \&c.}
\end{array}
$$

Si primus ſeriei terminus fiat $= 1$, reliqui $= 0$, provenient hæ determinationes coefficientium, qui determinandi veniunt, $A = \frac{1}{a}$, $B = \frac{-bA}{a}$, $C = \frac{-bB - cA}{a}$, $D = \frac{-bC - cB - dA}{a}$, $E = \frac{-bD - cC - dB - eA}{a}$, $F = \frac{-bE - cD - dC - eB - fA}{a}$ &c. Quod ſi numerator non eſſet unitas, ſed eſſet formula aut finitis, aut infinitis terminis conſtans, vel eodem modo calculum perfice, vel inventa ſerie æquali fractioni, cujus numerator eſt unitas, eamdem multiplica cum numeratore, prout præceptum eſt. Iuvenienda ſit ſeries, quæ æqualis ſit unitati diviſæ per ſeriem $1 - \frac{2x}{a} + \frac{3x^2}{a^2} - \frac{4x^3}{a^3} + \frac{5x^4}{a^4}$ &c. Ex calculo ſuperiore invenies $A = 1$, $B = \frac{2}{a}$, $C = \frac{1}{a^2}$, $D = 0$, $E = 0$, $F = 0$ &c.; ergo quotiens non eſt ſeries infinita, ſed trinomium $1 + \frac{2x}{a} + \frac{xx}{aa}$ ſive $\overline{1 + \frac{x}{a}}^2$.

10. Tranſeamus ad operationem maxime difficilem elevandi ſeriem infinitam ad quamlibet poteſtatem $p$: ob quam rem methodum ſeligo, in qua uſus calculi differentialis cognoſci incipiet. Itaque ſit ſeries $M$ elevanda ad poteſtatem $p$:

$M \mid a + bx + cx^2 + dx^3 + ex^4 + fx^5 + gx^6 + hx^7$ &c.

Huic elatæ ad poteſtatem $p$ fac æqualem ſeriem N

$N \mid A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + Gx^6 + Hx^7$ &c.

ut habeas æquationem $M^p = N$. Hæc æquatio valet quicumque ſit valor $x$; igitur etiamſi $x = 0$; ſed poſita $x = 0$, provenit $A = a^p$; ergo determinatus eſt valor primi coefficientis $A$. Æquationis $M^p = N$ ſume differentiam, ut habeas

$$p . M^{p-1} . \overline{bdx + 2cxdx + 3dx^2dx + 4ex^3dx + 5fx^4dx + 6gx^5dx + 7hx^6dx \text{ \&c.}}$$
$$= Bdx + 2Cxdx + 3Dx^2dx + 4Ex^3dx + 5Fx^4dx + 6Gx^5dx + 7Hx^6dx \text{ \&c.}$$

Ubi eſt duplex $d$, prima eſt coefficiens, altera nota differentiæ. Nunc divide primam partem per $M^p dx$, ſecundam per $Ndx$, quæ æquantur; & obtinebis

$$\frac{p.\,b+2cx+3dx^2+4ex^3+5fx^4+6gx^5+7hx^6\ \&c.}{a+bx+cx^2+dx^3+ex^4+fx^5+gx^6+hx^7\ \&c.} = \frac{B+2Cx+3Dx^2+4Ex^3+5Fx^4+6Gx^5+7Hx^6\ \&c.}{A+Bx+Cx^2+Dx^3+Ex^4+Fx^5+Gx^6+Hx^7\ \&c.}$$

11. Poſſem ſtatim determinare coefficientem $B$, facta $x=0$, quia proveniret $\frac{pb}{a}=\frac{B}{A}$; ergo $B=\frac{pbA}{a}$. Sed quoniam hac methodo alii coefficientes non determinantur, producentes calculum æquationem a diviſoribus liberemus, ut fiat

$$p.\ b+2cx+3dx^2+4ex^3+5fx^4+6gx^5+7hx^6\ \&c.$$

ducta in

$$A+Bx+Cx^2+Dx^3+Ex^4+Fx^5+Gx^6+Hx^7\ \&c.$$

æqualis

$$B+2Cx+3Dx^2+4Ex^3+5Fx^4+6Gx^5+7Hx^6\ \&c.$$

ducta in

$a+bx+cx^2+dx^3+ex^4+fx^5+gx^6+hx^7$ &c. Jam vero multiplicentur ſeries, quæ inter ſeſe ſunt multiplicandæ, ut proveniat æquatio

$$\begin{aligned}
p.\ bA+ bBx+ bCx^2+ bDx^3+ bEx^4+ bFx^5+ bGx^6+ bHx^7\ \&c.\\
+2cAx+2cBx^2+2cCx^3+2cDx^4+2cEx^5+2cFx^6+2cGx^7\ \&c.\\
+3dAx^2+3dBx^3+3dCx^4+3dDx^5+3dEx^6+3dFx^7\ \&c.\\
+4eAx^3+4eBx^4+4eCx^5+4eDx^6+4eEx^7\ \&c.\\
+5fAx^4+5fBx^5+\ fCx^6+5fDx^7\ \&c.\\
+6gAx^5+6gBx^6+6gCx^7\ \&c.\\
+7hAx^6+7hBx^7\ \&c.
\end{aligned}$$

æqualis

$$\begin{aligned}
aB+ bBx+ cBx^2+ dBx^3+ eBx^4+ fBx^5+ gBx^6+ hBx^7\ \&c.\\
+2aCx+2bCx^2+2cCx^3+2dCx^4+2eCx^5+2fCx^6+2gCx^7\ \&c.\\
+3aDx^2+3bDx^3+3cDx^4+3dDx^5+3eDx^6+3fDx^7\ \&c.\\
+4aEx^3+4bEx^4+4cEx^5+4dEx^6+4eEx^7\ \&c.\\
+5aFx^4+5bFx^5+5cFx^6+5dFu^7\ \&c.\\
+6aGx^5+6bGx^6+0cGx^7\ \&c.\\
+7aHx^6+7bHx^7\ \&c.
\end{aligned}$$

Hujus æquationis progreſſus ita patens eſt cum in columnis horizontalibus, tum in verticalibus, ut ſuperfluum ſit, de eo plura dicere.

12. Quoniam vero æquatio identica eſſe debet, ſingulorum terminorum collatio tot æquationes præbebit, quot ſunt litteræ $B$, $C$, $D$ &c., per quas coefficientes omnes determinationem accipient. Hujuſmodi autem valores proveniunt, quibus addendus eſt valor $A$ ſupra inventus $A=a^p$, $B=\frac{pbA}{a}$

$C=$

$$C = \frac{\overline{p-1}.bB + 2pcA}{2a},\ D = \frac{\overline{p-2}.bC + \overline{2p-1}.cB + 3pdA}{3a}$$

$$E = \frac{\overline{p-3}.bD + \overline{2p-2}.cC + \overline{3p-1}.dB + 4peA}{4a}$$

$$F = \frac{\overline{p-4}.bE + \overline{2p-3}.cD + \overline{3p-2}.dC + \overline{4p-1}.eB + 5pfG}{5a}$$

$$G = \frac{\overline{p-5}.bF + \overline{2p-4}.cE + \overline{3p-3}.dD + \overline{4p-2}.eC + \overline{5p-1}.fB + 6pgA}{6a}$$

$$F = \frac{\overline{p-6}.bG + \overline{2p-5}.cF + \overline{3p-4}.dE + \overline{4p-3}.eD + \overline{5p-2}.fC + \overline{6p-1}.gB + 7phA}{7a}$$

Patens eſt, quomodo horum coefficientium valores progrediantur, & quomodo ex progreſſu reliqui poſſint inveniri.

13. Methodus inſervire etiam poteſt elevando ad quamlibet poteſtatem $p$ polynomio, tametſi conſtet finito numero terminorum. Etenim ſatis erit ponere $=o$ coefficientes omnium terminorum, qui deſunt; quo in caſu ſi $p$. ſit numerus integer, ſeries abrumpitur, & formula continebit finitum numerum terminorum. Efferamus ad poteſtatem $p$ binomium $a+x$. Quando $b=1$, & $c$, $d$, $e$ &c. omnes $=o$, provenit $A=a^p$, $B=\frac{pA}{a}$, $C=\frac{\overline{p-1}.B}{2a}$, $D=\frac{\overline{p-2}.C}{3a}$, $E=\frac{\overline{p-3}.D}{4a}$, $F=\frac{\overline{p-4}.E}{5a}$ &c. Qui valores, ſi $A$, $B$, $C$ &c. ſucceſſive ſubſtituantur, in hos mutantur, $A=a^p$, $B=p.a^{p-1}$, $C=\frac{p.\overline{p-1}}{2}.a^{p-2}$, $D=\frac{p.\overline{p-1}.\overline{p-2}}{2.3}.a^{p-3}$, $E=\frac{p.\overline{p-1}.\overline{p-2}.\overline{p-3}}{2.3.4}.a^{p-4}$, $F=\frac{p.\overline{p-1}.\overline{p-2}.\overline{p-3}.\overline{p-4}}{2.3.4.5}.a^{p-5}$ &c. qui cohærent cum illis, quos diverſiſſima methodo in ſuperiori tomo invenimus. Elevemus ad ſecundam poteſtatem formulam $a-2x+\frac{3x^2}{a}+\frac{4x^3}{aa}$. Fiet $a=a$, $b=-2$, $c=\frac{3}{a}$, $d=\frac{-3}{aa}$, reliqui omnes $=o$, & $p=2$. Subductis rationibus invenies $A=aa$, $B=-4a$, $C=10$, $D=-\frac{20}{a}$, $E=\frac{25}{aa}$, $F=\frac{-24}{a^3}$, $G=\frac{16}{a^4}$; ergo quadratum propoſitæ formulæ erit $aa-4ax+10x^2-\frac{20x^3}{a}+\frac{25x^4}{aa}-\frac{24x^5}{a^3}+\frac{16x^6}{a^4}$.

14. Alicui ſeriei methodum applicemus. Ad poteſtatem $p$ elevanda ſit ſeries $a-x+\frac{x^2}{a}+\frac{x^3}{aa}+\frac{x^4}{a^3}+\frac{x^5}{a^4}$ &c. Computa valores coefficientium in-

de-

determinatorum poſitis $\alpha = a$, $b$, $c$, $d$, $e$ &c. omnibus $= 1$, & invenies

$A = a^p$, $B = pa^{p-1}$, $C = \frac{p.\overline{p+1}}{2}.a^{p-2}$

$D = \frac{p.\overline{p+1}.\overline{p+2}}{2.3}.a^{p-3}$, $E = \frac{p.\overline{p+1}.\overline{p+2}}{2.3.4}.a^{p-4}$: quare poteſtas ſeriei propoſitæ erit hujuſmodi

$$a^p + pa^{p-1}x + \frac{p.\overline{p+1}}{2}.a^{p-2}x^2 + \frac{p.\overline{p+1}.\overline{p+2}}{2.3}a^{p-3}x^3 + \frac{p.\overline{p+1}.\overline{p+2}.\overline{p+3}}{2.3.4}a^{p-4}x^4 \text{ \&c.}$$

Si $p = 2$, fiet quadratum ſeriei propoſitæ

$a^2 + 2ax + 3x^2 + \frac{4x^3}{a} + \frac{5x^4}{a^2}$ &c. Si $p = 3$; cubus erit

$a^3 + 3a^2x + 6ax^2 + 10x^3 + \frac{15x^4}{a} + \frac{21x^5}{a^2}$ &c. Si $p = 4$, quadrato quadratum erit $a^4 + 4a^3x + 10a^2x^2 + 35ax^2 + 56x^4 + \frac{84x^5}{a}$ &c.: atque ita deinceps. Si $p = \frac{1}{2}$, ut ſeriei propoſitæ invenienda ſit radix quadrata, prodibit

$a^{\frac{1}{2}} + \frac{x}{2.a^{\frac{1}{2}}} + \frac{3.x^2}{2^3.a^{\frac{3}{2}}} + \frac{3.5x^3}{2^4.3a^{\frac{5}{2}}} + \frac{3.5.7x^4}{2^5.3.4a^{\frac{7}{2}}}$ &c. Si $p = \frac{1}{3}$ erit

$a^{\frac{1}{2}} + \frac{x}{3.a^{\frac{2}{3}}} + \frac{4.x^2}{3^2.2a^{\frac{5}{3}}} + \frac{4.7.x^3}{3^3.2.3.a^{\frac{8}{3}}} + \frac{4.7.10.x^4}{3^4.2.3.4.a^{\frac{11}{3}}}$ &c., & ſic de reliquis.

15. Reſtat, ut de operatione maxime difficili agam, per quam, quum $z$ eſt æqualis ſeriei ex $x$ coaleſcenti, invenienda eſt $x$ per $z$. Sit æquatio

$y = a + bx + cx^2 + dx^3 + ex^4$ &c. Fac $y - a = z$, ut habeas

$M \mid z = bx + cx^2 + dx^3 + ex^4$ &c. Suppone jam

$N \mid x = Bz + Cz^2 + Dz^3 + Ez^4$ &c. Æquatio $M$ elevetur ad quadratum, ad cubum, ad quadrato quadratum, & ita deinceps; tum valores $z$, $z^2$, $z^3$ ſubſtituantur in æquatione $N$, quæ proveniet identica. Quare ſi primum coefficiens fiat $= 1$, reliqui omnes $= o$, determinabuntur valores $B$, $C$, $D$ &c., atque adeo dabitur $x$ per $z$. Ad exemplum ſit

$z = x + \frac{x^2}{a} + \frac{x^3}{a^2} + \frac{x^4}{a^3}$ &c., erit ex num. ſuperiore

$zz = x^2 + \frac{2x^3}{a} + \frac{3x^4}{a^2} + \frac{4x^5}{a^3}$ &c.

$z^3 = x^3 + \frac{3x^4}{a} + \frac{6x^5}{a^2} + \frac{15x^6}{a^3}$ &c.

$z^4 = x^4 + \frac{4x^5}{a} + \frac{10x^6}{a^2} + \frac{35x^7}{a^3}$ &c. Omnes iſti valores ſubſtituantur in ſerie $N$, ut fiat

$$x = Bx + B\frac{x^2}{a} + B\frac{x^3}{a^2} + B\frac{x^4}{a^3} + B\frac{x^5}{a^4} \text{ \&c.}$$
$$+ Cx^2 + \frac{2Cx^3}{a} + \frac{3Cx^4}{a^2} + \frac{4Cx^5}{a^3} \text{ \&c.}$$
$$+ Dx^3 + \frac{3Dx^4}{a} + \frac{6Dx^5}{a^2} \text{ \&c.}$$
$$+ Ex^4 + \frac{3Ex^5}{a} \text{ \&c.}$$
$$+ Fx^5 \text{ \&c.}$$

Facta, ut par eſt, comparatione, invenies $B = 1$, $C = \frac{-B}{a} = \frac{-1}{a}$, $D = \frac{-B}{aa} - \frac{2C}{a} = \frac{1}{aa}$, $E = \frac{-B}{a^3} - \frac{3C}{a^2} - \frac{3D}{a} = \frac{-1}{a^3}$, $F = \frac{-B}{a^4} - \frac{4C}{a^3} - \frac{6C}{a^2} - \frac{4E}{a} = \frac{1}{a^4}$, & ſic de reliquis. Itaque erit $x = z - \frac{z^2}{a} + \frac{z^3}{a^2} - \frac{z^4}{a^3} + \frac{z^5}{a^4}$ &c. Quare ſubſtituerim pro $y$ aliam ſpeciem $z = y - a$, quiſque per ſeſe videre poteſt. Etenim ſi poſuiſſem $y = A + Bx + Cx^2 + Dx^3$ &c., facta ſubſtitutione, in ſingulis terminis occurrerent ſeries verticales, ex quibus inveniendi eſſent ſucceſſive valores $A$, $B$, $C$ &c., atque hæc operatio in ſingulis repetenda. Analyſis hæc longa eſt, & difficilis, neque ad eam confugiendum eſt, niſi quum aliæ faciliores methodi deſunt.

16. Sed uſus ſerierum in integratione formularum differentialium oſtendendus. Nimirum formula transformanda eſt in ſeriem, cujus omnes termini inte-

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grationem recipiant. Exemplis res aperienda. Sit propoſita formula $\frac{a^3 dx}{a^3 - x^3}$; quæ integrationem algebraicam non ſuſcipit. Fractio $\frac{a^3}{a^3 - x^3}$ convertatur in ſeriem infinitam, ut fiat

$$\frac{a^3}{a^3 - x^3} = 1 + \frac{x^3}{a^3} + \frac{x^6}{a^6} + \frac{x^9}{a^9} \text{ \&c.; ergo}$$

$$\frac{a^3 dx}{a^3 - x^3} = dx + \frac{x^3 dx}{a^3} + \frac{x^6 dx}{a^6} + \frac{x^9 dx}{a^9} \text{ \&c. Itaque habebimus}$$

$$S\,\frac{a^3 dx}{a^3 - x^3} = x + \frac{x^4}{4a^3} + \frac{x^7}{7a^6} + \frac{x^{10}}{10a^9} \text{ \&c.}$$

Series hæc convergens eſt, donec ſit $a > x$; ſi vero $x > a$, poſt aliquot terminos ſpecie convergentes, evadit divergens, atque adeo inutilis. Quare in hoc caſu ſatius erit, ita diſtribuere formulam $\frac{-a^3 dx}{x^3 - a^3}$, & convertere in ſeriem fractionem $\frac{a^3}{x^3 - a^3}$. Fiet autem

$$\frac{a^3}{x^3 - a^3} = \frac{a^3}{x^3} + \frac{a^6}{x^6} + \frac{a^9}{x^9} + \frac{a^{12}}{x^{12}} \text{ \&c., \& multiplicando per } -dx$$

$$\frac{-a^3 dx}{x^3 - a^3} = \frac{-a^3 dx}{x^3} - \frac{a^6 dx}{x^6} - \frac{a^9 dx}{x^9} - \frac{a^{10} dx}{x^{10}} \text{ \&c., \& integrando}$$

$$S\,\frac{-a^3 dx}{x^3 - a^3} = \frac{a^3}{2x^2} + \frac{a^6}{5x^5} + \frac{a^9}{8x^8} + \frac{a^{12}}{11 . x^{11}} \text{ \&c.}$$

quæ convergens eſt, eoque magis, quo $x$ magis excedit $a$.

17. Ad alterum exemplum, quod poteſtates etiam imperfectas reſpicit, ſeligo formulam $x dx . \overline{a+x}^p$, quam docui eſſe integrabilem non ſolum exiſtente $p$ numero integro, ſed etiam fracto, & negativo. Supponens $a > x$, converto in ſeriem poteſtatem $p$ binomii $a+x$, hoc modo

$$\overline{a+x}^p = a^p + p a^{p-1} x + \frac{p . \overline{p-1}}{2} a^{p-2} x^2 + \frac{p . \overline{p-1} . \overline{p-2}}{2 . 3} a^{p-3} x^3 \text{ \&c.}$$

Facta multiplicatione per $x dx$

$$x dx . \overline{a+x}^p = a^p x dx + p a^{p-1} x^2 dx + \frac{p . \overline{p-1}}{2} a^{p-2} x^3 dx + \frac{p . \overline{p-1} . \overline{p-2}}{2 . 3} a^{p-3} x^4 dx \text{ \&c.}$$

Integra

$S x dx$:

$$S\,x\,dx.\overline{a+x}^{p} = \frac{a^{p}x^{2}}{2} + \frac{p.a^{p-1}x^{3}}{3} + \frac{p.\overline{p-1}.a^{p-2}x^{4}}{2.4} + \frac{p.\overline{p-1}.\overline{p-2}.a^{p-3}x^{5}}{2.3.5} \text{ \&c.}$$

Adverte, ſummam hanc ita acceptam eſſe, ut, ſi $x=o$, ipſa quoque nulleſcat. Si $p$ ſit numerus integer, & affirmativus, liquet ſeriem abrumpi, & finito conſtare numero terminorum, nam quum ſucceſſive inveniatur in terminis $p-2$, $p-3$, $p-4$ &c., tandem devinitur ad $o$, qui quum ſequentes terminos ſemper multiplicet, omnes evaneſcunt. Itaque in hoc caſu formula ſtatim cognoſcitur integrabilis. Verum exiſtente $p$ numero fracto, aut negativo, ſeries numquam abrumpitur, ſed abit in infinitum. Quapropter ſiquis ex ſerie infinita, in quam incidit, pronunciaret, formulam integrabilem algebraice non eſſe, in paralogiſmum profecto caderet, quum aliunde integrationem formulæ cognoſcamus. Hoc exemplum de induſtria attuli, ne quis adeo fidat methodo ſerierum, ut ex illa unice criterium deducat cognoſcendi formulas integrabiles. Nam ſi ſeries abrumpitur, ſine dubio formula recipit algebraicam integrationem, ſed, non abrupta ſerie, poteſt eſſe algebraice integrabilis, quum ſeries aut generatim, aut in infinitum producta poſſit in ſummam colligi.

18. Sed præſtat ſummam eam, quam modo per ſeriem invenimus, conferre cum illa, quam per ſubſtitutionem determinamus. Facta $a+x=z$ provenit $z^{p}.dz.\overline{z-a} = z^{p+1}dz - az^{p}dz$; ergo integrando cum additione conſtantis, quæ, ut mox apparebit, in præſens neceſſaria eſt, $S\,x\,dx.\overline{a+x}^{p}$

$$= \frac{z^{p+2}}{p+2} - \frac{az^{p+1}}{p+1} + A = \frac{\overline{a+x}^{p+2}}{p+2} - \frac{a.\overline{a+x}^{p+1}}{p+1} + A = \overline{a+x}^{p+1}.$$

$$\frac{a+x}{p+2} - \frac{a}{p+1} + A = \overline{a+x}^{p+1}.\frac{x}{p+2} - \frac{a}{\overline{p+1}.\overline{p+2}} + A.$$

Quoniam integratio num. ſuperioris ita facta eſt, ut nulleſcat, facta $x=o$, ut cohæreant, eodem modo iſta ſumenda eſt; ergo $a^{p+1}.\frac{-a}{\overline{p+1}.\overline{p+2}} + A = o$, ſeu $A = \frac{a^{p+2}}{\overline{p+1}.\overline{p+2}}$. Quare vera ſummatoria evadit $S\,x\,dx.\overline{a+x}^{p} = \overline{a+x}^{p+1}.$

$$\frac{x}{p+2} - \frac{a}{\overline{p+2}.\overline{p+1}} + \frac{a^{p+2}}{\overline{p+2}.\overline{p+1}}.$$

Ut palam fiat, utrum hæc cum ſuperiore conveniat, eleva ad poteſtatem $p+1$ binomium $a+x$, ut fiat

$$\overline{a+x}^{p+1} = a^{p+1} + \overline{p+1}.a^{p}x + \frac{\overline{p+1}.p}{2}a^{p-1}x^{2} + \frac{\overline{p+1}.p.\overline{p-1}}{2.3}a^{p-2}x^{3} \text{ \&c.}$$

Multiplica per $-\frac{a}{\overline{p+2}.\overline{p+1}} + \frac{x}{p+2}$, & habebis $\overline{a+x}^{p+1}.\frac{x}{p+2} - \frac{a}{\overline{p+2}.\overline{p+1}} =$

$$-\frac{a^{p+2}}{\overline{p+2}.\overline{p+1}} - \frac{a^{p+1}x}{p+2} - \frac{pa^{p}x^{2}}{2.\overline{p+2}} - \frac{p.\overline{p-1}.a^{p-1}x^{3}}{2.3.\overline{p+2}} - \frac{p.\overline{p-1}.\overline{p-2}.a^{p-2}x^{4}}{2.3.4.\overline{p+2}} \text{ \&c.}$$

$$+\frac{a^{p+1}x}{p+2} + \frac{\overline{p+1}.a^{p}x^{2}}{p+2} + \frac{\overline{p+1}.p.a^{p-1}x^{3}}{2.\overline{p+2}} + \frac{\overline{p+1}.p.\overline{p-1}.a^{p-2}x^{4}}{2.3.\overline{p+2}} \text{ \&c.}$$

Contrahantur termini facta ſubductione, & utrobique addatur $\frac{a^{p+2}}{\overline{p+2}.\overline{p+1}}$; ut reſultet æquatio $\overline{a+x}^{p+1}.\overline{\frac{x}{p+2}-\frac{a}{\overline{p+2}.\overline{p+1}}}+\frac{a^{p+2}}{\overline{p+2}.\overline{p+1}}=$ $\frac{a^p x^2}{2}+\frac{p a^{p-1} x^3}{3}+\frac{p.\overline{p-1}.a^{p-2} x^4}{2.4}$ &c. Quum hæc ſeries eadem ſit ac illa, quæ habetur num. 17, apparet duas integrationes exquiſite congruere; imo ſeriei in infinitum productæ per primam inventæ ſecunda integratio exhibet ſummam. Vides per hanc methodum plurium ſerierum in infinitum productarum poſſe ſummas determinari. Sed hoc ad præſens noſtrum inſtitutum non pertinet.

19. Exemplum tertium det formula $dx\,\frac{\overline{a+fxx}^p}{\overline{b+gx^2}^p}$. Eleva ad poteſtatem $p$ duo binomia $a+fx^2$, $b+gxx$, quæ ita elevata voca $M$, $N$, ut ſit

$$M=a^p+pa^{p-1}fx^2+\frac{p.\overline{p-1}}{2}a^{p-2}f^2x^4+\frac{p.\overline{p-1}.\overline{p-2}}{2.3}a^{p-3}f^3x^6+$$
$$\frac{p.\overline{p-1}.\overline{p-2}.\overline{p-3}}{2.3.4}.a^{p-4}f^4x^8 \text{ \&c.}$$

$$N=b^p+p.b^{p-1}gx^2+\frac{p.\overline{p-1}}{2}b^{p-2}g^2x^4+\frac{p.\overline{p-1}.\overline{p-2}}{2.3}b^{p-3}g^3x^6+$$
$$\frac{p.\overline{p-1}.\overline{p-2}.\overline{p-3}}{2.3.4}.b^{p-4}g^4x^8 \text{ \&c.}$$

Fac $\frac{M}{N}=A+Bx^2+Cx^4+Dx^6+Ex^8$ &c., quam ſeriem multiplica per ſeriem $N$, ut oriatur

$$\begin{aligned}M=b^pA+pb^{p-1}gAx^2&+\frac{p.\overline{p-1}}{2}b^{p-2}g^2Ax^4+\frac{p.\overline{p-1}.\overline{p-2}}{2.3}b^{p-3}g^3Ax^6+\frac{p.\overline{p-1}.\overline{p-2}.\overline{p-3}}{2.3.4}b^{p-4}g^4Ax^8\\
+b^pBx^2&+pb^{p-1}gBx^4+\frac{p.\overline{p-1}}{2}b^{p-2}g^2Bx^6+\frac{p.\overline{p-1}.\overline{p-2}}{2.3}b^{p-3}g^3Bx^8\\
&+b^pCx^4+pb^{p-1}gCx^6+\frac{p.\overline{p-1}}{2}b^{p-2}g^2Cx^8 \text{ \&c.}\\
&+b^pDx^6+pb^{p-1}gDx^8\\
&+b^pEx^8\end{aligned}$$

Ad determinandos valores coefficientium $A$, $B$, $C$ &c. confer hujus ſeriei terminos ſingulos cum ſerie $M$, & invenies $A=\frac{a^p}{b^p}$, $B=\frac{pa^{p-1}f-p.b^{p-1}gA}{b^p}$

C=

$$C=\frac{\frac{p.\overline{p-1}}{2}a^{p-2}f^2-\frac{p.\overline{p-1}}{2}.b^{p-2}g^2A-pb^{p-1}gB}{b^p}$$

$$D=\frac{\frac{p.\overline{p-1}.\overline{p-2}}{2.3}a^{p-3}f^3-\frac{p.\overline{p-1}.\overline{p-2}}{2.3}.b^{p-3}g^3A-\frac{p.\overline{p-1}}{2}.b^{p-2}g^2B-pb^{p-1}gC}{b^p}$$

$$E=\frac{p.\overline{p-1}.\overline{p-2}.\overline{p-3}}{2.3.4}a^{p-4}f^4-\frac{p.\overline{p-1}.\overline{p-2}.\overline{p-3}}{2.3.4}f^{p-4}g^4A-\frac{p.\overline{p-1}.\overline{p-2}}{2.3}b^{p-3}g^3B$$

$$-\frac{p.\overline{p-1}}{2}b^{p-2}g^2C-pb^{p-1}gD \text{ diviſæ per } b^p.$$

Ex progreſſu manifeſto fluit methodus determinandi reliquos coefficientes.

20. Res ipſa jubet, nos proferre theorema maxime utile, quod in epiſtola prima ad Joſephum Suzzium demonſtravit Vincentius Riccatus in primo Op. tomo opuſculo ſexto. In æquatione

(I) $du=\frac{adx}{z}$, ſi $z$ detur per $x$ in æquatione

(II) &c. $\int\frac{dx}{z}\int\frac{dx}{z}\int\frac{dx}{z}\int\frac{xdx}{z}=z$ invenitur $x$ per ſeriem ex ſola $u$, & coſtantibus coaleſcentem. In formula II pro $\frac{dx}{z}$ ſubſtitue $\frac{du}{a}$; æquales enim eſſe has quantitates formula I docet. Oritur æquatio

(III) &c. $\int\frac{du}{a}\int\frac{du}{a}\int\frac{du}{a}\int\frac{xdu}{a}=z$; atqui ex formula I $z=\frac{adx}{du}$;

ergo &c. $\int\frac{du}{a}\int\frac{du}{a}\int\frac{du}{a}\int\frac{xdu}{a}=\frac{adx}{du}$; igitur multiplicando per $\frac{du}{a}$, & integrando fiet

(IV) &c. $\int\frac{du}{a}\int\frac{du}{a}\int\frac{du}{a}\int\frac{du}{a}\int\frac{xdu}{a}=x$, in qua formula numerus ſignorum S unitate ſuperat numerum ſignorum eorumdem, quæ inſunt in formula II. Ex IV ope ſerierum facillime invenietur $x$ data per $u$. Fiat enim

(V) $x=A+Bu+Cu^2+Du^3+Eu^4+Fu^5$ &c.

21. Supponamus in formula II unicum adeſſe ſignum ſummatorium, duplex in IV, ut fiat

$\int\frac{du}{a}\int\frac{xdu}{a}=A+Bu+Cu^2+Du^3+Eu^4+Fu^5$ &c. Differentietur æquatio, & multiplicetur per $\frac{a}{du}$, ut fit

$S\frac{x\,du}{a} = aB + 2aCu + 3aDu^2 + 4aEu^3 + 5aFu^4$ &c. Facta iterum differentiatione, eademque multiplicatione, orietur

$x = 2a^2C + 2.3\,a^2Du + 3.4\,a^2Eu^2 + 4.5\,a^2Fu^3 + 5.6\,a^2Gu^4 + 6.7\,a^2Hu^5$ &c.

quæ ſeries debet eſſe identica cum ſerie præſuppoſita æquationis V. Si duarum ſerierum termini comparentur, invenietur

$$C = \frac{A}{2a^2};\ E = \frac{A}{2.3.4.a^4};\ G = \frac{A}{2.3.4.5.6.a^6} \text{ \&c.}$$

$$D = \frac{B}{2.3\,a^2};\ F = \frac{B}{2.3.4.5.a^4};\ H = \frac{B}{2.3.4.5.6.7.a^6} \text{ \&c.}$$

Quapropter habebimus

$$x = \begin{array}{l} A + \frac{Au^2}{2a^2} + \frac{Au^4}{2.3.4.a^4} + \frac{Au^6}{2.3.4.5.6.a^6} \text{ \&c.} \\ Bu + \frac{Bu^3}{2.3\,a^2} + \frac{Bu^5}{2.3.4.5.a^4} + \frac{Bu^7}{2.3.4.5.6.7.a^6} \text{ \&c.} \end{array}$$

Determinationes effectæ oſtendunt, duas quantitates $A$, $B$ indeterminatas manere, quæ in ſingulis caſibus erunt determinandæ.

22. Supponamus deinde in æquatione II duo ineſſe ſigna ſummatoria, tria in IV. Factis iiſdem operationibus, quibus in caſu ſuperiore uſi ſumus, perveniemus ad æquationem

$S\frac{x\,du}{a} = 2a^2C + 2.3\,a^2Du + 3.4\,a^2Eu^2 + 4.5\,a^2Fu^3 + 5.6\,Gu^4$ &c.; ergo differentiando, & dividendo per $\frac{du}{a}$, orietur

$x = 2.3\,a^3D + 2.3.4.Eu + 3.4.5\,a^3Fu^2 + 4.5.6\,a^3Gu^3 + 5.6.7\,a^3Hu^4$ &c.,

quæ ſeries identica ſit, oportet, cum ſerie V. Quare ſi comparentur termini, obtinebuntur ſequentes determinationes

$$D = \frac{A}{1.2.3.a^3};\ G = \frac{A}{1.2.3.4.5.6.a^6};\ K = \frac{A}{1.2.3.4.5.6.7.8.9\,a^9}$$

$$E = \frac{B}{2.3.4.a^3};\ H = \frac{B}{2.3.4.5.6.7.a^6};\ L = \frac{B}{2.3.4.5.6.7.8.9.10\,a^9}$$

$$F = \frac{C}{3.4.5.a^3};\ I = \frac{C}{3.4.5.6.7.8.a^6};\ M = \frac{C}{3.4.5.6.7.8.9.10.11\,a^9},$$

ex quibus determinationibus conſtat, tres quantitates $A$, $B$, $C$ indeterminatas manere. Itaque erit

$A +$

$$x = \begin{cases} A + \dfrac{Au^3}{1.2.3.a^3} + \dfrac{Au^6}{1.2....5.6.a^6} + \dfrac{Au^9}{1.2....8.9.a^9} \ \&c. \\ Bu + \dfrac{Bu^4}{2.3.4.a^3} + \dfrac{Bu^7}{2.3....6.7.a^6} + \dfrac{Bu^{10}}{2.3....9.10.a^9} \ \&c. \\ Cu^2 + \dfrac{Cu^5}{3.4.5.a^3} + \dfrac{Cu^8}{3.4....7.8.a^6} + \dfrac{Cu^{11}}{3.4....10.11.a^9} \ \&c. \end{cases}$$

23. Si tria adſint ſigna ſummatoria in æquatione II, quatuor in IV, eadem inſtituta analyſi inveniemus

$$x = \begin{cases} A + \dfrac{Au^4}{1.2.3.4.a^4} + \dfrac{Au^8}{1.2.3...7.8.a^8} + \dfrac{Au^{12}}{1.2.3....11.12.a^{12}} \ \&c. \\ Bu + \dfrac{Bu^5}{2.3.4.5.a^4} + \dfrac{Bu^9}{2.3.4...8.9.a^8} + \dfrac{Bu^{13}}{2.3.4....12.13.a^{12}} \ \&c. \\ Cu^2 + \dfrac{Cu^6}{3.4.5.6.a^4} + \dfrac{Cu^{10}}{3.4.5...9.10.a^8} + \dfrac{Cu^{14}}{3.4.5....13.14.a^{12}} \ \&c. \\ Du^3 + \dfrac{Du^7}{4.5.6.7.a^4} + \dfrac{Du^{11}}{4.5.6...10.11.a^8} + \dfrac{Du^{15}}{4.5.6....14.15.a^{12}} \ \&c. \end{cases}$$

in qua quatuor quantitates A, B, C, D adhuc indeterminatæ ſuperſunt. Methodus jam ſatis aperte docet, quo pacto coefficientes ſeriei V ſuppoſitæ in caſibus magis compoſitis determinandi veniant.

24. Idem dicas velim de æquatione

(VI) $du = -\frac{a\,dx}{z}$, ſi $u$ data ſit per $x$ in formula

(VII) &c. $\mathrm{S}\frac{-dx}{z}\,\mathrm{S}\frac{-dx}{z}\,\mathrm{S}\frac{-dx}{z}\,\mathrm{S}\frac{-x\,dx}{a} = z$. Namque eamdem uſurpans methodum invenies

(VIII) &c. $\mathrm{S}\frac{du}{a}\,\mathrm{S}\frac{du}{a}\,\mathrm{S}\frac{du}{a}\,\mathrm{S}\frac{du}{a}\,\mathrm{S}\frac{x\,du}{a} = -x$, ex qua, præſuppoſita iterum ſerie V, iidem prorſus coefficientes exurgent, ſed cum aliqua varietate ſignorum. Namque coefficientes, qui indeterminati remanent, & qui in noſtris ſeriebus conſtituunt primam columnam verticalem, ſi ſigno + afficiantur, qui formant ſecundam columnam verticalem, afficientur ſigno —, qui tertiam, ſigno +, atque ita deinceps, alternantibus ſemper ſignis columnarum verticalium ita, ut in illis, quæ tenent ſedes impares, præfigendum ſit ſignum +; illæ vero, quæ occupant ſedes pares, ſigno — ſint afficiendæ.

25. Ex his, quæ hactenus explicata ſunt dilucide, proclive eſt cognitu, æqua-

æquationis II utramque indeterminatam $x$, $z$ datam reperiri per ſeries ex tertia indeterminata $u$ coaleſcentes. Etenim poſita $du = \frac{a\,dx}{z}$ inventa eſt ſupra $x$ æqualis ſeriei expreſſæ per $u$. Sit porro $x = A + Bu + Cu^2 + Du^3 + Eu^4$ &c. cujus ſeriei coefficientes in ſingulis caſibus determinavimus. Fiat differentiatio, & erit $dx = B\,du + 2Cu\,du + 3Du^2\,du + 4Eu^3\,du$ &c.; ergo $\frac{a\,dx}{du} = z = aB + 2aCu + 3aDu^2 + 4aEu^3$ &c., per quam ſeriem $z$ definita eſt. Quæ quum ita ſint, ſi unicum dumtaxat ſignum ſummatorium habeatur in æquatione II, fiet

$$x = \begin{matrix} + \frac{Au}{a} + \frac{Au^3}{2.3.a^3} + \frac{Au^5}{2.3.4.5.a^5} \text{ \&c.} \\ aB + \frac{Bu^2}{2a} + \frac{Bu^4}{2.3.4.a^3} + \frac{Bu^6}{2.3.4.5.6.a^5} \text{ \&c.} \end{matrix}$$

Si in eadem æquatione duo inexiſtant ſigna ſummatoria, habebimus

$$z = \begin{matrix} + \frac{Au^2}{1.2.a^2} + \frac{Au^5}{1.2.3.4.5.a^5} + \frac{Au^8}{1.2\ldots7.8.a^8} \text{ \&c.} \\ aB + \frac{Bu^3}{2.3.a^2} + \frac{Bu^6}{2.3.4.5.6.a^5} + \frac{Bu^9}{2.3\ldots8.9.a^8} \text{ \&c.} \\ + 2aCu + \frac{Cu^4}{3.4.a^2} + \frac{Cu^7}{3.4.5.6.7.a^5} + \frac{Cu^{10}}{3.4\ldots9.10.a^8} \text{ \&c.} \end{matrix}$$

In caſibus autem magis compoſitis ſeries nanciſcemur ſecundum eam, quæ conſpicua eſt, progreſſionem. Idem dicendum de æquatione VII, in qua ſuppoſita $x$ eidem ſeriei æquali, invenietur $z = -aB - 2aCu - 3aDu^2 - 4aEu^3$ &c. Per quam facile cognoſces, in ſingulis caſibus eaſdem prorſus ſeries oriri, cum aliqua tamen varietate ſignorum. Nam in terminis, qui conſtituunt primam, tertiam, quintam & alias impares columnas verticales, habebitur ſignum $-$, in illis, qui formant ſecundam, quartam, & alias columnas pares, relinquendum eſt ſignum $+$. Hoc theoremate utemur deinceps, quum ſinus & coſinus per arcus circulares, & logarithmos analogos erunt inveniendi.

26. Adeſt alia methodus propria calculi integralis, qua elegantiæ, ac perſpicuitatis cauſa utuntur melioris notæ Analyſtæ, ut vel integrent per ſeries formulas differentiales, vel ut eaſdem reducant ad formulas ſimpliciores. Methodus in hoc omnis ſita eſt. Dividatur formula differentialis in duos factores, quorum unus differentiam contineat, & ſit integrabilis, alter finitam variabilem includat. Primum integra, perinde ac ſi alter eſſet conſtans, tum formulam ortam ex non vera integratione differentia: prodibit formula propoſita addita alia formula, quam alterius factoris differentia ſuppeditat. Hæc ſi addatur propoſitæ, & detrahatur, coaleſcit formula ex tribus, quarum duæ primæ ſunt abſo-

abſolute integrabiles. Si tertia habeat cum propoſita affinitatem, & eadem methodo tractari poſſit, pariter in tres mutabitur, quarum duæ primæ integrabiles erunt, tertia eodem modo tractabitur, atque ita ſucceſſive operando ſeries efformabitur, quæ requiritur. Verum elegantius fiet, ſi aſſumatur formula generalis, quæ differentiata duas affines formulas exhibeat; per hanc enim definitis coefficientibus indeterminatis ſeries ſimplicior prodit. Quam regulæ iſtæ univerſales habent obſcuritatem, exempla diſcutient.

27. Sit evolvenda in ſeriem formula $\frac{dx}{a^3+x^3}$. Spectata tamquam conſtante $a^3+x^3$, integro, & provenit $\frac{x}{a^3+x^3}$. Hanc non veram ſummam differentio, ut ſit $\frac{dx}{a^3+x^3} - \frac{3x^3dx}{(a^3+x^3)^2}$. Hujus vera ſumma eſt $\frac{x}{a^3+x^3}$. Itaque ſi propoſitam formulam ita diſtribuam $\frac{dx}{a^3+x^3} - \frac{3x^3dx}{(a^3+x^3)^2} + \frac{3x^3dx}{(a^3+x^3)^2}$; primi duo termini ſimul ſumpti habent integrationem algebraicam; tertius ſimilis eſt formulæ propoſitæ, neque quidquam differt niſi in exponentibus, in numeratore enim per tres unitates creſcit, in denominatore per unam. Integretur itaque, ſuppoſita eadem conſtante, & fiet $\frac{3x^4}{4\cdot(a^3+x^3)^2}$; hæc non vera ſumma differentietur, & prodibit $\frac{3x^3dx}{(a^3+x^3)^2} - \frac{2\cdot 3^2 x^6 dx}{(a^3+x^3)^3}$. Quare formula ita exponenda $\frac{3x^3dx}{(a^3+x^3)^2} - \frac{2\cdot 3^2 x^6 dx}{(a^3+x^3)^3} + \frac{2\cdot 3^2 x^6 dx}{(a^3+x^3)^3}$. Primæ duæ ſimul ſumptæ algebraice integrantur, tertia ſimili methodo tractatur. Itaque, iteratis hiſce operationibus, ſeries, quæ quæritur, invenietur.

28 Verum ut elegantiori quodam modo ſeriem producam, aſſumo formulam genaralem $\frac{x^{3p+1}}{(a^3+x^3)^q}$, cujus differentiam accipio hoc modo

$$D\frac{x^{3p+1}}{(a^3+x^3)^q} = \frac{\overline{3p+1}\,x^{3p}dx}{(a^3+x^3)^q} - \frac{3qx^{3p+3}dx}{(a^3+x^3)^{q+1}};\ \text{ergo}$$

$$S\frac{x^{3p}dx}{\overline{a^3+x^3}^{\,q}} = \frac{1}{3p+1}\cdot\frac{x^{3p+1}}{\overline{a^3+x^3}^{\,q}} + \frac{3q}{3p+1}S\frac{x^{3p+3}dx}{\overline{a^3+x^3}^{\,q+1}}.$$ Hoc veluti lemmate conſtituto, pono $p=0$, $q=1$, ut fiat

$$S\frac{dx}{a^3+x^3} = \frac{x}{a^3+x^3} + 3S\frac{x^3dx}{\overline{a^3+x^3}^{\,2}}.$$ Tum fiat $p=1$, $q=2$.

$$S\frac{x^3dx}{\overline{a^3+x^3}^{\,2}} = \frac{1}{4}\;\frac{x^4}{\overline{a^3+x^3}^{\,2}} + \frac{3\cdot 2}{4}S\frac{x^6dx}{\overline{a^3+x^3}^{\,3}}.$$ Pone $p=2$, $q=3$:

$$S\frac{x^6dx}{\overline{a^3+x^3}^{\,3}} = \frac{1}{7}\cdot\frac{x^7}{\overline{a^3+x^3}^{\,3}} + \frac{3\cdot 3}{7}S\frac{x^9dx}{\overline{a^3+x^3}^{\,4}}.$$ Facta item $p=3$, $q=4$:

$$S\frac{x^9dx}{\overline{a^3+x^3}^{\,4}} = \frac{1}{10}\cdot\frac{x^{10}}{\overline{a^3+x^3}^{\,4}} + \frac{3\cdot 4}{10}S\frac{x^{12}dx}{\overline{a^3+x^3}^{\,5}};$$ atque ita gradatim progredere, quouſque opus fuerit. Denique in prima ſubſtitue pro $S\frac{x^3dx}{\overline{a^3+x^3}^{\,2}}$ ejus valorem, quem habes ex ſecunda, tum pro $S\frac{x^6dx}{\overline{a^3+x^3}^{\,3}}$ ſcribe ejus valorem, quem dat tertia, atque ita ſucceſſive ſubſtituendo, ſeries orietur

$$S\frac{dx}{a^3+x^3} = \frac{x}{a^3+x^3} + \frac{3}{4}\cdot\frac{x^4}{\overline{a^3+x^3}^{\,2}} + \frac{3^2\cdot 2}{4\cdot 7}\cdot\frac{x^7}{\overline{a^3+x^3}^{\,3}} + \frac{3^3\cdot 2\cdot 3}{4\cdot 7\cdot 10}\cdot\frac{x^{10}}{\overline{a^3+x^3}^{\,4}} + \frac{3^4\cdot 2\cdot 3\cdot 4}{4\cdot 7\cdot 10\cdot 13}\cdot\frac{x^{13}}{\overline{a^3+x^3}^{\,5}}$$ &c., cujus progreſſum operatio effecta ſatis patefacit.

29. Methodus hæc utilis ſæpe eſt reducendæ formulæ ad ſimpliciorem. Exemplum parum diſſimile affero in formula $dx.\overline{a^3+x^3}^{\,5}$, quam, elevato binomio ad poteſtatem quintam, apparebit eſſe integrabilem. Attamen, ut præſentem methodum uſurpem, aſſumo formulam generalem $x^p.\overline{a^3+x^3}^{\,q}$, ejuſque accipio differentiam, ut habeam

$$D\,x^p.\overline{a^3+x^3}^{\,q} = px^{p-1}dx.\overline{a^3+x^3}^{\,q} + 3qx^{p+2}dx.\overline{a^3+x^3}^{\,q-1};$$ ergo

Sx

$$S\, x^{p-1} dx \,.\, \overline{a^3+x^3}^{\,q} = \frac{x^p \,.\, \overline{a^3+x^3}^{\,q}}{p} - \frac{3q}{p}\, S\, x^{p+2} dx \,.\, \overline{a^3+x^3}^{\,q-1}$$

Fac $p=1$, $q=5$, & oritur

$$S\, dx \,.\, \overline{a^3+x^3}^{\,5} = x \,.\, \overline{a^3+x^3}^{\,5} - 3.5 \,.\, S\, x^3 dx \,.\, \overline{a^3+x^3}^{\,4}$$

Pone $p=4$, $q=4$, & fiet

$$S\, x^3 dx \,.\, \overline{a^3+x^3}^{\,4} = \frac{x^4 \,.\, \overline{a^3+x^3}^{\,4}}{4} - \frac{3.4}{4}\, S\, x^6 dx : \overline{a^3+x^3}^{\,3}.$$

Fac $p=7$, $q=3$, & habebis

$$S\, x^6 dx \,.\, \overline{a^3+x^3}^{\,3} = \frac{x^7 \,.\, \overline{a^3+x^3}^{\,3}}{7} - \frac{3.3}{7}\, S\, x^9 dx \,.\, \overline{a^3+x^3}^{\,2}.$$

Poſita $p=10$, $q=2$ reſultat

$$S\, x^9 dx \,.\, \overline{a^3+x^3}^{\,2} = \frac{x^{10} \,.\, \overline{a^3+x^3}^{\,2}}{10} - \frac{3.2}{10}\, S\, x^{12} dx \,.\, \overline{a^3+x^3}.$$

Poſſem hic ſubſiſtere; nam ultima formula aperte integrabilis eſt; ſed producens calculum pono $p=13$, $q=1$, ut fiat

$$S\, x^{12} dx \,.\, \overline{a^3+x^3}^{\,1} = \frac{x^{13} \,.\, \overline{a^3+x^3}^{\,1}}{13} - \frac{3.1}{13}\, S\, x^{15} dx.$$

Quando vero $S\, x^{15} dx = \frac{x^{16}}{16}$, factis neceſſariis ſubſtitutionibus habebimus

$$S\, dx \,.\, \overline{a^3+x^3}^{\,5} = x \,.\, \overline{a^3+x^3}^{\,5} - \frac{3.5}{4} \,.\, x^4 \,.\, \overline{a^3+x^3}^{\,4} + \frac{3^2.5.4}{4.7} \,.\, x^7 \,.\, \overline{a^3+x^3}^{\,3}$$

$$- \frac{3^3.5.4.3}{4.7.10}\, x^{10} \,.\, \overline{a^3+x^3}^{\,2} + \frac{3^4.5.4.3.2}{4.7.10.13}\, x^{13} \,.\, \overline{a^3+x^3} - \frac{3^5.5.4.3.2.1}{4.7.10.13.16}\, x^{16}.$$

Duo hæc exempla ſatis ſint. Nam methodi uſus, qui frequentiſſimus erit in hoc libro, eamdem deinceps declarabit, ejuſque fœcunditatem, atque elegantiam oſtendet.

## CAPUT QUINTUM

### De quadratura curvarum, ejuſque uſu in conſtruendis formulis, quæ carent integratione Algebraica.

1. Antequam exponam uſum, & compendium calculi integralis in quadraturis curvarum inveniendis, liceat applicare eam methodum, quam adhibui Cap. 1. num. 7. omnibus parabolis, & hyperbolis, ſi unam excipias hyperbolam apollonianam. Hæc enim applicatio & oſtendet utilitatem calculi integralis, & eidem perſpicuitatem, & evidentiam impertiet. Ibi relata parabola ad tangentem diviſimus ſpatium in infinita rectangula circumſcripta, quæ exponuntur a terminis ſeriei numero infinitis. Series, quæ ſeſe offert habet ſorte fortuna ſummam generalem algebraicam: quare hac ſumma inventa facile fuit invenire ſummam rectangulorum, adeoque ſpatium parabolicum. Verum ſi relata fuiſſet parabola ad axem, res non ita feliciter proceſſiſſet. Ubi poſita ſit difficultas, breviter aperiam. Sit parabola (*Fig.* 1.) AD relata ad axem AB, ut vocatis AB $= x$, ordinatis BD $= y$, æquatio fit $ax = yy$. Dividatur abſciſſa AB in partes æquales numero infinitas
Ae, e2e, 2e3e, 3e4e &c., quarum ſingulæ $= q$, ut ſit $nq = x$. Deinde ducantur ordinatæ
ei, 2e2i, 3e3i, 4e4i, &c. palam eſt, has exprimi per terminos ſequentis ſeriei
$\sqrt{aq}, \sqrt{2aq}, \sqrt{3aq}, \sqrt{4aq} \ldots\ldots \sqrt{naq}$. Igitur rectangula
ieA, 2i2ee, 3i3e2e, 4i4e3e &c., quorum ſummam adæquat area parabolica ABD repræſentantur a terminis ſeriei
$q\sqrt{aq}, q\sqrt{2aq}, q\sqrt{3aq}, q\sqrt{4aq} \ldots\ldots q\sqrt{naq}$. Igitur area ABD æqualis eſt quantitati $q\sqrt{aq}$ multiplicatæ per ſummam ſeriei
$\sqrt{1}, \sqrt{2}, \sqrt{3}, \sqrt{4} \ldots\ldots \sqrt{n}$.

2. Hinc difficultas exoritur maxima. Nam ſeries expoſita, cujus terminus generalis $= \sqrt{n}$, caret ſumma generali algebraica. Quare videtur, per methodum capiendi ſummas generales ſerierum, non poſſe hoc modo obtineri quadraturam ſpatii parabolici ABD. Ut me liberem ab hac non contemnenda difficultate, duplicem methodum in uſum traduco. Utramque autem exponam eodem adhibito exemplo parabolæ apollonianæ. Quod ſpectat ad primam: Quis me jubet dividere abſciſſam AB in partes æquales. Ad rem meam ſatis eſt, ut dividam in partes infiniteſimas. Quare eo modo partibor, ut reſpondentes ordinatæ ejuſmodi valorem induant, qui in terminis ſeriei expellat irrationalia, & ſeriem producat præditam ſumma generali. In vulgari parabola voti compos fiam ſumens ſpatiola infiniteſima
Ae, e2e, 2e3e, 3e4e &c. (*Fig.* 2) in ratione numerorum imparium ita, ut vocato primo Ae $= q$, ſint ſucceſſive
$q, 3q, 5q, 7q$ &c. Agantur ordinatæ
ei, 2e2i, 3e3i, 4e4i &c., quæ ex natura parabolæ ita exprimentur
$\sqrt{aq}, 2\sqrt{aq}, 3\sqrt{aq}, 4\sqrt{aq}$ &c. Quare rectangula
ieA, 2i2ee, 3i3e2e, 4i4e3e &c., quæ adæquat area parabolica, repræſen-

ſentantur ab hac ſerie

$q\sqrt{aq}$, $2.3.q\sqrt{aq}$, $3.5.q\sqrt{aq}$, $4.7.q\sqrt{aq}$ &c. Ergo area parabolica ABD invenitur æqualis $q\sqrt{aq}$ ductæ in ſummam ſeriei

1.1, 2.3, 3.5, 4.7 &c.

3. Series hæc formatur a multiplicatione duarum ſerierum

| | |
|---|---|
| 1, 2, 3, 4 &c. | quarum prima eſt numerorum naturalium, & habet |
| 1, 3, 5, 7 &c. | terminum generalem $= n$; altera eſt numerorum |

imparium & habet terminum generalem $= 2n - 1$; ergo ſeries, a cujus ſumma dependet parabolæ quadratura, habebit terminum generalem $= 2nn - n$. Si hujuſce ſeriei ex tomi primi libri tertii capite quarto ſummam quæras, reperies eam eſſe $= \frac{2}{3} n^3 + \frac{1}{2} n^2 - \frac{1}{2.3}$. Hæc autem facta $n$ infinita, evaneſcentibus duobus ultimis terminis, fit $= \frac{2}{3} n^3$. Itaque ſpatium parabolicum ABD $= \frac{2}{3} n^3 q\sqrt{aq}$. Verum obſervandum eſt, abſciſſam $x$ in facta hypotheſi non æquare $nq$, ſed æquare ſummam ſeriei

$q, 3q, 5q, 7q$ &c. uſque ad terminum $n^{eſimum}$, quæ ſumma $= n^2 q$; ergo $x = n^2 q$, & $x\sqrt{x} = n^3 q\sqrt{q}$. Igitur area parabolica erit $= \frac{2}{3} x\sqrt{ax}$; ſed $y = \sqrt{ax}$; ergo eadem area $= \frac{2}{3} xy$. Q. E. D.

4. Quod ſi aveas uti ea ſerie, quæ oritur ex diviſione abſciſſæ AB in partes æquales, cujus terminus generalis $= \sqrt{n}$, alia longe diverſa adhibenda eſt methodus, quæ majorem poſcit induſtriam, quia ſeries prædita termino generali $= \sqrt{n}$ caret ſumma generali algebraica. Quapropter dabo operam, ut eamdem ſeriem, cui eſt terminus generalis $= \sqrt{n}$, conſtituam mediam inter duas ſeries, quæ præditæ ſint conditionibus duabus; primum ut ſummam generalem admittant, deinde ut facta $n$ infinita, ſumptiſque terminis numero infinitis, ſerierum ſummæ æquales evadant, licet generatim ſint inæquales. Quomodo autem hoc præſtare liceat, enitar, ut dilucide exponam. Adverto quantitatem $\sqrt{n}$, exiſtente $n$ numero integro, & poſitivo, mediam eſſe inter has duas quantitates

| | |
|---|---|
| $\frac{2}{3} n^{\frac{3}{2}} - \frac{2}{3} . \overline{n-1}^{\frac{3}{2}}$ | majorem ſcilicet eſſe prima, minorem altera; quod |
| $\frac{2}{3} \overline{n+1}^{\frac{3}{2}} - \frac{2}{3} n^{\frac{3}{2}}$ | tibi conſtabit, ſi duo binomia $n-1$, $n+1$ ele- |

ves ad poteſtatem $\frac{3}{2}$. Itaque ſi tres ſeries formes, quarum termini generales ſint $\frac{2}{3} n^{\frac{3}{2}} - \frac{2}{3} . \overline{n-1}^{\frac{3}{2}}$, $\sqrt{n}$, $\frac{2}{3} . \overline{n+1}^{\frac{3}{2}} - \frac{2}{3} n^{\frac{3}{2}}$, ſinguli termini ſe-

cundæ ſeriei erunt medii inter analogos terminos aliarum; ergo etiam ſumma ſecundæ media erit inter ſummas aliarum.

5. Præterea ſeries, cujus terminus generalis $=\frac{2}{3}\, n^{\frac{3}{2}} - \frac{2}{3} . \overline{n-1}^{\frac{3}{2}}$, habet ſummam $= \frac{2}{3}\, n^{\frac{3}{2}}$; illa, cujus terminus generalis $=\frac{2}{3}\,\overline{n+1}^{\frac{3}{2}} - \frac{2}{3} n^{\frac{3}{2}}$, habet ſummam $= \frac{2}{3} . \overline{n+1}^{\frac{3}{2}} - \frac{2}{3}$; igitur inter duas haſce quantitates $\frac{2}{3}\, n^{\frac{3}{2}}$, $\frac{2}{3}\,\overline{n+1}^{\frac{3}{2}} - \frac{2}{3}$ media eſſe debet ſumma ſeriei, quam gignit terminus generalis $\sqrt{n}$. Quantitates iſtæ duæ inæquales ſunt, donec $n$ fuerit numerus finitus, ſed facto $n$ infinito evadunt æquales, & utraque $= \frac{2}{3}\, n^{\frac{3}{2}}$; ergo ſeriei mediæ præditæ termino generali $\sqrt{n}$ ſumma fiet $= \frac{2}{3}\, n^{\frac{3}{2}}$. Quoniam autem paullo ante inveni ſpatium parabolicum ABD (*Fig.* 1) æquale eſſe $q^{\frac{3}{2}}\, \sqrt{a}$ ductæ in ſummam ſeriei

$\sqrt{1}$, $\sqrt{2}$, $\sqrt{3}$, $\sqrt{4}$ . . . . . $\sqrt{n}$, erit area

$$ABD = \frac{2}{3}\, n^{\frac{3}{2}}\, q^{\frac{3}{2}} . \sqrt{a};$$ ſed $nq = x$; ergo area

$$ABD = \frac{2}{3}\, x^{\frac{3}{2}} . \sqrt{a} = \frac{2}{3}\, x\sqrt{ax} = \frac{2}{3}\, xy,$$ ut antea.

6. Ut methodos applicem altioribus omnibus parabolis, & hyperbolis excepta hyperbola apolloniana, neceſſe eſt, ut, ad evitandam calculi prolixitatem, atque moleſtiam, doceam modum, quo ſeries in infinitum producta facilius colligatur in ſummam. Loquar hic de ſeriebus, quarum terminus generalis ſit functio algebraica integra. Harum ſerierum ſumma in hunc modum invenitur. Formatur formula algebraica data per $n$, quæ ſpecies terminorum numerum indicat, uno gradu altior, quam ſit terminus generalis, prædita omnibus terminis excepto ultimo, quorum coefficientes determinandi ſunt in operationis progreſſu. In hac pro $n$ ſcribitur $n-1$, & quæ exurgit formula a ſuperiore detrahitur. Facta deductione nova ſe prodit formula ejuſdem gradus ac terminus generalis, atque cum hoc idemica ſit oportet. Itaque, comparatione rite inſtituta, fit coefficientium determinatio, atque ſeriei ſumma detegitur. Hujus analyſeos progreſſum opportunum erit exemplo ad mentem revocare. Inveniendà ſit ſumma generalis ſeriei modo memoratæ, cujus terminus generalis

lis $= 2nn - n$. Finge hujus ſummam eſſe $= An^3 + Bn^2 + Cn$; quæ formula & caret ultimo termino, & eſt uno gradu altior, quam terminus generalis. In hac pro $n$ ſcribe $n - 1$, & novam, quæ exoritur, formulam fac deducas ex ſuppoſita, ut fit

$$\begin{array}{rrrr} An^3 + & Bn^2 + & Cn & \\ -An^3 + & 3An^2 - & 3An + & A \\ & -Bn^2 + & 2Bn - & B \\ & & -Cn + & C \\ \hline \end{array}$$

Facta actuali ſubductione, reliqua erit formula

$$\begin{array}{rrr} 3An^2 - & 3An + & A \\ & +2Bn - & B \\ & & +C \end{array}$$

quæ ad determinandos valores coefficientium A, B, C, conferenda eſt cum termino generali: quæ collatio dabit

$A = \frac{2}{3}$, $B = \frac{1}{2}$, $C = \frac{-1}{2.3}$; ergo ſumma erit $= \frac{2}{3}n^3 + \frac{1}{2}n^2 - \frac{1}{2.3}n$; quemadmodum antea ſuppoſita eſt.

7. In hoc progreſſu ex formula ſummæ ſuppoſita ſi deducatur ea, quæ oritur ex ſubſtitutione $n - 1$ pro $n$, ſemper evaneſcit primus terminus ex contrarietate ſignorum, & formula deſcendit ad eum gradum, in quo ſitus eſt terminus generalis. In altero termino, licet ſecundus terminus formulæ ſuppoſitæ cum ſuo coefficiente elidatur, tamen reliquus eſt ſecundus terminus binomii $n - 1$ elati ad poteſtatem, & affecti coefficiente primi termini formulæ ſuppoſitæ. Si autem $n$ fiat infinitus, reliqui termini omnes reſpectu hujus evaneſcunt. Quare ad inveniendam ſummam ſeriei in infinitum productæ ſatis erit, terminum hunc, qui reſiduus eſt, comparare cum termino primo termini generalis, & determinare coefficientis valorem. Ita in exemplo adducto facta deductione evaneſcit terminus $An^3$, & in ſequenti termino evaneſcit $Bn^2$, & ſolum remanet $3An^2$; qua de re coefficiens $B$ ex ſecundo termino abit. Qui reliquus eſt $3An^2$ nihil aliud eſt, quam ſecundus terminus binomii $A.\overline{n-1}^3$ ſigno mutato. Quare ſi $3An^2$ conferatur cum $2n^2$, invenietur $A = \frac{2}{3}$. Igitur ſumma ſeriei in infinitum productæ erit $= \frac{2}{3}n^3$, exiſtente $n$ infinito.

Quæ quum ita ſint, hæc ſtatuatur regula ad inveniendam ex dato termino generali ſummam ſeriei in infinitum productæ: Eleva binomium $n - 1$ ad poteſtatem uno gradu altiorem, quam ſit poteſtas termini generalis; terminum alterum mutato ſigno multiplica per indeterminatam $A$, tum compara cum termini generalis termino primo, & determina valorem $A$. Hunc multiplica per poteſtatem $n$ uno gradu altiorem, quam ſit poteſtas termini generalis, & invenies quæſitam ſummam ſeriei in infinitum productæ.

8. His explicatis redeo in viam. Ex tribus methodis, quas applicavi parabolæ apollonianæ, ajo primam, de qua Cap. primo, inſervire quadrandis parabolis, quæ hac æquatione continentur, $a^{m-1}y = x^m$, exiſtente $m$ numero integro, & poſitivo. Sit hæc parabola A D, (*Fig.* 3) cujus abſciſſæ $AB = x$, ordinatæ $BD = y$. Diviſa AB in infinitas partes æquales, retentiſque ſuperio-

rioribus denominationibus, conſtat, ſucceſſivas ordinatas fore

$$\frac{q^m}{a^{m-1}},\ \frac{2^m q^m}{a^{m-1}},\ \frac{3^m q^m}{a^{m-1}},\ \frac{4^m q^m}{a^{m-1}} \ldots\ldots \frac{n^m q^m}{a^{m-1}}.$$ Igitur ſuceſſiva rectangula, quorum ſumma adæquat aream prabolicam A B D, erunt

$$\frac{q^{m+1}}{a^{m-1}},\ \frac{2^m q^{m+1}}{a^{m-1}},\ \frac{3^m q^{m+1}}{a^{m-1}},\ \frac{4^m q^{m+1}}{a^{m-1}} \ldots\ldots \frac{n^m q^{m+1}}{a^{m-1}};$$ ergo area A B D æquabit quantitatem $\frac{q^{m+1}}{a^{m-1}}$ ductam in ſequentem ſeriem

$1,\ 2^m,\ 3^m,\ 4^m \ldots\ldots n^m$. Itaque ſi quis hujuſce ſeriei ſummam invenerit, aream parabolicam obtinebit. Series autem eſt algebraica ordinis $m^{eſimi}$, cujus ſcilicet differentiæ $m^{eſimæ}$ conſtantes ſunt, & habet pro termino generali $n^m$. Facta $n$ infinita, ſummam ex præcedenti animadverſione ita invenies. Eleva ad poteſtatem $m+1$ binomium $n-1$, ſecundus terminus erit mutato ſigno $\overline{m+1}\,.\,n^m$. Hunc multiplica per $A$, & compara cum $n^m$ primo imo unico termino termini generalis, & invenies $A = \frac{1}{m+1}$; ergo ſumma ſeriei in infinitum productæ fiet $= \frac{n^{m+1}}{m+1}$; ergo area parabolica A B D $= \frac{n^{m+1} q^{m+1}}{\overline{m+1}\,.\,a^{m-1}}$; ſed $nq = x$; ergo A B D $= \frac{x^{m+1}}{\overline{m+1}\,.\,a^{m-1}} = \frac{xy}{m+1}$. Q. E. I,

9. Secunda methodus inſervit quadrandis omnibus parabolis, quibus eſt æquatio $a^{m-1}x = y^m$, exiſtente $m$ numero integro, & poſitivo. Etenim abſciſſa A B (*Fig.* 2.) non dividatur in partes æquales, ſed ita ſucceſſive creſcentes, ut ſervent inter ſe eam rationem, quam termini ſequentis ſeriei

$1,\ 2^m - 1,\ 3^m - 2^m,\ 4^m - 3^m \ldots\ldots n^m - \overline{n-1}^{\,m}$, ita ut, vocata prima parte A e $= q$, partes

A e, e 2 e, 2 e 3 e, 3 e 4 e &c. ſucceſſive exprimantur per terminos ſeriei

$q,\ \overline{2^m - 1}\,.\,q,\ \overline{3^m - 2^m}\,.\,q,\ \overline{4^m - 3^m}\,.\,q \ldots\ldots \overline{n^m - \overline{n-1}^{\,m}}\,.\,q$. Manifeſtum eſt abſciſſas integras

A e, A 2 e, A 3 e, A 4 e &c. fore

$q,\ 2^m q,\ 3^m q,\ 4^m q \ldots\ldots n^m q$. Quapropter ordinatæ

e i, 2 e 2 i, 3 e 3 i, 4 e 4 i &c. invenientur

$$\overline{a^{m-1} q}^{\,\frac{1}{m}},\ \overline{a^{m-1}\,.\,2^m q}^{\,\frac{1}{m}},\ \overline{a^{m-1}\,.\,3^m q}^{\,\frac{1}{m}},\ \overline{a^{m-1}\,.\,4^m q}^{\,\frac{1}{m}} \ldots\ldots \overline{a^{m-1} n^m q}^{\,\frac{1}{m}},$$

ſive

$$1.\overline{a^{m-1}q}^{\frac{1}{m}},\ 2.\overline{a^{m-1}q}^{\frac{1}{m}},\ 3.\overline{a^{m-1}q}^{\frac{1}{m}},\ 4.\overline{a^{m-1}q}^{\frac{1}{m}} \ldots\ldots n.\overline{a^{m-1}q}^{\frac{1}{m}}.$$

Itaque rectangula omnia
j e A, 2 i 2 e e, 3 i 3 e 2 e, 4 i 4 e 3 e &c., quibus area parabolica per adæquationem æquivalet, fient ſucceſſive æqualia

$$q.\overline{a^{m-1}q}^{\frac{1}{m}},\ 2.\overline{2^m-1}.q.\overline{a^{m-1}q}^{\frac{1}{m}},\ 3.\overline{3^m-2^m}.q.\overline{a^{m-1}q}^{\frac{1}{m}},$$

$$4.\overline{4^m-3^m}.q.\overline{a^{m-1}q}^{\frac{1}{n}} \ldots\ldots n.\overline{n^m-(\overline{n-1}^m}.q.\overline{a^{m-1}q}^{\frac{1}{n}}.$$ Qua de re area parabolica invenitur æqualis quantitati

$$q.\overline{a^{m-1}q}^{\frac{1}{m}}$$ ductæ in ſummam ſeriei.

$$1,\ 2.\overline{2^m-1},\ 3.\overline{3^m-2^m},\ 4.\overline{4^m-3^m} \ldots\ldots n.\overline{n^m-(\overline{n-1}^m}.$$

10. Series hæc, cujus terminus generalis $= n.n^m - (\overline{n-1}^m$, quamquam primo aſpectu videtur eſſe gradus $m+1$, tamen quia primus terminus ex ſignorum contrarietate eliditur, eſt tantum gradus $m$, & primus terminus ejus termini generalis eſt $m.n^m$. Ut itaque ſeriei in infinitum productæ invenias ſummam, ex ſuperiori animadverſione eleva ad poteſtatem $m+1$ binomium $n-1$, ejuſque ſecundum terminum $\overline{m+1}.n^m$, in quo ſignum mutatum eſt, multiplica per $A$, & confer cum $m.n^m$ primo termino termini generalis, & invenies $A = \frac{m}{m+1}$; ergo ſeriei in infinitum productæ ſumma $= \frac{m}{m+1}.n^{m+1}$. Igitur area parabolica A B D $= \frac{m}{m+1}.n^{m+1}q.\overline{a^{m-1}q}^{\frac{1}{m}}$. Nemo unus non videt abſciſſam $x$ æqualem eſſe quantitati $q$ ductæ in ſummam ſeriei
$1,\ 2^m-1,\ 3^m-2^m,\ 4^m-3^m \ldots\ldots n^m-(\overline{n-1}^m$. Hujus autem ſeriei ſumma $= n^m$; ergo $x = n^m q$. Igitur facta ſubſtitutione fiet area parabolica A B D $=$
$\frac{m}{m+1}x.\overline{a^{m-1}x}^{\frac{1}{m}}$; ſed $\overline{a^{m-1}x}^{\frac{1}{m}} = y$: ergo A B D $= \frac{m}{m+1}xy$.

11. Methodus tertia utilis eſt ad quadrandas parabolas omnes, & hyperbolas relatas ad aſymptota cujuſcumque gradus, excepta hyperbola apolloniana. Harum curvarum omnium æquatio eſt $\frac{x^m}{a^{m-1}} = y$, in qua $m$ poteſt eſſe

qui-

quilibet numerus poſitivus vel negativus, integer vel fractus. Si $m$ poſitivus ſit, æquatio eſt ad parabolas, ſi $m$ eſt negativus, eſt ad hyperbolas, & ſi $m = -1$, æquatio eſt ad hyperbolam apollonianam. Diviſa abſciſſa A B (*Fig.* 1. 4.) in partes minimas æquales, quarum ſingulæ $= q$, reſpondentes ordinatæ ei, 2e2i, 3e3i, 4e4i &c. erunt

$$\frac{q^m}{a^{m-1}},\ \frac{2^m q^m}{a^{m-1}},\ \frac{3^m q^m}{a^{m-1}},\ \frac{4^m q^m}{a^{m-1}} \dots \frac{n^m q^m}{a^{m-1}}$$

; ergo rectangula 1eA, 2i2ee, 3i3e2e, 4i4e3e &c., quorum ſummæ æqualis eſt area curvilinea, fient

$$\frac{q^{m+1}}{a^{m-1}},\ \frac{2^m q^{m+1}}{a^{m-1}},\ \frac{3^m q^{m+1}}{a^{m-1}},\ \frac{4^m q^{m+1}}{a^{m-1}} \dots \frac{n^m . q^{m+1}}{a^{m-1}}$$

; ergo area curvilinea æquabit quantitatem $\frac{q^{m+1}}{a^{m-1}}$ ductam in ſummam ſeriei

$1, 2^m, 3^m, 4^m \dots n^m$, cujus terminus generalis $= n^m$. Hujus ſeriei ſumma invenienda eſt pro hypotheſi $n$ infiniti.

12. Hanc ob rem præmittendum eſt theorema. Quicumque ſit numerus $n$, dummodo integer, & poſitivus, quantitas $n^m$ ſemper media eſt inter duas quantitates

$$\frac{1}{m+1}.n^{m+1} - \frac{1}{m+1}.\overline{n-1}^{m+1}$$
$$\frac{1}{m+1}.\overline{n+1}^{m+1} - \frac{1}{m+1}.n^{m+1}$$

Si $m$ ſit poſitiva, $n^m$ ſemper major eſt prima ex duabus hiſce quantitatibus, minor ſecunda; ſi vero $m$ eſt negativa, minor eſt prima, major altera. Ex hoc theoremate conſtat, omnes terminos ſeriei, cui eſt terminus generalis $= n^m$, medios eſſe inter reſpectivos terminos ſerierum, quarum termini generales ſunt

$$\frac{1}{m+1}.n^{m+1} - \frac{1}{m+1}.\overline{n-1}^{m+1}$$
$$\frac{1}{m+1}.\overline{n+1}^{m+1} - \frac{1}{m+1}.n^{m+1}$$

quod item dicendum eſt de ſummis. Atqui conſtat ſeries præditas his terminis generalibus habere ſummas

$$\frac{1}{m+1}.n^{m+1}$$
$$\frac{1}{m+1}.n^{m+1} - \frac{1}{m+1}$$

Igitur ſumma ſeriei, cujus terminus generalis $= n^m$ media eſt inter ſtatutas quantitates

$\frac{1}{m+1}.n^{m+1}$, $\frac{1}{m+1}.\overline{n+1}^{m+1} - \frac{1}{m+1}$. Quæ duæ quantitates, donec $n$ finitus eſt, ſemper ſunt inæquales. Verum ſi $n$ creſcat in infinitum, æquales fiunt, & ſingulæ evadunt $\frac{1}{m+1}\ n^{m+1}$; ergo pro hac hypotheſi $n$ infiniti ſe-

ti ſeries termini generalis $n^m$ habet ſummam $= \frac{1}{m+1}\, n^{m+1}$. Igitur area curvilinea A B D $= \frac{n^{m+1}\, q^{m+1}}{m+1 \,.\, a^{m-1}}$, ſed $nq = x$; ergo A B D $= \frac{x^{m+1}}{m+1 \,.\, a^{m-1}} = \frac{xy}{m+1}$. Q. E. I.

13. Quod ſpectat ad parabolas, hæc ſufficiunt, nec quidquam addendum eſt. Sed aliqua adnotanda ſunt de hyperbolis, quæ negotium non leve facere poſſunt. Æquatio pertinet ad hyperbolas, ſi $m$ (*Fig. 4.*) ſit negativa, quare, mutato ſigno ſpeciei $m$, ut poſitiva fiat, æquatio erit $\frac{a^{m+1}}{x^m} = y$. Duplex eſt caſus; vel enim $m$ minor eſt unitate, vel major. Si $m < 1$, nihil mutandum in calculo, & omnia rite procedunt ſicut in parabolis. Verum adverſus demonſtrationem allatam hoc opponi poteſt, quod inventa quidem eſt ſumma ultima rectangulorum A e i, e 2 e 2 i &c.; ſed hæc ſumma non probatur ultimo æqualis ſpatio hyperbolico H A B D, quia differentia inter hoc ſpatium & rectangulorum ſummam licet ſit minor ſpatio infinite longo H A e i, tamen hoc non probatur minus quocumque dato. Objectio hæc nullo modo eſt ſpernenda, & omnino probandum ſpatium infinite longum H A e i fieri minus quocumque dato, ut demonſtratio ſubſiſtat. Ac primum accepta A e quacumque data minore $= q$, rectangulum A e i, erit quocumque dato minus. Etenim ordinata e i $= \frac{a^{m+1}}{q^m}$, quæ ducta in $q$ dabit rectangulum A e i $= a^{m+1} q^{1-m}$; ſed hæc quantitas, poſita $m < 1$, eſt minor quacumque data; ergo rectangulum A e i ſit minus quocumque dato.

14. Præterea aſſumpta A E infiniteſima, ejus ſpatium infinite longum H A E I (*Fig. 5.*) evadere infiniteſimum. Dividatur A E bifariam in 2 E, A 2 E in 3 E, A 3 E in 4 E, & ita deinceps, ductiſque, ordinatis circumſcribantur hyperbolæ rectangula 2 I 2 E E, 3 I 3 E 2 E, 4 I 4 E 3 E &c.; patet ſucceſſivas ordinatas 2 E 2 I, 3 E 3 I, 4 E 4 I &c. fore

$$\frac{2^m \,.\, a^{m+1}}{q^m},\ \frac{2^{2m} \,.\, a^{m+1}}{q^m},\ \frac{2^{3m} \,.\, a^{m+1}}{q^m} \,\ldots\ldots\, \frac{2^{nm} \,.\, a^{m+1}}{q^m}.$$

Ut habeantur ſucceſſiva rectangula, iſtæ ducendæ ſunt in

$$\frac{q}{2},\ \frac{q}{2^2},\ \frac{q}{2^3} \,\ldots\ldots\, \frac{q}{2^n};$$

ergo ſucceſſiva rectangula erunt

$$\frac{a^{m+1} q^{1-m}}{2^{1-m}},\ \frac{a^{m+1} q^{1-m}}{2^{2-2m}},\ \frac{a^{m+1} q^{1-m}}{2^{3-3m}} \,\ldots\ldots\, \frac{a^{m+1} q^{1-m}}{2^{n-nm}};$$

ergo ſumma omnium rectangulorum erit æqualis quantitati $a^{m+1} . q^{1-m}$ ductæ in ſummam ſeriei $\frac{1}{2^{1-m}},\ \frac{1}{(2^{1-m})^2},\ \frac{1}{(2^{1-m})^3} \,\ldots\ldots\, \frac{1}{(2^{1-m})^n}$. Hujus

ſeriei ſumma, quæ geometrica eſt, ex traditis tomo ſuperiore invenitur $= \frac{1}{2^{1-m}-1} \cdot \overline{1 - \frac{1}{(2^{1-m})^{n}}}$, quæ facto $n$ infinito, ultimo fit $= \frac{1}{2^{1-m}-1}$; ergo ſumma rectangulorum ultimo $= \frac{a^{m+1}q^{1-m}}{2^{1-m}-1}$, quæ quantitas minor eſt quacumque data: atqui ſpatium hyperbolicum infinite longum HAEI minus eſt ſumma rectangulorum circumſcriptorum: ergo hoc ſpatium tandem fit minus quocumque dato. Quare calculus effectus hyperbolis, de quibus agimus, tuto applicari poteſt. Sed hic primum illud te advertere volo, quod ſemper memoria retinendum erit, ubi agetur de ſpatiis infinite longis, nimirum quantitatem $\frac{a^{m+1}q^{1-m}}{2^{1-m}-1}$, cui ultimo æqualem eſſe diximus ſummam rectangulorum circumſcriptorum, atque adeo ſpatium hyperbolicum infinite longum, nihil aliud eſſe niſi quantitatem limitis, quantitatem ſcilicet, ad quam numquam poſſumus pervenire additis quibuſcumque rectangulis, poſſumus tamen accedere, ultra quamcumque menſuram ita, ut defectus ab ipſa minor evadat quacumque data quantitate.

15. Verumtamen ſi $m$ major ſit unitate, termini generales ſerierum, quæ inventam mediam tenent, ſunt hujuſmodi

$$\frac{-1}{\overline{m-1}\,.\,n^{m-1}} + \frac{1}{\overline{m-1}\,.\,\overline{n-1}^{m-1}}$$

$$\frac{-1}{\overline{m-1}\,.\,\overline{n+1}^{m-1}} + \frac{1}{\overline{m-1}\,.\,n^{m-1}}$$

In his $m$ mutato ſigno facta eſt poſitiva. Serierum autem, cujus hi ſunt termini generales, ex primo tomo inveniuntur ſummæ

$$\frac{1}{\overline{m-1}\,.\,0^{m-1}} - \frac{1}{\overline{m-1}\,.\,n^{m-1}}$$

$$\frac{1}{m-1} - \frac{1}{\overline{m-1}\,.\,\overline{n+1}^{m-1}}$$

Advertere neceſſe eſt, formulas has ita eſſe inæquales, ut etiamſi $n$ evadat infinitus, inæquales remaneant, quia termini conſtantes, & independentes ab $n$ inæquales ſunt, reliquis præ iſtis evaneſcentibus. Qui termini, facto $n$ infinito, ad æqualitatem ultra quemcumque limitem accedunt, hi duo ſunt $\frac{1}{\overline{m-1}\,.\,n^{m-1}}$, $\frac{1}{\overline{m-1}\,.\,\overline{n+1}^{m-1}}$, qui a conſtantibus quantitatibus deducuntur. Quare etiam ſumma ſeriei, quæ media eſt, & quæ habet terminum generalem $= \frac{1}{n^{m}}$, coaleſcet ex termino conſtante, a quo deducendus erit terminus variabilis dependens ab $n$, qui facto $n$ infinito $= \frac{1}{\overline{m-1}\,.\,n^{m-1}}$. Terminus conſtans vocetur $= B$; ergo ſumma ſe-

riei

riei in infinitum productæ exprimetur a formula $= B - \frac{1}{\overline{m-1}.n^{m}}$.

16. Itaque area hyperbolica ABD ( *Fig.* 4. ) fiet $=$

$\frac{B.a^{m+1}}{q^{m-1}} - \frac{a^{m+1}}{\overline{m-1}.n^{m-1}q^{m-1}}$; ſed $nq = x$; ergo area ABD $=$

$\frac{B.a^{m+1}}{q^{m-1}} - \frac{a^{m+1}}{\overline{m-1}.x^{m-1}}$; atqui facta $x = o$, area hæc debet nulleſcere; ergo fiet $B = \frac{q^{m-1}}{\overline{m-1}.o^{m-1}}$, quo valore ſubſtituto, habebimus aream ABD $=$

$\frac{a^{m+1}}{\overline{m-1}.o^{m-1}} - \frac{a^{m+1}}{\overline{m-1}.x^{m-1}}$. Hæc vero formula clariſſime indicat aream eſſe infinitam. Quod ſi poſita AB $= a$, velis invenire aream DBFG, quæ evaneſcat, facta $x = a$, tum per eamdem methodum determinabis $B =$

$\frac{q^{m-1}}{\overline{m-1}.a^{m-1}}$, & facta ſubſtitutione, obtinebis aream

DBFG $= \frac{a^{2}}{m-1} - \frac{a^{m+1}}{\overline{m-1}.x^{m-1}}$. Quando vero eſt $x^{m} = \frac{a^{m+1}}{y}$, fiet area

$$ABD = \frac{a^{m+1}}{\overline{m-1}.o^{m-1}} - \frac{xy}{m-1}$$

$$DBFG = \frac{a^{2}}{m-1} - \frac{xy}{m-1}.$$

17. Ex his apparet, regulam ſupra traditam ad inveniendam ſummam ſeriei in infinitum productæ, quoties $m$ eſt numerus integer, & poſitivus eodem modo valere, tametſi ſit negativus, & fractus. Nempe accipiendus eſt ſecundus terminus binomii $n - 1$ elati ad poteſtatem $m + 1$; tum mutato ſigno, multiplicandus per $A$; & comparandus cum primo termino termini generalis, ut determinetur valor $A$; & ſumma ſeriei erit $An^{m+1}$. Quod ſi contingat, ut hæc ſumma proveniat negativa, tunc præmittenda eſt quantitas conſtans. Superfluum eſt adnotare, methodum hanc eſſe omnino inutilem quadrandæ hyperbolæ apollonianæ, in qua, mutato ſigno ſpeciei $m$, habetur $m = 1$. Hoc tamen ſilentio præterire non poſſum, ſummam ſeriei

$1, \frac{1}{2^{m}}, \frac{1}{3^{m}}, \frac{1}{4^{m}}, \frac{1}{5^{m}} \ldots\ldots \frac{1}{n^{m}}$ in infinitum productæ, ſi $m > 1$, obtineri quidem, quantum ſatis eſt, ad inveniendam hyperbolæ quadraturam, ut patet ex calculo, quia non eſt neceſſe, definire valorem indeterminatæ $B$ per quantitates finitas. Verum ſi abſolute ſeriei ſummam quæras, fruſtra præſenti metho-

do uteris: numquam enim per quantitates finitas licebit tibi indeterminatæ B valorem invenire, cui valori æqualis eſt ſumma quæſita ſeriei in infiitum productæ.

18. Præmiſi methodum quadrandi parabolas, & hyperbolas per ſummas ſerierum in infinitum productarum, tum ut ex ea calculus ſummatorius evidentiam, & perſpicuitatem accipiat; tum ut cognoſcas, hunc calculum per compendium eaſdem ſerierum ſummas exhibere. Ut hoc calculo utamur in quadrandis parabolis æquationis $\frac{x^m}{a^{m-1}} = y$, augenda eſt abſciſſa A B $= x$ ( *Fig.* 1. ) infiniteſima differentia B b $= dx$. Differentia ſpatii hoc eſt rectangulum D B b $= y\,dx = \frac{x^m dx}{a^{m-1}}$; ergo integrando area

$$\mathrm{ADB} = \frac{x^{m+1}}{\overline{m+1}\,.\,a^{m-1}}.$$ Nulla conſtans addenda, ſi ſpatium in A incipiat, quia facta $x = o$, ipſum quoque nulleſcit. Verum ſi ſpatium incipiat ab ordinata HK, cujus abſciſſa A H $= b$, addita conſtante, fiat

$$\mathrm{HKDB} = \frac{x^{m+1}}{\overline{m+1}\,.\,a^{m-1}} + A,$$ quod, facta $x = b$, debet nulleſcere; ergo

$$\frac{b^{m+1}}{\overline{m+1}\,.\,a^{m-1}} + A = o, \;\&\; A = \frac{-b^{m+1}}{\overline{m+1}\,.\,a^{m-1}},$$ & ſpatium

$$\mathrm{HBDK} = \frac{x^{m+1} - b^{m+1}}{\overline{m+1}\,.\,a^{m-1}}.$$

19. Eadem methodus uſurpanda in hyperbolis æquationis $\frac{a^{m+1}}{x^m} = y$. Nam elementum ſpatii B D d b $= y\,dx = \frac{a^{m+1} dx}{x^m}$; ( *Fig.* 4. ) ergo integrando

$$A + \frac{a^{m+1} x^{-m+1}}{-m+1}.$$ Si $m < 1$, & velis aream nulleſcere, evaneſcente abſciſſa, fiet A $= o$; ſi vero area incipiat, ubi abſciſſa $= b$, fiet

$$A + \frac{a^{m+1} b^{-m+1}}{-m+1} = o;$$ ergo $$A = \frac{-a^{m+1} b^{-m+1}}{-m+1};$$ ergo area erit

$$\frac{a^{m+1} x^{-m+1}}{-m+1} - \frac{a^{m+1} b^{-m+1}}{-m+1}.$$ Adverte in hac hyperbola ſpatium infinite longum ad partem H finitum eſſe, ad partem vero K infinitum. Si ſit $m > 1$, ita exponatur formula $A - \frac{a^{m+1}}{\overline{m-1}\,.\,x^{m-1}}$, quæ ſi debeat evaneſcere

facto

facto $x=o$, dabit $A=\frac{a^{m+1}}{\overline{m-1}\,.\,o^{m-1}}$, quæ infinita eſt. Hoc autem evenit, quia hyperbolicum ſpatium ad plagam H infinitum eſt. Itaque ne integratio fruſtretur, ſumendum eſt areæ initium ab abſciſſa $x=b$, quo in caſu fiet

$A=\frac{a^{m+1}}{\overline{m-1}\,.\,b^{m-1}}$, & area $=\frac{a^{m+1}}{\overline{m-1}\,.\,b^{m-1}}-\frac{a^{m+1}}{\overline{m-1}\,.\,x^{m-1}}$. Si $x$ fiat infinita, ultimus terminus evaneſcet, & area infinite longa ad partes K erit finita, & $=\frac{a^{m+1}}{\overline{m-1}\,.\,b^{m-1}}$. Adverte, in quacumque hyperbola, ſumptis abſciſſis in uno aſymptoto, $m$ eſſe $<1$, acceptis in alio erit $m>1$. Quare quælibet hyperbola, ſi apollonianam excipias, habet ad unam partem ſpatium infinite longum finitum, ad aliam infinitum.

20. Areæ omnes parabolicæ, & hyperbolicæ dependent ab integratione formulæ $x^m dx$, ad quam, quotquot integrantur formulæ, reducuntur. Quum autem per methodum ſummandi ſeries geometrice exhibuerim areas parabolicas, & hyperbolicas, inveni item geometrice ſummatoriam formulæ differentialis $x^m dx$ atque adeo principia calculi ſummatorii clariſſime demonſtravi.

21. Quæ dicta ſunt de quadratura parabolica, & hyperbolica ſatis oſtendunt, qua methodo in aliarum curvarum quadraturas debeat inquiri. Sint lineæ abſciſſarum AC, BD, (*Fig. 6.*) quarum primæ curva FEG obvertat concavum, alteri convexum. Ducta qualibet ordinata CED agatur ei infinite proxima ced, & parallelæ abſciſſis Em, en. Minimum rectangulum Emen ad rectangula CEmc, DEmd habet minorem rationem quacumque data; eſt enim ut me infiniteſima ad finitam CE ſeu DE; ergo rectangulum CEmc, DEmd adæquat elementum areæ, a quo non differt niſi quantitate minore quacumque data; ergo vocatis ordinatis $=y$, abſciſſis $=x$ erit elementum areæ curvilineæ $=y\,dx$, & area curvilinea $=\mathrm{S}x\,dy$. Jam vero ſi in hac formula pro $x$ ſubſtituas ejus valorem, ex natura curvæ datum per $y$, vel pro $dy$ ejus valorem datum per $x$, $dx$, habebis aream curvilineam ab unica variabili expreſſam, quam formulam ſi rite ad integrationem perducas, de quantitate areæ curvilineæ tibi conſtabit. Si ordinatæ CE non eſſent normales abſciſſis, ſed facerent angulum quemcumque $=p$, tunc area curvilinea $=\frac{Cc\,.\,p}{r}\,\mathrm{S}y\,dx$; ſed niſi aliter moneamus, rectum eſſe angulum ſupponemus.

21. Si habeas paraboloidem, cujus æquatio ſit

$y=a+bx^m+cx^n$ &c. erit

$y\,dx=a\,dx+bx^m dx+cx^n dx$ &c.; ergo integrando

$\mathrm{S}y\,dx=ax+\frac{bx^{m+1}}{m+1}+\frac{cx^{n+1}}{n+1}$ &c. Quare integratio ſemper valet, excepto caſu, ubi aliquis exponens æquaret $-1$. Hæc ſummatoria ita accepta eſt, ut nulla ſit area, quum nulla eſt abſciſſa. Quod ſi velis aream nulleſcere poſita ab-

abſciſſa $=f$, debes addere in integratione conſtantem $A$, tum in formula æquali areæ ponere $f$ pro $x$, eamdemque æquare $o$. Hæc æquatio verum valorem quantitatis $A$ addendæ tibi ſuppeditabit. Contra ſi data curvæ æquatio ſit

$x = a + by^m + cy^n$, accipe differentiam

$dx = mby^{m-1}dy + ncy^{n-1}dy$. Valorem $dx$ ſubſtite in formula $ydx$, ut ſit

$ydx = mby^m dy + ncy^n dy$. Integra

$\mathrm{S}ydx = \frac{mby^{m+1}}{m+1} + \frac{ncy^{n+1}}{n+1}$, quæ integratio fruſtratur, ſi aliquis exponens $= -1$. Si evaneſcente $y$, area non debeat nulleſcere, adde conſtantem, eamque ut ſupra determina. Plura hic exempla non addo, quia poſt expoſitam theoriam majore cum utilitate proponam.

22. Hac methodo quadrantur curvæ, quum earum æquatio more carteſiano data eſt ordinatis inter ſe parallelis exiſtentibus. Sed ſi alio modo detur curvæ æquatio, alia erit formula quadraturæ. Diſcedant ordinatæ C B $=y$ ex foco C (*Fig.* 7). Faciat C b cum C B angulum infiniteſimum. Centro C intervalo C B deſcribatur minimus arcus B m $= dx$. Data ſit æquatio inter C B $=y$, ejus differentiam b m $= dy$, & B m $= dx$. Differentia areæ A C B erit ſector C B m $= \frac{ydx}{2}$; ergo eadem area $= \frac{1}{2}\mathrm{S}ydx$. In hac pro $dx$ ſubſtituendus ejus valor datus per $y$, $dy$, & integratio facienda, ut area quæſita inveniatur. Dixi arcendam ex formula $dx$, ejuſque valorem introducendum datum per $y$, quia, quum $dx$ ſit arcus minimus deſcriptus radio variabili, nullius lineæ eſt differentia; ergo ejus integratio nequit haberi. Quod ſi velles removere $y$ a formula, neceſſe eſſet eamdem reducere ad arcum deſcriptum radio conſtanti, qui arcus eſſet differentia arcus finiti, ut deinceps ad ſummatoriam poſſit deveniri. Hac autem methodo uti neceſſe eſt. Dato radio C A $= a$ deſcribe circulum A D ſecantem C B, C b, in D, d. Voca A D $= z$, D d $= dz$.

Erit $y : a :: dx : dz$; ergo $dx = \frac{ydz}{a}$; ergo $\frac{1}{2}\mathrm{S}ydx = \frac{1}{2}\mathrm{S}\frac{y^2dz}{a}$. In hac potes pro $y$ ſubſtituere ejus valorem datum per $z$, & deinde integrationem perficere, quæ exprimetur per arcum circuli dati.

23. Sit curva A B ejus naturæ, ut valeat æquatio $\frac{dy\sqrt{y}}{\sqrt{a}} = dx$; ergo

$\frac{y^{\frac{3}{2}}dy}{2\sqrt{a}} = \frac{1}{2}ydx$ & integrando $A + \frac{y^{\frac{5}{2}}}{5\sqrt{a}} = \frac{1}{2}\mathrm{S}ydx$. Si area, facta

$y = \mathrm{CA} = a$, debeat evaneſcere, erit $A + \frac{a^2}{5} = o$; ergo $A = \frac{-a^2}{5}$; igitur

area æquabit $\frac{y^{\frac{5}{2}}}{5\sqrt{a}} - \frac{a^2}{5}$, quæ poſitiva erit, ſi $y > a$; negativa, ſi $y < a$;

&

& poſito $y=o$, fiet area $=\frac{a^2}{5}$. Si velis arcere $y$, deſcripto ut antea circulo radii $=a$, pone in æquatione curvæ pro $dx$ ejus valorem $\frac{y\,dz}{a}$; ergo $\frac{dy}{\sqrt{y}}=\frac{dz}{\sqrt{a}}$, & integrando ut poſita $y=a$, ſit $z=o$, fiet $2\sqrt{y}-2\sqrt{a}=\frac{z}{\sqrt{a}}$;

ergo $2\sqrt{y}=\frac{z+2a}{\sqrt{a}}$, & $y^2=\frac{\overline{z+2a}^4}{16aa}$; ergo $\frac{1}{2}S\frac{y^2dz}{a}=$

$\frac{1}{32}S\frac{dz.\overline{z+2a}^4}{a^3}=\frac{\overline{z+2a}^5}{5.32a^3}-\frac{a^2}{5}$, ita enim, poſita $z=o$, nulleſcit. Modus hic quadrandi curvas omittendus non erat, quia ſæpiſſime analyſtæ utuntur æquatione curvarum, quæ referuntur ad focum. Cæterum ex geneſi, aliiſque proprietatibus curvarum non raro quadraturæ quoque eruuntur non ineleganter, quod nos deinceps, ubi res poſcet, opportunis exemplis patefaciemus.

24. Nunc autem dicendum eſt de uſu, quem præſtant quadraturæ ad conſtructionem earum formularum, quæ reſpuunt algebraicam integrationem. Omnes enim conſtruuntur ſuppoſita quadratura alicujus curvæ algebraicæ. Sit quælibet formula $X\,dx$, in qua $X$ data eſt unice per $x$, & conſtantes. Reducatur ſemper $X$ ad poteſtatem linearem eam multiplicando, vel dividendo, ſi opus eſt, per conſtantes, ut ſi X ſit tertiæ poteſtatis dividenda erit per $a^2$. Ita præparata formula ponamus X linearem. Deſcribe curvam, cujus abſciſſæ ſint $=x$, ordinatæ autem $y=X$. Hæc curva ſit FE (*Fig. 6.*) erit $S\,X\,dx=$ ſpatio curvilineo AFEC; ergo habito hoc ſpatio, quod haberi poteſt ſaltem mechanice, determinatur ſummatoria quæſita, cujus conſtructio dependet a quadratura curvæ algebraicæ.

25. Sit $dx\sqrt{2ax-xx}$. Quum quantitas multiplicans $dx$ ſit dimenſionis linearis, nulla multiplicatio, aut diviſio facienda eſt. Pone $y=\sqrt{2ax-xx}$, & deſcribe hujus æquationis curvam, quæ erit circulus, cujus radius CA $=a$ (*Fig. 8.*) ſinus verſus AB $=x$, ſinus BD $=y$; ergo $S\,dx\sqrt{2ax-xx}=$ ſemiſegmento circulari ABD, quo ſuppoſito data formula ad integrationem perducitur. Si formula fuiſſet $dx\sqrt{2ax+xx}$, deſcripta hyperbola æquilatera AD, (*Fig. 9.*) cujus ſemiaxis CA $=a$, $S\,dx\sqrt{2ax+xx}$ æquaret ſpatium ABD.

26. Sit $\frac{dx\sqrt{x}}{\sqrt{a-x}}$, quæ multiplicetur per $a$, ut fiat $\frac{a\,dx\sqrt{x}}{\sqrt{a-x}}$. Pone $\frac{a\sqrt{x}}{\sqrt{a-x}}=y$, quæ curva eſt tertii gradus. Hanc deſcribe, & ſit ADH (*Fig. 10.*) Ducta qualibet ordinata BD erit $S\frac{a\,dx\sqrt{x}}{\sqrt{a-x}}=$ ABD, &



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$S\frac{dx\sqrt{x}}{\sqrt{a-x}} = \frac{ABD}{a}$. Si proponatur formula $\frac{dx}{a^2+x^2}$, ea ducatur in $a^3$, ut fiat $\frac{a^3dx}{a^2+x^2}$. Pone $\frac{a^3}{a^2+x^2}=y$, & deſcribe curvam HCK (*Fig.* 11.) hujus æquationis poſita $AB=x$, $BC=y=\frac{a^3}{a^2+x^2}$. Erit $S\frac{a^3dx}{a^2+x^2}=$ ſpatio HABC; ergo $S\frac{dx}{a^2+x^2}=\frac{HABC}{a^3}$.

27. Verum non eſt neceſſe ad integrandam formulam conſtantem ex indeterminata $x$, conſtruere curvam, cujus abſciſſa ſit $x$. Immo elegans, & utile erit plerumque, ſeligere ejus factorem differentialem, qui algebraicam integrationem admittat; ejuſque integrale ponere $=z$ abſciſſæ curvæ deſcribendæ, reliquum factorem facere $=y$ ejuſdem curvæ ordinatæ. Ita in formula ſuperiore $\frac{dx\sqrt{x}}{\sqrt{a-x}}$, facta multiplicatione per $\frac{a}{2}$, ut fiat $\frac{adx\sqrt{x}}{2\sqrt{a-x}}$; accipe $\frac{dx\sqrt{a}}{2\sqrt{a-x}}=dz$, factaque integratione $A-\sqrt{aa-ax}=z$. Addita $A$ pro libito aſſumi poteſt; ergo $A-z=\sqrt{aa-ax}$ ſive $\frac{\overline{A-z}^2}{a}=a-x$. Præterea fac $\sqrt{ax}=y$: ergo $x=\frac{yy}{a}$. Igitur æquatio curvæ, cujus abſciſſa $=z$, ordinata $=y$, proveniet $\frac{\overline{A-z}^2}{a}=\frac{aa-yy}{a}$, ſive $AA-2Az+zz=aa-yy$, factaque $A=a$, erit $2az-zz=yy$. Quare radio $CA=a$ (*Fig.* 8.) deſcripto circulo, ſeca $AB=z=a-\sqrt{aa-ax}$; & duc ordinatam $BD=y=\sqrt{ax}$, erit $S\frac{adx\sqrt{x}}{2\sqrt{a-x}}=ABD$; ergo $S\frac{dx\sqrt{x}}{\sqrt{a-x}}=\frac{2.ABD}{a}$. Quando formulæ $\frac{dx\sqrt{x}}{\sqrt{a-x}}$ integratio dependet a quadratura circuli, ab hoc quoque dependebit quadratura curvæ ADH, (*Fig.* 10.) per quam Num. 25. eadem formula conſtructa eſt. Itaque ſpatium infinite longum ACH erit æquale duplo quadranti circulari, cujus radius $=a$. Similiter propoſita æquatione $-\frac{a^3dx\sqrt{a^2-x^2}}{x^3}$, ſelige factorem algebraice integrabilem $-\frac{a^2dx}{x^2}$, cujus integrale $\frac{a^2}{x}=z$;

ſeu

ſeu $\frac{a^2}{z}=x$. Tum fac $\frac{a}{x}\sqrt{a^2-x^2}=y$. Deſcribe curvam coordinatarum $z$, $y$, & hujus quadratura formulam propoſitam integrabit. Ut naturam curvæ determines, ſubſtitue in ſecunda valorem $x$ datum per primam, & fiet

$\frac{z}{a}\sqrt{a^2-\frac{a^2}{zz}}=\sqrt{zz-aa}=y$, quæ eſt ad hyperbolam æquilateram. Quare cum ſemiaxe $CA=a$ (*Fig. 9.*) deſcripta hyperbola æquilatera, in axe primo abſcinde $CB=z=\frac{aa}{x}$, & duc ordinatam BD, fiet $S\frac{-a^3 dx\sqrt{aa-xx}}{a^3}=ABD$.

28. Interdum elegantius fiet, ſi non propoſitam formulam conſtruas per quadraturas, ſed aliam, quæ ſimul cum propoſita integrationem algebraicam recipiat. Si enim hanc ab algebraica detrahas, propoſitam obtinebis. Generatim hoc artificium explico. Sit propoſita formula $ydx$; eam integra perinde ac ſi $y$ eſſet conſtans, ut fiat $xy$. Hujus differentiam accipe, & erit $Dxy=xdy+ydx$; ergo $xy-Sxdy=Sydx$. Si itaque per quadraturas invenias $Sxdy$, eamque a rectangulo $xy$ detrahas, habebis integrale propoſitæ $ydx$.

Ad exemplum ſit formula $\frac{-x^4dx}{\overline{x^2+a^2}^{\frac{3}{2}}}$, quam ita diſpone $x^3.\frac{-xdx}{\overline{xx+aa}^{\frac{3}{2}}}$.

Hæc integrata quaſi $x^3$ eſſet conſtans dabit $\frac{x^3}{\sqrt{xx+aa}}$. Hujus autem differentia fiet

$$D\frac{x^3}{\sqrt{xx+aa}}=\frac{-x^4dx}{\overline{x^2+aa}^{\frac{3}{2}}}+\frac{3x^2dx}{\sqrt{xx+aa}}, \text{ ſive}$$

$$\frac{x^3}{\sqrt{xx+aa}}-3S\frac{x^2dx}{\sqrt{xx+aa}}=S\frac{-x^4dx}{\overline{x^2+a^2}^{\frac{3}{2}}}.$$ Integremus per quadraturas

$\frac{x^2dx}{\sqrt{xx+aa}}$. Pone $\frac{xdx}{\sqrt{xx+aa}}=dz$; ergo $\sqrt{xx+aa}=z$, & $x=\sqrt{zz-aa}$. Fac $x=y$, & curvam deſcribe coordinatarum $z$, $y$, cujus æquatio $zz-aa=yy$, quæ eſt ad hyperbolam æquilateram. Semiaxe $CA=a$ deſcripta hiperbola HD, (*Fig. 9.*) abſcinde $CB=z=\sqrt{xx+aa}$, & ducta ordinata CD habebis $\frac{x^2dx}{\sqrt{xx+aa}}=ABD$. Quare ſi ſpatium hoc ter ſumptum demas ex quantitate algebraica $\frac{x^3}{\sqrt{xx+aa}}$ habebis integrale quæſitum formulæ $\frac{-x^4dx}{\overline{xx+aa}^{\frac{3}{2}}}$.

 29. Uſus

29. Uſus ſubſtitutionum, per quas formulæ in ſimpliciores redigantur, frequentiſſimus eſt in analyſtis. Exemplum habe in formula

$n^{\frac{1}{3}}dn\sqrt{n^{\frac{4}{3}}+2a^{\frac{2}{3}}n^{\frac{2}{3}}+2a^{\frac{4}{3}}}$. Pone $n^{\frac{2}{3}}+a^{\frac{2}{3}}=\frac{x}{a^{\frac{1}{3}}}$, ut fiat

$n^2=\frac{\overline{x-a}^{3}}{a}$, & acceptis differentiis $2ndn=\frac{3dx.\overline{x-a}^{2}}{a}$. Divide primam partem per $n^{\frac{2}{3}}$, ſecundam per $\frac{x-a}{a^{\frac{1}{3}}}$, ut ſit $n^{\frac{1}{3}}dn=\frac{3}{2}.\frac{dx.\overline{x-a}}{a^{\frac{2}{3}}}$;

peractiſque ſubſtitutionibus, in hanc formulam mutatur propoſita

$\frac{3}{2}.\frac{xdx-adx}{a}.\sqrt{xx+aa}$. Formula $\frac{xdx\sqrt{xx+aa}}{a}$ integrabilis eſt algebraice; ejuſque ſummatoria $=\frac{\overline{xx+aa}^{\frac{3}{2}}}{3a}$; Secunda $dx\sqrt{xx+aa}$ conſtruitur ſuppoſita hyperbolæ quadratura. Sit hyperbola æquilatera AD, (*Fig.9.*) cujus ſemiaxis CA $=a$. In ſecundo axe abſcinde CE $=x$, & duc ordinatam ED, erit $S\,dx\sqrt{xx+aa}$ $=$ CADE; ergo ſummatoria quæſita

$$\frac{3}{2}.\frac{xdx-adx}{a}.\sqrt{xx+aa}=n^{\frac{1}{3}}dn\sqrt{n^{\frac{4}{3}}+2a^{\frac{2}{3}}n^{\frac{2}{3}}+2a^{\frac{4}{3}}}=$$

$$\frac{\overline{xx+aa}^{\frac{3}{2}}}{2a}-\frac{3}{2}.\,CADE=\frac{\overline{n^{\frac{4}{3}}+2a^{\frac{2}{3}}n^{\frac{2}{3}}+2a^{\frac{4}{3}}}^{\frac{3}{2}}}{2}-\frac{3}{2}\,CADE,$$

exiſtente CE $=a^{\frac{1}{3}}.\overline{n^{\frac{2}{3}}+a^{\frac{2}{3}}}$.

30. Methodus conſtruendi formulas per quadraturas curvarum non deficit, tametſi formulæ differentiales implicent ſignum ſummatorium unum aut multiplex. Verum opus erit uti quadraturis curvarum traſcendentium. Nihilominus ſi unicum adſit ſignum ſummatorium, & quantitas differentialis hoc multiplicans ſit algebraice integrabilis, res conficietur, ut antea per quadraturas curvarum algebraicarum. Ut hoc generatim demonſtrem, aſſumo formulam $dzSydx$, in qua $y$ ſit data per $x$, $dz$ præterea per $dx$ isa tamen ut algebraicam integrationem recipiat. Perſpicuum eſt $DzSydx=dzSydx+zydx$; ergo $zSydx-Szydx=SdzSydx$; atqui utraque $Sydx$, $Szydx$ integratur ſuppoſita quadratura curvæ algebraicæ, niſi ſecunda ſit ſummabilis algebraice, quod aliquando accidit; ergo $SdzSydx$ integratur unice quadratis curvis alge-

gebraicis. Exemplum det formula $\overline{a+x}.dx\mathrm{S}\frac{dx}{x}$. Etenim habebimus

$\frac{2ax+xx}{2}\mathrm{S}\frac{dx}{x}=\mathrm{S}\overline{\overline{a+x}.dx\mathrm{S}\frac{dx}{x}}+\mathrm{S}\overline{\frac{2a+x.dx}{2}}$, quæ ultima algebraice integrabilis eſt; ergo $\frac{2ax+xx}{2}\mathrm{S}\frac{dx}{x}-\frac{4ax-xx}{4}=\mathrm{S}\overline{\overline{a+x}.dx\mathrm{S}\frac{dx}{x}}$, quæ proinde integratur, poſita integratione formulæ $\frac{dx}{x}$, quæ dependet a quadratura hyperbolæ apollonianæ relate ad aſymptota.

31. Formula differentialis $dt\mathrm{S}\overline{dz\mathrm{S}ydx}$ implicans duo ſigna ſummatoria integrabitur ſuppoſitis quadraturis curvarum algebraicarum, ſi $dt$, $dz$ ita datæ fuerint per $x$, $dx$, ut $dt$, $dz$, $tdz$ ſint algebraice integrabiles. Etenim $t\,\mathrm{S}\overline{dz\mathrm{S}ydx}-\mathrm{S}\overline{tdz\mathrm{S}ydx}=\mathrm{S}\overline{dt\,\mathrm{S}\overline{dz\mathrm{S}ydx}}$; atqui $\mathrm{S}\overline{dz\mathrm{S}ydx}$, $\mathrm{S}\overline{tdz\mathrm{S}ydx}$ obtinentur per quadraturas curvarum algebraicarum ex num. 28. ſi $dz$, $tdz$ algebraice integrentur; ergo etiam $\mathrm{S}dt\mathrm{S}\overline{dz\mathrm{S}ydx}$, ſi $dt$, $dz$, $tdz$ algebraicam recipiant integrationem. Similiter $ds\mathrm{S}\overline{dt\mathrm{S}\overline{dz\mathrm{S}ydx}}$ involvens tria ſigna ſummatoria ad ſui conſtructionem requiret ſolum quadraturas curvarum algebraicarum, ſi $ds$, $dt$, $tdz$, $sdt$, $dz\mathrm{S}sdt$, algebraice integrabiles fuerint. Nam $s\mathrm{S}\overline{dt\mathrm{S}\overline{dz\mathrm{S}ydx}}-\mathrm{S}\overline{sdt\mathrm{S}\overline{dz\mathrm{S}ydx}}=\mathrm{S}\overline{ds\mathrm{S}\overline{dt\mathrm{S}\overline{dz\mathrm{S}ydx}}}$; atqui quadratis curvis algebraicis habetur prima ſi $dt$, $tdz$, algebraice integrentur, obtinetur ſecunda, ſi algebraice ſummentur $sdt$, $dz\mathrm{S}sdt$; ergo quadratis algebraicis curvis invenietur tertia, ſi algebraice ſint integrabiles $ds$, $dt$, $tdz$, $sdt$, $dz\mathrm{S}sdt$. Ita aucto ſignorum S numero, creſcunt conditiones neceſſariæ, ut formula integretur per quadraturas curvarum algebraicarum.

32. Verum ubi multiplicatur ſignorum S numerus, elegantior evadet calculus, ſi initium ſumas a ſignis interioribus; & paullatim eadem adaugeas. Hoc clariſſime conſtabit in formula, quam in exemplum propono &c.

$\mathrm{S}\overline{dx\,\mathrm{S}\overline{dx\,\mathrm{S}\overline{dx\mathrm{S}\overline{dx.\mathrm{S}ydx}}}}$. Ita enim calculum inſtituam, ut eam ad integrationem perducam; $x\mathrm{S}ydx-\mathrm{S}yxdx=\mathrm{S}\overline{dx\mathrm{S}ydx}$. Hanc multiplica per $dx$; & integra, ut fiat

$\mathrm{S}\overline{xdx\mathrm{S}ydx}-\mathrm{S}\overline{dx\,\mathrm{S}yxdx}=\mathrm{S}\overline{dx\mathrm{S}\overline{dx\mathrm{S}ydx}}$; atqui

$\frac{x^2}{2}\mathrm{S}ydx+\frac{1}{2}\mathrm{S}yx^2dx=\mathrm{S}\overline{xdx\mathrm{S}ydx}$, &

$x\mathrm{S}yxdx-\mathrm{S}yx^2dx=\mathrm{S}dx\mathrm{S}yxdx$; Ergo

$\frac{x^2}{2}\mathrm{S}ydx-x\mathrm{S}yxdx+\frac{1}{2}\mathrm{S}yx^2dx=\mathrm{S}\overline{dx\,\mathrm{S}\overline{dx\mathrm{S}ydx}}$. Hanc multiplica per $dx$, & integra

$\frac{1}{2}\mathrm{S}x^2dx\mathrm{S}ydx-\mathrm{S}xdx\mathrm{S}yxdx+\frac{1}{2}\mathrm{S}\overline{dx\mathrm{S}yx^2dx}=\mathrm{S}\overline{dx\mathrm{S}\overline{dx\,\mathrm{S}\overline{dx\,\mathrm{S}ydx}}}$

atqui

$$\frac{x^3}{2.3}Sydx + \frac{1}{2.3}Syx^3dx = \frac{1}{2}S\overline{x^2dxSydx}$$

$$\frac{x^2}{2}Syxdx - \frac{1}{2}Syx^3dx = S\overline{xdxSyxdx}$$

$$\frac{x}{2}Syx^2dx - \frac{1}{2}Syx^3dx = \frac{1}{2}S\overline{dxSyx^2dx}.$$ Ergo

$$\frac{x^3}{2.3}Sydx - \frac{x^2}{2}Syxdx + \frac{x}{2}Syx^2dx - \frac{1}{2.3}Syx^3dx = S\overline{dxS\overline{dxS\overline{dxSydx}}}$$

Hanc, ut ſupra, multiplica per $dx$. & integra

$$\frac{1}{2.3}Sx^3dxSydx - \frac{1}{2}Sx^2dxSyxdx + \frac{1}{2}SxdxSyx^2dx - \frac{1}{2.3}SdxSyx^3dx$$

$$= S\overline{dxS\overline{dxS\overline{dxS\overline{dxSydx}}}};$$ atqui

$$\frac{x^4}{2.3.4}Sydx - \frac{1}{2.3.4}Syx^4dx = \frac{1}{2.3}Sx^3dxSydx$$

$$\frac{x^3}{2.3}Syxdx - \frac{1}{2.3}Syx^4dx = \frac{1}{2}S\overline{x^2dxSyxdx}$$

$$\frac{x^2}{2.2}Syx^2dx - \frac{1}{2.2}Syx^4dx = \frac{1}{2}S\overline{xdxSyx^2dx}$$

$$\frac{1x}{2.3}Syx^3dx - \frac{1}{2.3}Syx^4dx = \frac{1}{2.3}S\overline{dxSyx^3dx};$$ ergo

$$\frac{x^4}{2.3.4}Sydx - \frac{x^3}{2.3}Syxdx + \frac{x^2}{2.2}Syx^2dx - \frac{x}{2.3}Syx^3dx$$

$$+\frac{1}{2.3.4}Syx^4dx = S\overline{dxS\overline{dxS\overline{dxS\overline{dxSydx}}}};$$ atque ita progrediens ad quadraturas curvarum deduces formulam, quotcumque implicet ſigna ſummatoria.

33. Cæterum ſi generatim loquamur, formula differentialis implicans ſigna ſummatoria ad ſui conſtructionem indiget quadratura curvæ traſcendentis, aut mechanicæ, quæ eo altior erit, quo augetur ſignorum numerus. Qua vero methodo conſtructio peragenda ſit, paucis oſtendam. Sit conſtruenda formula $\frac{zdx}{a^2}S\frac{ydx}{a}$; quæ ſemper redigenda eſt ad dimenſionem linearem facta multiplicatione, aut diviſione per $a$. Conſtrue primum curvam algebraicam AD (*Fig.* 12.) cujus abſciſſæ $AB = x$, ordinatæ $BD = y$; ejus area $ADB = Sydx$. Pone $BE = \frac{ABD}{a} = S\frac{ydx}{a}$, & orietur curva mechanica AE. Facta $BE = m$; noſtra formula in hanc mutabitur $\frac{zm}{a} . \frac{dx}{a}$. Inveni quartam

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proportionalem poſt $a$, $z$, & $BE = m$, quam pone in BF, & orietur alia curva mechanica AF, cujus area $ABF = S\frac{zm\,dx}{a}$. Huic diviſæ per $a$ fac æqualem BG, erit $BG = S\frac{zm\,dx}{aa} = S\left(\frac{z\,dx}{aa}\,S\frac{y\,dx}{a}\right)$, quæ erat invenienda. Si plura adſint ſigna ſummatoria, eadem methodo debes progredi. Hæc methodus aperte docet, formulas differentiales una tantum variabili præditas per quadraturas curvarum numquam conſtrui non poſſe. Sed advocans ea artificia, quæ paullo ante indicata ſunt, ad elegantiores ſæpe conſtructiones devenies.

34. Requiret fortaſſe aliquis, utrum formula differentialis conſtrui poſſit per quadraturam curvæ relatæ ad focum, facto uſu earum formularum, quas invenimus num. 22. Conſtrui poſſe reſpondeo; ſed niſi uſurpes artificia recondita, & difficilem calculum requirentia, conſtructio perficietur per quadraturam curvæ mechanicæ. Hæc omnia breviter declarabo. Sit conſtruenda formula $Y\,dx$, in qua Y data ſit per $y$. Hanc diviſam per $\frac{1}{2}y$ fac æqualem $dx$, quæ $dx$ ſit minimus arcus circuli radio $y$ deſcriptus, ut proveniat æquatio curvæ relatæ ad focum $\frac{2Y\,dy}{y} = dx$. Sit hæc curva AB, (*Fig.* 7.) ubi $CB = y$, $Bm = dx$; ergo fiet $Y\,dy = \frac{y\,dx}{2}$, ſeu $SY\,dy = \frac{1}{2}Sy\,dx$; atqui $\frac{1}{2}Sy\,dx$ æquat aream ACB; ergo $SY\,dy = ACB$: quare conſtructa eſt formula propoſita per quadraturam. Verum curva AB hoc modo inventa, cujus quadratura exhibet noſtræ formulæ integrationem, plerumque mechanica eſt, atque traſcendens.

35. Quod ſi eas æquationes ſpectes, quæ comparant radios $CB = y$, & arcus $AD = z$, qui conſtanti radio $CA = a$ præditi ſunt, ut areæ differentia $CBb = \frac{1}{2}Sy^2\,dz$, hac methodo uti poteris ad conſtruendam æquationem $X\,dx$, in qua X data eſt per $x$. Pone $s\,dx = dz$, in qua $s$ data eſt quomodocumque per $x$, & abſcinde arcum $AD = Ss\,dx$. Deinde ducta CD, abſcinde $CB = \sqrt{\frac{2aX}{s}} = y$; erit area $ACB = S\frac{y^2\,dz}{a} = SX\,dx$. Hæc quoque conſtructio integrat datam formulam per quadraturam curvæ æquationis $s\,dx = dz$, quæ plerumque eſt traſcendens. Verum ſi indeterminatam $s$ datam per $x$ ita accipias, ut ſummatoria $s\,dx$ exprimat arcum circuli, verus radius ſit $a$, & qui ſit ad $z$ in ratione numeri ad numerum, curva, quæ orietur, erit algebraica, & formula differentialis integrabitur per quadraturam curvæ algebraicæ. Quænam autem ſint formulæ exprimentes per ſinus, coſinus, tangentes, & alias lineas, arcus circulares, & quomodo ex æquationibus, quibus inſunt conditiones ante poſitæ, ſiat tranſitus ad æquationes algebraicas inter ordinatas orthonogonales, ſuo loco oportebit fuſius explicare. Ad rem autem præſentem hæc ſatis ſint, ut cognoſcat analyſta, quadraturas curvarum, quæ referuntur ad focum, poſſe aliquando non ineleganter formularum integrationem exhibere.

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## CAPUT SEXTUM.

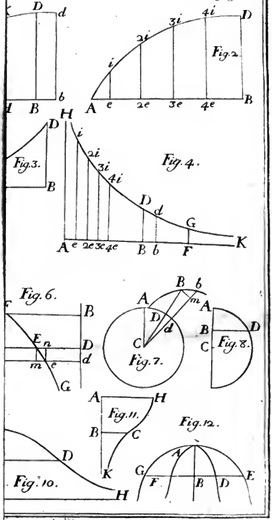
### De quadratura hyperbolæ apollonianæ inter aſymptota, de logarithmis, & de multiplici eorum ſiſtemate.

QUamquam formulæ differentiales ſemper conſtrui poſſunt quadratis curvis algebraicis; tamen rectus ordo poſtulat, ut ſi fieri poteſt, per quadraturas maxime ſimplices integrentur. Sectiones conicæ ſeſe offerant. Parabola, ut oſtenſum eſt, recipit algebraicam quadraturam. Reliquæ ſunt ellypſis, & hyperbola. Itaque a recta metodo diſcedet, qui advocet altiorum curvarum quadraturas ad conſtruendas formulas, quæ integrari poſſunt quadratis ellypſi, & hyperbola. Quænam ſint hujuſmodi formulæ, diligenter eſt inſpiciendum. Incipiamus ab hyperbola relata ad aſymptota.

1. Sit inter aſymptota CH, CN (*Fig.* 1.) in angulo recto concurrentia conſtituta hyperbola æquilatera EPR, ex cujus præcipuo vertice demitte BA normalem aſymptoto CN. Duc ordinatam PI, eique infinite proximam pi. Vocatis $CA = AB = a$, $CI = x$, $IP = y = \frac{a}{x}$, erit rectangulum PIi, quod eſt elementum areæ hiperbolicæ $= \frac{a^2 dx}{x}$. Incidimus in eam formulam, quæ licet contineatur in generali $x^m dx$; tamen, ut ſæpe monuimus, algebraicam reſpuit integrationem. Ea igitur dependet ab hyperbolæ quadratura. Sed de hac plura æque maximi momenti dicenda ſunt.

2. Si in aſymptoto CN acceptis quibuſcumque CI, CL, ſecentur aliæ duæ CM, CN, quæ ſint cum primis in eadem ratione, & ducantur ordinatæ IP, LQ, MR, NS; ajo æqualia eſſe ſpatia hyperbolica PILQ, RMNS. Acceptis elementis Ii, Mm ita, ut ſervent rationem CI, CM, agantur ordinatæ ip, mr. Quoniam ex conſtructione Ii : Mm :: CI : CM, & ex natura hyperbolæ CI : CM eſt in ratione reciproca IP : MR, erit Ii : Mm in ratione reciproca IP : MR; ergo rectangula PIi, RMm reciprocantia latera erunt æqualia. Si ſumantur denuo duo nova elementa, quæ ſint ut CI : CM ſive ut Ci : Cm, eadem ratione rectangula provenient æqualia; ergo progreſſu hoc in infinitum producto palam fiet, omnia rectangula exhaurientia ſpatium PL æqualia eſſe omnibus rectangulis exhaurientibus ſpatium RN; ergo ſpatia hyperbolica PL, RN erunt æqualia. Q.E.D. Ex quocumque puncto D ducatur ordinata DE. Quoniam differentia inter ſpatia EDIP, EDLQ eadem eſt ac differentia inter ſpatia EDMR, EDNS, apparet inter hæc quatuor ſpatia intercedere proportionem arithmeticam. Itaque dum abſciſſæ ſumptæ in aſymptoto, & incipientes a centro ſunt in ratione geometrica, ſpatia hyperbolica incipientia a qualibet DE, & iiſdem abſciſſis reſpondentia tenent rationem arithmeticam; igitur ſi accipiantur in aſymptoto lineæ conſtituentes ſeriem geometricam, ſpatia hyperbolica conſtituent ſeriem arithmeticam.

3. Ex hac proprietate colligimus ſpatia duo infinite longa EDK, EDCH ambo eſſe infinita. Nam quoad primum, accipiantur CD, CA, CI, CL &c.


D
d
b
B
Fig. 2.
A
e
2e
3e
4e
i
2i
3i
4i
D
B
H
Fig. 3.
D
B
Fig. 4.
i
2i
3i
4i
D
d
G
K
A
e
2e
3e
4e
B
b
F
Fig. 6.
B
D
d
E
n
m
e
G
Fig. 7.
A
B
b
m
D
d
C
Fig. 8.
A
B
D
C
Fig. 11.
A
H
B
C
K
Fig. 12.
A
G
E
F
B
D
Fig. 10.
D
H

&c. continue proportionales, conſtat æqualia eſſe ſpatia EDAB, BAIP, PILQ, &c. atqui numerus continue proportionalium eſt infinitus: ergo numerus ſpatiorum æqualium eſt infinitus: atqui primum ex his EDAB, exiſtente DA finita, finitum eſt: ergo eorum ſumma, adeoque ſpatium hyperbolicum EDK erit infinitum. Similis ratio valet pro altero, quia acceptis poſt CD, Ca continue proportionalibus, earum numerus eſt infinitus: ergo ſpatia ab ordinatis clauſa ſunt numero infinita: atqui hæc omnia ſunt æqualia inter ſe, & primo ſpatio EDa, quod finitum ſumitur: igitur eorum ſumma; atque adeo ſpatium infinite longum EDCH eſt infinitum. Q.E.D. Quapropter ſi ad conſtruendam S $\frac{aadx}{x}$, quæ ſine dubio æqualis eſt areæ hyperbolicæ, initium areæ ſumeretur ab aſymptoto CH, fruſtraretur omnino conſtructio, quia cuilibet abſciſſæ $x$ finitæ reſponderet quantitas infinita. Quare ne fruſtretur, neceſſe eſt ſumere areæ initium ab ordinata fixa DE, quæ pro libito accipi poteſt. Itaque exiſtente CI $= x$, erit S $\frac{aadx}{x}$ æqualis areæ hyperbolicæ EDIP.

4. Quoniam acceptis abſciſſis in continua ratione geometrica, areæ ſervant continuam proportionem arithmeticam, ſpectari poſſunt abſciſſæ ut numeri, areæ hyperbolicæ ut logarithmi. Sed ſi ita acciperentur, eſſent logarithmi ſecundæ dimenſionis. Juvabit autem eos ad primam dimenſionem perducere hac methodo. Intelligatur deſcripta curva HDY, tranſiens per punctum fixum D, ea conditione, ut quælibet ordinata IT ducta in conſtantem $= b$ æquet aream hyperbolicam EDIP; ergo ordinatæ IT erunt ut reſpondentes areæ hyperbolicæ incipientes a fixa DE: ergo creſcentibus abſciſſis in ratione geometrica, noſtræ curvæ ordinatæ creſcent in ratione arithmetica; illæ itaque erunt numeri, iſtæ logarithmi. Qua de re curva HDY, vocatur logiſtica, ſeu logarithmica. Ut autem expeditis notis utamur, ad indicandum logarithmum numeri cujuslibet CI, ſive rectam IT ſcribemus $l$. CI; ita ut littera $l$ lineæ præfixa ejus lineæ deſignet logarithmum.

5. Ex ſola inſpectione figuræ apparet $l$. CD $= o$. Numerus vero CD ex arbitrato determinari poteſt. Logarhitmus autem numeri ſuperantis CD erit affirmativus, negativus logarithmus numeri minoris quam CD. Ex quibus propoſitio converſa ſtatim colligitur: nimirum logarithmus $= o$ habet numerum $=$ CD, logarithmus affirmativus habet numerum majorem quam CD, negativus autem minorem. Quemadmodum conſideravi tamquam poſitivos logarithmos poſitos ſub CN, negativos, qui ſupra ſiti ſunt; ita hos potuiſſem conſiderare ut affirmativos, illos ut negativos. Arbitraria eſt, ut diximus, determinatio numeri CD habentis logarithmum $= o$. Verum facta hac determinatione nondum logarithmica HDY determinata eſt; ſed præterea determinanda eſt conſtans $= b$, quæ per logarithmos multiplicata exhibet ſpatia hyperbolica, aut alia quælibet conſtans, per quam ipſa $b$ determinata remaneat: ut ſi quis determinaret numerum, cujus logarithmus aut eſſet æqualis CD, aut ad CD datam haberet proportionem. Hoc autem ſilentio prætereundum non erat; quia ex determinatione unius potius, quam alterius conſtantis oriuntur diverſa logarithmorum ſiſtemata.

6. Si fuerint quatuor numeri geometrice proportionales, logarithmi mediorum æquabunt logarithmos extremorum. Nam numeri geometrice proportionales habent logarithmos proportionales arithmetice: atque notum eſt, quatuor

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quantitates arithmetice proportionales ita eſſe affectas, ut ſumma mediarum æquet ſummam extremarum; ergo ſi numeri ſint geometrice proportionales, habebunt eorum logarithmi ſummam extremorum æqualem ſummæ mediorum. Hinc deſcendit fore $l\frac{bk}{f} = lb + lk - lf$, quia $\frac{bk}{f}$ eſt quarta proportionalis poſt $f, b, k$, ergo ejus logarithmus, addito $lf = lb + lk$; igitur $l\frac{bk}{f} = lb + lk - lf$. Si $b = k$ haberetur $l\frac{b^2}{f} = 2lb - lf$; ſi $k = \frac{b^2}{f}$, erit $l\frac{b^3}{f^2} = 3lb - 2lf$. Imo generatim per hanc methodum inveniemus $l\frac{b^p}{f^{p-1}} = plb - (\overline{p-1}.lf$. Si $f = CD$, ut ſit $lf = o$, erit $l\frac{b^p}{f^{p-1}} = plb$. Generalius etiam $l\frac{b^p k^q}{f^{p+q-1}} = plb + qlk - (\overline{p+q-1}.lf$; ergo ſi $lf = o$, hoc omiſſo, fiet $l\frac{b^p k^q}{f^{p+q-1}} = plb + qlk$; & ſi fuerit $lb = o$, orietur $l\frac{b^p k^q}{f^{p+q-1}} = qlk - (\overline{p+q-1}.lf$. In his continentur principia, per quæ ab æqualitate inter logarithmos fit tranſitus ad æqualitatem inter numeros, & vice verſa.

7. Exhibeamus jam logarithmicæ æquationem differentialem. Ex vertice hyperbolæ B demittatur BA normalis aſymptoto CN, & fiat $BA = CA = a$, $CD = f$. Sumatur $CL = mf$; $m$ eſt numerus conſtans determinandus prout uſus veniet. Numeri autem CL logarithmus eſt LV, quem vocabimus $= g$ ita, ut ſit $g = lmf$. Ducta vero VZ parallela CM, fiat $LM = VZ = x$, $MR = y$, $ZX = z$; quare $mf + x$ erunt numeri, $g + z$ logarithmi. Ex æquatione hyperbolæ eſt $\frac{a^2}{mf + x} = y$; ergo facta multiplicatione per $dx$ fiet $\frac{a^2 dx}{mf + x} = ydx$. Quum autem ſpatium hyperbolicum $EDMR = b.MX = b.\overline{g+z}$, elementum ejuſdem ſpatii nempe $ydx$ debet æquare $bdz$: igitur $\frac{aadx}{mf + x} = bdz$, per quam æquationem oportet determinare proportionem inter numeros $mf + x$, & logarithmos $g + z$.

8. Quum $\frac{a^2}{b} = \overline{mf + x}.\frac{dz}{dx}$, deinceps demonſtrabimus, ſubtangentem logiſticæ acceptam in aſymptoto CH fore conſtantem, & $= \frac{a^2}{b}$. Quare liceat vocare ſubtangentem logiſticæ quantitatem $\frac{a^2}{b}$, quam deinceps dicemus $= c$, ut ſit æquatio $\frac{cdx}{mf + x} = dz$. Hinc deducere poſſumus per quamlibet logarithmicam, ſeu per logarihmicam cujuſcumque ſubtangentis quadrari poſſe quamlibet

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libet hyperbolam. Etenim ſi oporteat quadrare hyperbolam æquilateram, cujus potentia $= a$, inveniatur tertia proportionalis poſt $c, a$, quæ vocetur $= b$, ita ut ſit $b = \frac{aa}{c}$, ſpatium hyperbolicum, erit $= b.\overline{g+z}$ addita, demptave conſtante, prout opus fuerit. Quod ſi hyperbola non ſit æquilatera, neque angulus aſymptotorum rectus, tunc ſpatium hyperbolicum æquabit non rectangulum $b.g+z$, ſed parallelogrammum, cujus latus unum ſit $b$, alterum $g+z$, quæ duo latera angulum efficiant æqualem angulo aſymptotorum.

9. Nunc dato numero $mf+x$ inveniamus per ſeriem logarithmum $g+z$. In æquatione $dz = \frac{cdx}{mf+x}$, ſi fractionem evolvamus in ſeriem, proveniet

$$dz = \frac{cdx}{mf} - \frac{cxdx}{m^2f^2} + \frac{cx^2dx}{m^3f^3} - \frac{cx^3dx}{m^4f^4} \text{ \&c. ergo integrando}$$

$$g+z = A + \frac{cx}{mf} - \frac{cx^2}{2m^2f^2} + \frac{cx^3}{3m^3f^3} - \frac{cx^4}{4m^4f^4} \text{ \&c.}$$ Ad determinandam conſtantem $A$, opus eſt advertere, fieri $z = o$, ſi $x = o$; ergo $A = g = lmf$; igitur $g+z$, ſive $l.\overline{mf+x} = lmf + c.\frac{x}{mf} - \frac{x^2}{2m^2f^2} + \frac{x^3}{3.m^3f^3} - \frac{x^4}{4.m^4f^4}$ &c.

Si $x$ foret negativa, ſeu ſi quærendus eſſet $l.\overline{mf-x}$, orirentur

$$l.\overline{mf-x} = lmf - c.\frac{x}{mf} + \frac{x^4}{2m^2f^2} - \frac{x^3}{3.m^3f^3} + \frac{x^4}{4.m^4f^4} \text{ \&c.}$$

10. Ut uſus harum ſerierum palam fiat, ſupponamus primum $m = 1$, ita ut puncta L, V coincidant cum puncto D. Quia $lf = o$, habebimus

$$l.\overline{f+x} = c.\frac{x}{f} - \frac{x^2}{2f^2} + \frac{x^3}{3f^3} - \frac{x^4}{4f^4} \text{ \&c.}$$ Hæc ſeries convergens erit, donec $x = f$; ſi enim vel tantillum ſuperet $f$, tum poſt aliquos terminos ſpecie convergentes ſequuntur termini divergentes. Quare ſeries utilis erit ad inveniendos logarithmos numerorum ab $f$ uſque ad $2f$. Inveniemus autem

$l2f = c.1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}$. &c. Hiſce inventis ponamus $m = 2$, & proveniet

$$l.\overline{2f+x} = l2f + c.\frac{x}{2f} - \frac{x^2}{2^2.2f^2} + \frac{x^3}{2^3.3f^3} - \frac{x^4}{2^4.4f^4} \text{ \&c.}$$ Quæ ſeries convergens erit, donec $x = 2f$, quo in caſu habebimus

$l4f = l2f + c.1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4}$ &c. ſive

$l4f=2l2f=2c.\overline{1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}\ \&c.}$ Series inventa determinabit logarithmos omnium numerorum ab $2f$ uſque ad $4f$. Ponamus $m=4$, erit

$l.\overline{4f+x}=l4f+c.\overline{\frac{x}{4f}-\frac{x^2}{2.4^2f^2}+\frac{x^3}{3.4^3f^3}-\frac{x^4}{4.4^4f^4}\ \&c.}$ Series hæc erit convergens, donec $x=4f$, quo in caſu

$l8f=l4f+c.\overline{1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}\ \&c.}$, ſive

$l8f=3l2f=3c.\overline{1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}\ \&c.}$ Atque ita deinceps progrediendo, omnium numerorum logarithmos per ſeries convergentes inveniemus. Quod attinet ad logarithmos numerorum minorum quam $f$; ſi ponatur $m=1$, & $x$ negativa; invenientur eadem ratione logarithmi numerorum a $f$ ad $\frac{f}{2}$; tum ſi ponatur $m=\frac{1}{2}$, invenientur logarithmi numerorum a $\frac{f}{2}$ uſque ad $\frac{f}{4}$; atque ita deinceps.

11. In ſuperioribus quædam adnotanda ſunt, quæ praxim reddunt difficillimam. Series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}$ &c. ex duabus ſeriebus componitur, quarum prima ex illis terminis conſtat, quibus præfigitur ſignum $+$, altera ex illis, quibus ſignum $-$ præponitur. Utraque ex iſtis ſeriebus infinita eſt, licet earum differentia ſit finita. Huic incommodo facile medemur, ſi reapſe ſingulorum parium differentiam ſumamus, & ſeriem formemus hujuſmodi. $\frac{1}{2}+\frac{1}{3.4}+\frac{1}{5.6}+\frac{1}{7.8}$ &c. Verum hæc ſeries lentiſſime convergit ita, ut oporteat ſumere terminos ſeptem & decem, ut ſolum ad partes milleſimas perveniamus. Quemadmodum proceſſimus per ſeriem $f$, $2f$, $4f$, $8f$ &c., ut inveniremus primum logarithmos numerorum inter $f$, & $2f$, tum inter $2f$ & $4f$; atque eadem ratione deinceps: ita poſſumus progredi per ſeriem, cujus termini minus inter ſe differant, ex cauſ. $f$, $\frac{3}{2}f$, $\frac{9}{4}f$, $\frac{27}{8}f$, $\frac{81}{16}f$ &c., ex qua ratione inveniemus numerorum logarithmos per ſeries magis convergentes. Facta itaque $m=1$, inveniemus logarithmos numerorum ab $f$ uſque ad $\frac{3}{2}f$; quo in caſu extremo, quum ſit $x=\frac{1}{2}f$, habebimus

$l\frac{3}{2}f=c.\overline{\frac{1}{2}-\frac{1}{2.2^2}+\frac{1}{3.2^3}-\frac{1}{4.2^4}+\frac{1}{5.2^5}}$ &c. quæ ſeries aliquanto magis convergit. Tum fiat $m=\frac{3}{2}$, atque in hac ſuppoſitione inveniemus logarithmos numerorum a $\frac{3}{2}f$ uſque ad $\frac{9}{4}f$, inter quos jacet $2f$, cujus logarithmus inve-

invenietur $l\,2f = l\,\frac{3}{2}f + c.\overline{\frac{1}{3} - \frac{1}{2.3^2} + \frac{1}{3.3^3} - \frac{1}{4.3^4} + \frac{1}{5.3^5}}$ &c.

Deinde inveniemus logarithmum numeri $\frac{9}{4}f$, nempe

$l\,\frac{9}{4}f = 2l\,\frac{3}{2}f = 2c.\overline{\frac{1}{2} - \frac{1}{2.2^2} + \frac{1}{3.2^3} - \frac{1}{4.2^4}}$ &c. Præterea faciamus $m = \frac{9}{4}$, & progrediamur uſque ad logarithmum numeri $\frac{27}{8}f$, qui erit triplus $l\,\frac{3}{2}f$. Inter hos numeros jacet $3f$, cujus logarithmus erit

$l\,3f = l\,\frac{9}{4}f + c.\overline{\frac{3}{4} - \frac{3^2}{2.4^2} + \frac{3^3}{3.4^3} - \frac{3^4}{4.4^4}}$ &c. Tum fiat $m = \frac{27}{9}$, & inveniantur logarithmi numerorum uſque ad $\frac{81}{27}f$, atque ita deinceps. Series autem, ut cuiquam patens eſt, convergerent magis magiſque, ſi progrederemur per ſeriem, cujus numeri primi magis ad æqualitatem accederent, ut $f, \frac{10}{11}f$. Hæc methodus utilis eſſe poteſt, ſed plures operationes inſtituendæ ſunt ad inveniendos logarithmos numerorum integrorum quam in illa, quam modo paucis expono.

12. Quum $l\,mf.\frac{mf+x}{mf-x} = l\,mf + l\,\overline{mf+x} - l\,\overline{mf-x}$, habebimus

$l\,mf.\frac{mf+x}{mf-x} = l\,mf + 2c.\overline{\frac{x}{f} + \frac{x^3}{3f^3} + \frac{x^5}{5f^5} + \frac{x^7}{7f^7}}$ &c. Hæc magis convergens eſt, quam ſuperiores. Si ponamus $m = 1$, inveniemus

$l\,f.\frac{f+x}{f-x} = 2c.\overline{\frac{x}{f} + \frac{x^3}{3f^3} + \frac{x^5}{5f^5} + \frac{x^7}{7f^7}}$ &c. Hujuſmodi ſeries, cujuſcumque numeri logarithmus quæratur, ſemper convergens eſt. Nam ſi generatim quæratur $l\,pf$, fieri debet $\frac{f+x}{f-x} = p$; igitur $\frac{x}{f} = \frac{p-1}{p+1}$; quæ quantitas quum minor ſit, quam unitas, conſequitur terminos poſteriores in ſerie, ſemper anterioribus eſſe minores, & ultimos evadere infiniteſimos. Si ex hac ſerie quæramus $l\,10f$, fiet $\frac{x}{f} = \frac{9}{11}$; quare habebimus

$l\,10f = 2c.\overline{\frac{9}{11} + \frac{9^3}{3.11^3} + \frac{9^5}{5.11^5} + \frac{9^7}{7.11^7}}$, &c. Verum magis convergentes ſeries inveniemus, ſi per partes progrediemur. Pone $m = 1$, & ad invenien-

niendum $l2f$, ſtatue $\frac{x}{f} = \frac{1}{3}$, & ſeriem hujuſmodi elicies

$l2f = 2c.\overline{\frac{1}{3} + \frac{1}{3\cdot 3^3} + \frac{1}{5\cdot 3^5} + \frac{1}{7\cdot 3^7}}$ &c., quæ maxime convergit. Invento $l2f$, nanciſcemur $l4f = 2l2f$, $l8f = 3l2f$, $l16f = 4l2f$, atque ita deinceps. Si in noſtra formula $m = 2$, inveniemus logarithmos numerorum, qui ſiti ſunt inter $2f, 4f$, ut $l3f$. In hoc caſu inveniemus $\frac{x}{2f} = \frac{1}{5}$; quare erit

$l3f = l2f + 2c.\overline{\frac{1}{5} + \frac{1}{3\cdot 5^3} + \frac{1}{5\cdot 5^5} + \frac{7}{7\cdot 5^7}}$ &c. Similiter facto $m = 4$, inveniemus logarithmos numerorum, qui ſiti ſunt inter $8f$, $16f$. Ut autem $l10f$ inveniamus, neceſſe eſt ponere $\frac{x}{8f} = \frac{2}{18} = \frac{1}{9}$; igitur

$l10f = l8f + 2c.\overline{\frac{1}{9} + \frac{1}{3\cdot 9^3} + \frac{1}{5\cdot 9^5} + \frac{1}{7\cdot 9^7}}$ &c. Atque hac ratione progredientes logarithmos numerorum per ſeries maxime convergentes detegemus.

13. Non ſum neſcius, in pleriſque logarithmis inveniendis compendio non uti poſſe. Nam dato cujuslibet numeri logarithmo, omnium numerorum, qui conſtituunt ſeriem geometricam poſt $f$, & datum numerum, logarithmi inveniuntur. Imo datis duorum numerorum logarithmis, tertii, quarti &c. proportionales logarithmi inveniuntur. Numerus, qui oritur ex multiplicatione duorum, quorum logarithmi dati ſunt, habet logarithmum datum: numeri, qui oritur per diviſionem, ſi diviſor, & dividendus habet logarithmos datos, logarithmus cognoſcetur. Hæc & ſimilia compendia geometræ cuilibet obvia ſunt. Quare ad ſeries non erit confugiendum, niſi ubi habeantur numeri primi. Series autem ſemper magis convergentes invenientur, ſi $mf$ illi numero ponatur æqualis, qui proxime præcedit numerum, cujus quæritur logarithmus.

14. Converſa propoſitio tractanda eſt: ſcilicet dato logarithmo $g + z$, invenire numerum $mf + x$. Hanc ob rem fiat

$mf + x = A + Bz + Cz^2 + Dz^3 + Ez^4$ &c.; & differentiis acceptis

$dx = Bdz + 2Czdz + 3Dz^2dz + 4Ez^3dz$ &c., & utramque partem multiplicando per $c$, & dividendo primam per $mf + x$, ſecundam per ſeriem æqualem, nanciſcemur

$$\frac{cdx}{mf + x} = c.\frac{Bdz + 2Czdz + 3Dz^2dz + 4Ez^3dz \text{ \&c.}}{A + Bz + Cz^2 + Dz^3 + Ez^4 \text{ \&c}} = dz;$$

igitur facta multiplicatione per ſeriem dividentem, orietur

$cBdz + 2cCzdz + 3cDz^2dz + 4cEz^3dz + 5cFz^4dz =$

$Adz + Bzdz + Cz^2dz + Dz^3dz + Ez^4dz$. Ex comparatione ſingulorum terminorum coefficientium valores hujuſmodi determinantur $B = \frac{A}{c}$,

$C =$

$C = \frac{A}{2c^2}$, $D = \frac{A}{2.3c^3}$, $E = \frac{A}{2.3.4.c^4}$, $F = \frac{A}{2.3.4.5.c^5}$, atque ita deinceps. Igitur fiet $mf + x = A + \frac{Az}{c} + \frac{Az^2}{2c^2} + \frac{Az^3}{2.3.c^3} + \frac{Az^4}{2.3.4.c^4}$ &c. Ad determinandum conſtantis A valorem, adverte fieri $x = o$, ſi $z = o$; ergo evadet $A = mf$. Quare facta ſubſtitutione, & in utraque æqualitatis parte deletis æqualibus, erit $x = mf.\ \frac{z}{c} + \frac{z^2}{2c^2} + \frac{z^3}{2.3c^3} + \frac{z^4}{2.3.4c^4}$ &c. Fac advertas $mf$ eſſe numerum, cujus logarithmus $= g$.

15. Hiſce poſitis ſit primo $g = o$, ut ejus numerus $= f$, & $m = 1$; orietur $x = f.\ \frac{z}{c} + \frac{z^2}{2c^2} + \frac{z^3}{2.3c^3} + \frac{z^4}{2.3.4c^4}$ &c., ex qua æquatione, quæcumque ponatur $z$, quæ eſt logarithmus numeri $f + x$, invenietur $x$; adeoque $f + x$. Verum ſatius erit per partes procedere, & ex ultima æquatione invenire numerum, cujus logarithmus ſit $z$, quæ ſatis parva ſumatur: tum hoc numero facto $= mf$, cujus logarithmus $= g$, ſumatur alia $z$ exigua, & ſic inveniemus numerum $mf + x$, cujus logarithmus eſt $g + z$; hoc definito ad alios deinceps progrediemur. Numeri, qui ex datis logarithmis inveniuntur, dependent tum a ſubtangente $c$, tum etiam a quantitate $f$: quarum quantitatum, niſi altera per alteram determinetur per partes progredi non licebit. Ut autem exemplum clarum proponamus rationis, qua per partes progredi oportet, ſupponamus $c = f$, quæ hypotheſis eſt omnium maxime ſimplex. Hac ſtatuta determinatione, fiet $x = mf.\ \frac{z}{f} + \frac{z^2}{2f^2} + \frac{z^3}{2.3f^3} + \frac{z^4}{2.3.4f^4}$ &c. Pono $m = 1$, & invenio $x = f.\ \frac{z}{f} + \frac{z^2}{2f^2} + \frac{z^3}{2.3f^3} + \frac{z^4}{2.3.4f^4}$ &c. Hæc ſeries exhibet numeros $f + x$ omnium logarithmorum, qui poſiti ſunt inter $o$, & $f$. Quapropter poſita $z = f$, invenietur $f + x = f + f.\ 1 + \frac{1}{2} + \frac{1}{2.3} + \frac{1}{2.3.4}$ &c. Series hæc vocetur $= A$, ut ſit $f + x = f + fA$. Supponatur jam $m = 1 + A$, ut fiat $x = \overline{1 + A}.\ f.\ \frac{z}{f} + \frac{z^2}{2f^2} + \frac{z^3}{2.3f^3} + \frac{z^4}{2.3.4f^4}$ &c. Series iſta utilis erit inveniendis numeris, qui reſpondent logarithmis ſitis inter $f$ & $2f$, ſi $z$ indatur valor poſitus inter $o$, & $f$. Quare ſi ponatur $z = f$, fiet $x = \overline{1 + A}.f.\ 1 + \frac{1}{2} + \frac{1}{2.3} + \frac{1}{2.3.4}$ &c. $= \overline{1 + A}.Af$. Itaque $\overline{1 + A}.f + x = \overline{1 + A}.f + \overline{1 + A}.Af = \overline{AA + 2A + 1}.f = \overline{A + 1}^2.f$, quæ eſt

eſt numerus, cujus logarithmus $= 2f$. Si ſtatuas $m = \overline{A+1}^{2}$, eodem modo invenies numerum, cujus logarithmus $= 3f$, eſſe $= \overline{A+1}^{3}.f$. Per ſeriem autem elicies numeros omnium logarithmorum, qui ſiti ſunt inter $2f$ & $3f$, atque ita in infinitum. Ita ſi deſideras numerum, cujus logarithmus $= \frac{5}{2}f$, fac invenias primum numerum, cujus logarithmus $= 2f$, qui numerus demonſtratus eſt $= \overline{A+1}^{2}.f$; tum fac $m = \overline{A+1}^{2}$, & $z = \frac{1}{2}f$; ergo

$$x = \overline{A+1}^{2}.f.\overline{\frac{1}{2} + \frac{1}{2^{2}.2} + \frac{1}{2^{3}.2.3} + \frac{1}{2^{4}.2.3.4}} \text{ \&c.},$$

quæ ſeries ſi vocetur $= B$, erit $x = \overline{A+1}^{2}.B.f$; ergo $\overline{A+1}^{2}.f + x = \overline{A+1}^{2}.\overline{B+1}.f$, qui eſt numerus logarithmi dati.

16. His præmiſſis advertendum eſt, logarithmorum ſyſtema non minus dependere a quantitate $f$, cujus logarithmus $= o$, quam deinceps vocabimus protonumerum, quam a quantitate $c$, quæ eſt ſubtangens logiſticæ, & ſubtangens ſyſtematis ex analogia appellari poteſt. Quoniam ſyſtema mutari poteſt ex utriuſque quantitatis mutatione, conſiderandum eſt, quibuſnam conditionibus hujuſmodi ſyſtematum mutationes peragantur. Primum mutetur protonumerus eadem manente logiſticæ ſubtangente. Conſtituta ut ſupra, hyperbola EBR, (*Fig.* 2.) & ſumpto protonumero $CD = f$, deſcribatur logiſtica DX, cujus ordinata MX ducta in $b = \frac{aa}{c}$ æquet ſpatium hyperbolicum EDMR. Deinde ſumpto protonumero $CF = F$, deſcribatur alia logiſtica FZ, cujus ordinata MZ in eamdem $b$ ducta æquet ſpatinm hyperbolicum GFMR. Igitur $b$. XZ æquabit ſpatium hyperbolicum EDFG: atqui hoc ſpatium conſtans eſt: ergo XZ ubique conſtans erit, atque adeo æqualis FH. Quare logarithmi ſyſtematis, in quo protonumerus minor eſt, ſuperabunt eorumdem numerorum logarithmos in ſyſtemate, ubi major eſt protonumerus, per quantitatem conſtantem æqualem FH. Quantitas autem FH eſt logarithmus numeri $CF = F$ ſumptus in ſyſtemate, in quo protonumerus eſt $CD = f$. Quamobrem dato quocumque logarithmorum ſyſtemate, cujus protonumerus $= f$, invenientur facillime logarithmi alterius ſyſtematis, cujus protonumerus $= F$. Etenim in ſyſtemate protonumeri $f$ inveniatur logarithmus numeri $F$, qui logarithmus dematur ab omnibus logarithmis ſyſtematis protonumeri $f$, & provenient logarithmi ſyſtematis protonnmeri $F$. Si $F > f$, in ſyſtemate protonumeri $f$, $lF$ erit poſitivus, adeoque ſubtractio facienda eſt. Si vero $F < f$, $lF$ erit negativus; quare mutato ſigno ſubtractio in additionem tranſibit.

17. Si unus idemque ſervetur protonumerus, & ſubtangens logiſticæ mutetur, videndum eſt, quamnam mutationem ſubeat logarithmorum ſyſtema. Iiſdem prorſus ſuppoſitis cum eodem protonumero $CD = f$ deſcribatur primum logiſtica DX, (*Fig.* 3.) cujus ordinata MX dncta in $b$ æquet ſpatium hyperbolicum EDMR. Item alia logiſtica deſcribatur DZ, cujus ordinata MZ ducta in $B$ æquet idem ſpatium EDMR; ergo $b.MX = B.MZ$; igitur $MX : MZ :: \frac{1}{b} : \frac{1}{B}$; ergo duorum ſyſtematum logarithmi ſunt in ratione conſtan-

ti,

ti, hoc eſt in ratione reciproca earum quantitatum, quæ in ipſos ductæ æquant ſpatium hyperbolicum; ſed iſtæ quantitates ſunt reciproce ut logarithmicarum ſubtangentes: ergo logarithmi duorum ſyſtematum, de quibus agimus, ſunt directe ut ſubtangentes. Quare datis unius ſyſtematis logarithmis facile eſt, alterius cujuſcumque ſyſtematis per regulam, quam vocant auream, logarithmos invenire; dummodo datæ ſint ſubtangentes.

18. Quod ſi datum ſit ſyſtema, cujus protonumerus ſit *f*, ſubtangens vero *c*, & alterum ſyſtema inveniendum ſit, cujus protonumerus ſit *F* ſubtangens *C*: primo ex dato ſyſtemate inveniatur ſyſtema protonumeri *F*, & ſubtangentis *c*; tum fiat *c*: *C* ut ſinguli logarithmi modo inventi ad eos, qui quæruntur. Quod ad praxim in dato ſyſtemate inveni logarithmum numeri *F*; Hunc deme ex datis logarithmis; demum inveni *f* quartas proportionales poſt *c*, *C* & logarithmos datos et quantitate minutos; & quæſitos logarithmos determinabis.

19. Problema hoc: dato logarithmorum ſyſtemate aliud ſyſtema invenire, in quo manentibus cæteris mutetur protonumerus, eſt problema plerumque traſcendens. Etenim FH, (*Fig.* 2.) quæ detrahenda eſt ex logarithmis dati ſyſtematis, non datur plerumque niſi traſcendenter per protonumerum novi ſyſtematis CF. Addidi plerumque, quia ſi in ſyſtemate protonumeri CD logarithmus numeri CF, nimirum FH, habeat rationem effabilem ad CD, quod aliquando contingit, problema algebraicum eſt. Verum alterum hoc problema, manente eodem protonumero, & dato uno logarithmorum ſyſtemate, invenire aliud ſyſtema diverſæ ſubtangentis, eſt pure algebraicum; dummodo ſubtangentes ſint altera per alteram algebraice datæ. Adverte hanc conditionem; quam appoſuimus, ſi ex ſubtangentibus altera per alteram data ſit. Nam ſi habeatur ſyſtema logarithmorum, in quo præter protonumerum data ſit ſubtangens, & quæritur ſyſtema logarithmorum, in quo, poſito eodem protonumero, datus ſit dati numeri logarithmus, hoc non niſi traſcendenter obtinebitur; & vice verſa ſi ex hoc primum quæratur logarithmorum ſyſtema. Etenim ſubtangens ex dato logarithmo dati numeri non niſi traſcendenter habetur. Quod ſi poſito eodem protonumero, unius ejuſdemque numeri in uno ſyſtemate logarithmus $= q$, in alio $Q$, omnes logarithmi eorumdem numerorum in duobus ſyſtematibus erunt in directa ratione $q : Q$. Quare dato uno ſyſtemate algebraice alterum determinatur. Hinc oritur duplex genus logarithmorum: vel enim datur protonumerus, & ſubtangens ſyſtematis; vel datur protonumerus cum logarithmo dati numeri. Primum vocabimus ſyſtema logarithmorum generis hyperbolici; alterum generis vulgaris. Eodem poſito protonumero nullus eſt tranſitus ab uno ſyſtemate ad aliud niſi per quantitates traſcendentes.

20. Verum ut ejus ſyſtematis, cujus tabulæ exhibent logarithmos, conditiones cognoſcamus, quid ſit baſis logarithmica explicandum eſt. Baſis logarithmica dicitur numerus ille, cujus logarithmus æquat protonumerum. Quapropter ſi datus ſit protonumerus cum baſi logarithmica, logarithmorum ſyſtema erit generis vulgaris; neque ab hoc fiet tranſitus ad ſyſtema generis hyperbolici, niſi per quantitates traſcendentes. Itaque traſcendentia erunt hujuſmodi problemata: data baſi logarithmica invenire ſubtangentem ſyſtematis; data ſubtangente invenire baſim logarithmicam. Nunc de duobus ſyſtematibus, quæ in uſu poſita ſunt diſtinctius verba faciamus. Syſtema logarithmorum vulgarium ponit logarithmum quantitatis cujuslibet *f*, quam ſtatuit protonumerum, eſſe $= o$, baſim autem logarithmicam $= 10 f$, cujus ſcilicet logarithmus æquat protonumerum *f*. Syſtema vero logarithmorum hyperbolicorum, quo utuntur paſſim

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ſcriptores, convenit cum vulgari in hoc, quod ponit $lf = o$, ſed ejuſdem ſubtangentem ponit $= f$.

21 Quo pacto alterum ſyſtema redigatur ad alterum, oportet cognoſcere: Quamobrem hoc primum inveſtigemus neceſſe eſt ut in ſyſtemate vulgari ſubtangentem determinemus. Ex ſupradictis invenimus.

$$l\,10f = l\,8f + 2c.\left(\frac{1}{9} + \frac{1}{3\cdot 9^3} + \frac{1}{5\cdot 9^5} + \frac{1}{7\cdot 9^7}\ \&c.\right);\ \text{atqui}$$

$$l\,8f = 3\,l\,2f,\ \&\ l\,2f = 2c.\left(\frac{1}{3} + \frac{1}{3\cdot 3^3} + \frac{1}{5\cdot 3^5} + \frac{1}{7\cdot 3^7}\ \&c.\right)\ \text{ergo}$$

$$l\,10f = 6c.\left(\frac{1}{3} + \frac{1}{3\cdot 3^3} + \frac{1}{5\cdot 3^5} + \frac{1}{7\cdot 3^7}\ \&c.\right)$$

$$+ 2c.\left(\frac{1}{9} + \frac{1}{3\cdot 9^3} + \frac{1}{5\cdot 9^5} + \frac{1}{7\cdot 9^7}\ \&c.\right)$$

Prima ex his ſeriebus fiat $= A$, altera $= B$, ut fit $l\,10f = 6Ac + 2Bc$; atqui $l\,10f = f$; ergo $f = 6Ac + 2Bc$, atque adeo $c = \frac{f}{6A + 2B}$, quæ eſt ſubtangens ſyſtematis vulgaris. Series duæ $A$, $B$ maxime convergunt, & nullo negotio ad fractiones decimales rediguntur. Quum ſemper poſito eodem protonumero $f$, logarithmi eorumdem numerorum in diverſis ſyſtematibus ſint inter ſe, ut ſyſtematum ſubtangentes, logarithmi hyperbolici ad vulgares erunt ut $f : \frac{f}{6A + 2B}$, ſive ut $6A + 2B : 1$.

22. Jam ad logarithmos hyperbolicos convertendus eſt animus, & in illis invenienda eſt baſis logarithmica; quam ob rem ſolvendum eſt hoc problema: poſito non minus protonumero, quam ſubtangente $= f$ invenire numerum, cujus logarithmus $= f$. Solutio ex ſupradictis eſt per quam facilis. Namque ſi ponamus $\frac{z}{f}$, ſeu $\frac{z}{c} = 1$, inveniemus $x = f.\left(1 + \frac{1}{2} + \frac{1}{2.3} + \frac{1}{2.3.4} + \frac{1}{2.3.4.5}\ \&c.\right)$, atque adeo $f + x$ numerus reſpondens $lf$ erit æqualis

$$f + f.\left(1 + \frac{1}{2} + \frac{1}{2.3} + \frac{1}{2.3.4} + \frac{1}{2.3.4.5}\ \&c.\right),$$

quæ ſeries ſatis convergit. Quod ſi quæretur numerus, cujus logarithmus æquaret 2, 3, aut $4f$, tum per partes eſſet procedendum, quemadmodum ſupra ſatis docuimus. Adverte, in tabulis cum logarithmorum hypetbolicorum, tum vulgarium protonumerum $f$ æqualem poni unitati. Tabulæ logarithmorum vulgarium conſtructæ ſunt, & abſolutæ a multis, præſertim a Briggio, & Ulacquio, qui uſi ſunt ſubſidiis, quæ maxime diſtant a noſtris ſeriebus. Tabulæ vero logarithmorum hyperbolicorum deſiderantur. Optandum eſſet, ut aliquis hunc laborem ſuſciperet, & eas præſertim, quæ deſunt, tabulas per computum perficeret, atque communi utilitati in lucem emitteret.

23. Hæc omnia deſumpſimus ex Vincentii Riccati T. 1. Opuſ. opuſculo tertio, quod inſcribitur *De multiplici logarithmorum ſyſtemate*. Verum idem Riccatus in ejuſdem tomi opuſculo quarto parte ſecunda contemplatus eſt aliud ſyſtema logarithmorum, quos vocavit analogos ex analogia, quam habent cum arcubus

cubus circularibus. Quum deinceps horum logarithmorum uſus ſit frequentiſſimus, neceſſe eſt, ut hoc quoque ſyſtema accurate illuſtremus. Sit hyperbola æquilatera AF, (*Fig.* 4.) cujus ſemiaxis $CA = r$, quem ſinum totum appellare ſolemus. Agatur ejus aſymptotum CH, in quod cadat ex vertice A ordinata AK, & quælibet alia FH, & jungatur CF. Primum demonſtrare oportet ſectorem hyperbolicum CAF æqualem eſſe ſpatio AKHF. Notetur punctum O, ubi CF ſecat AK. Ex natura hyperbolæ æqualia ſunt rectangula AKC, FHC; ergo æqualia erunt triangula AKC, FHC, quæ ſunt rectangulorum dimidia. Dematur commune triangulum COK, remanebit triangulum COA æquale quadrilatero OKHF. Addatur triangulum AOF, & fiet ſector ACF æqualis ſpatio AKHF. Q. E. D.

24. His præmiſſis quemadmodum poſito protonumero CK orietur ſyſtema logarithmicum ſi ſpatium AKHF dividatur per quamlibet conſtantem; ita eodem modo orietur, ſi per conſtantem dividatur ſector CAF. Notiſſimum eſt, ſectorem circularem diviſum per dimidium radii, ſeu ſinus totius exhibere arcum circularem. Quare ſi dividatur ſector hyperbolicus per dimidium ſemiaxis, ſeu ſinus totius, obtinebimus logarithmos, qui arctiſſimam analogiam habebunt cum arcubus circularibus. Hos vocamus logarithmos analogos, quia determinantur per ſectores hyperbolicos eo pacto, quo arcus determinantur per ſectores circulares. De his etiam in tomo primo ſæpe verba fecimus, eorumque ſinus, & coſinus conſideravimus; ſed in præſentia conſtat, quomodo ex ipſius hyperbolæ natura naſcantur, quod promiſimus nos ſuo loco eſſe demonſtraturos. Ex his cognoſcis protonumerum ſyſtematis logarithmorum analogorum eſſe $f = \frac{r}{\sqrt{2}}$; quantitatem, quæ multiplicata per logarithmos dat ſpatium hyperbolicum AKHF, ſeu ſectorem ACF eſſe $b = \frac{r}{2}$, rectangulum hyperbolæ $aa = \frac{rr}{2}$; ergo ſubtangens ſyſtematis $= c = \frac{aa}{b} = r$. Quare logarithmi hyperbolici, & analogi in hoc conveniunt, quod habeant eamdem ſubtangentem $= r$; at ſyſtematis hyperbolici protonumerus æqualis eſt ſubtangenti $= r$, analogi vero protonumerus $= \frac{r}{\sqrt{2}}$.

25. Si in ſyſtemate logarithmorum analogorum quæras $lr$, per ſeries ſupra traditas facile invenies $lr = 2r.\ \frac{\sqrt{2}-1}{\sqrt{2}+1} + \frac{\overline{\sqrt{2}-1}^{3}}{3.\overline{\sqrt{2}+1}^{3}} + \frac{\overline{\sqrt{2}-1}^{5}}{5.\overline{\sqrt{2}+1}^{5}}$ &c., quam quantitatem ſi demas ex omnibus logarithmis analogis, proveniet ſyſtema logarithmorum hyperbolicorum; contra ſi addas omnibus logarithmis hyperbolicis; orietur ſyſtema analogorum. Nihil aliud reſtat, niſi ut ſyſtematis analogi inveniamus baſim logarithmicam, hoc eſt numerum cujus logarithmus æqualis ſit protonumero; In expoſitis ſeriebus facto tum protonumero $f$, tum $g = \frac{r}{2}$ invenies

$$x = \frac{r}{\sqrt{2}} \cdot \overline{\frac{1}{\sqrt{2}} + \frac{1}{2.2} + \frac{1}{2\sqrt{2.2.3}} + \frac{1}{2.2.2.3.4} + \frac{1}{2.2.\sqrt{2.2.3.4.5}}} \text{ \&c.}$$

ergo

$$\frac{r}{\sqrt{2}} + x = \frac{r}{\sqrt{2}} \cdot \overline{1 + \frac{1}{\sqrt{2}} + \frac{1}{2.2} + \frac{1}{2.\sqrt{2.2.3}} + \frac{1}{2.2.2.3.4} + \frac{1}{2.2.\sqrt{2.2.3.4.5}}} \text{ \&c.}$$

quæ quantitas eſt baſis logarithmica ſyſtematis analogi; ſeu numerus cujus logarithmus æquat protonumerum $\frac{r}{\sqrt{2}}$.

26. Poſtquam ſatis diximus de logarithmis numerorum poſitivorum, conſiderandum eſt diligenter, quos logarithmos habeant numeri negativi, ne in calculo logarithmico aliqua ambiguitas relinquatur. Rectæ H h, K k (*Fig.* 5.) ſe invicem ſecent ad angulos rectos, & deſcribantur hyperbolæ oppoſitæ Y E R, y e r. Sumatur protonumerus C D, & ducatur ordinata D E; tum abſcindatur C d = C D, & ordinetur d e, quæ æquabit D E. Spatia hyperbolica E D M R, quæ initium ducunt a linea D E, quæq; poſita ſunt ad partes K, accipiuntur poſitiva; ergo logiſticæ ordinatæ M X, ſeu logarithmi numerorum C M, qui majores ſunt protonumero C D, poſitivi ſunt. Spatia hyperbolica E D N Y ſita ad plagam oppoſitam, & jacentia ſuper aſymptotum C K negativa erunt; ergo ordinatæ N S, ſive logarithmi numerorum C N < C D negativi ſunt. Si progrediamur ultra punctum C, & accipiamus numeros C n negativos, liquet ſpatium hyperbolicum h C n y ſitum infra aſymptotum K k fieri poſitivum; ſed ſpatium E D C H eſt negativum; ergo logarithmicæ ordinata, ſeu logarithmus fiet æqualis differentiæ inter ſpatia H C D E, h c n y. Si C n ſit minor C d, conſtat negativum prævalere poſitivo; quare n s ordinata logyſticæ, ſeu logarithmus numeri C n erit negativus. Si numerus ſit C d, ſpatium negativum æquale erit poſitivo, adeoque logarithmus numeri C d = o. Contra ſi numerus ſit C m major D C, ſpatium poſitivum prævalebit negativo; ergo m x ordinata logyſticæ, ſeu logarithmus numeri C m erit poſitivus. Abſcinde C n = C N, & C m = C M. Perſpicuum eſt, differentiam ſpatiorum H C D E, h C n y æqualem eſſe ſpatio E D N Y; ergo logarithmi numerorum C N, C n æquales ſunt. Idem dic de logarithmis numerorum C m, C n, quia differentia ſpatiorum h C m r, H C D E æquat ſpatium E D M R. Ex his patet, numeros negativos eoſdem habere logarithmos, quos numeri poſitivi. Quare logarithmica prædita eſt duobus ramis prorſus æqualibus H D X, H d x. Uterque accedit ad aſymptotum ad partes H, ad partes oppoſitas ab eodem in infinitum recedit.

27. Nunc dicendum eſt de integratione earum formularum, in cujus numeratore continetur differentia denominatoris. Ordiar a facilioribus. Sit primo integranda formula $\frac{c\,dx}{x}$; ejus integrale expreſſum per logarithmos $lx$, qui logarithmus ſumitur in ſyſtemate ſubtangentis $= c$. Protonumerus nondum eſt determinatus; determinari autem poteſt, ut ſupleat additionem conſtantis, quæ fieri poteſt in integratione. Contra ſi differentiare oporteat $lx$ ſumptum in ſyſtemate ſubtangentis $= c$, ejus differentia erit $= \frac{c\,dx}{x}$. Quoniam docuimus numero ſuperiore cuilibet logarithmo duos numeros convenire æquales quidem,

ſed

ſed unum poſitivum, alterum negativum, idcirco $\frac{c\,dx}{x}$ habet etiam pro integrale $l-x$. Reapſe hujus differentia $= \frac{c.-dx}{-x} = \frac{c\,dx}{x}$. Ex his collige diverſas eſſe quantitates $l-x$, $-lx$; prima habet pro differentia $\frac{c\,dx}{x}$, ſecunda $\frac{-c\,dx}{x}$, exiſtente ſubtangente ſyſtematis $=c$. Novi $\frac{-c\,dx}{x}$ habere pro integrali $lx$ in ſyſtemate ſubtangentis $=-c$. Verum hoc ſyſtema diverſum eſt a ſuperiore per hoc, quod habet logarithmos majores protonumero negativos, minores poſitivos. Idem dicas volo de formula $\frac{c\,dx}{x\pm a}$, cujus integrale $=l\pm\overline{(x\pm a)}$ in ſyſtemate ſubtangentis $=c$. Contra integrale $\frac{c\,dx}{a-x}$ eſt $-l\pm\overline{(a-x}$, & $\frac{-c\,dx}{a-x}$ habet integrale $=l\pm\overline{(a-x}$ in eodem ſyſtemate ſubtangentis $=c$. Præpoſui nunc numeris ſignum $\pm$, quia vel accipiantur poſitive, vel negative, idem eſt logarithmus. Verum deinceps, ut confuſio vitetur, integrationem perficiens ſignum $\pm$ omittam; attamen moneo, memoria retinendum eſſe, numerum accipi poſſe & poſitive, & negative.

28. In ſuperiorum formularum integratione ſubtangens ſyſtematis ſumi poterat ex libito, & ſtatim apparebat, quænam ſubtangens foret eligenda, ut formula evaderet ſimpliciſſima. Verum docere neceſſe eſt, quomodo facienda ſit integratio in ſyſtemate logarithmorum ſubtangentis datæ. Sit integranda formula $\frac{a\,dx}{x}$ in ſyſtemate ſubtangentis $=c$, quæcumque ſit quantitas $a$. In hunc modum diſtribuenda eſt formula $\frac{a}{c}\cdot\frac{c\,dx}{x}$, tum integratio provenit $\frac{a}{c}\,lx$. Eodem modo de formulis $\frac{a\,dx}{x\pm b}$, $\frac{a\,dx}{b-x}$, quæ habent integralia $\frac{a}{c}\,l\overline{x\pm b}$, $\frac{-a}{c}\,l\overline{b-x}$, ſumptis logarithmis in ſyſtemate ſubtangentis $=c$. Quandoquidem ut ſupra docuimus, generatim $m\,lx = l\frac{x^m}{f^{m-1}}$, propterea formulæ ſuperiores integrari poſſunt in hunc modum

$$l\frac{x^{\frac{a}{c}}}{f^{\frac{a}{c}-1}},\; l\frac{\overline{x+b}^{\frac{a}{c}}}{f^{\frac{a}{c}-1}},\; l\frac{\overline{b-x}^{\frac{-a}{c}}}{f^{\frac{-a}{c}-1}} = l\frac{f^{\frac{a}{c}+1}}{\overline{b-x}^{\frac{a}{c}}}$$

exiſtente $f$ protonumero. Ex quibus ſequitur formulæ $\frac{-c\,dx}{x}$ integrale $= l\frac{ff}{x}$, poſito protonumero $f$, ſubtangente $c$.

29. Satis de quantitatibus, in quibus incognita in diviſore poteſtatem obtinet linearem. In aliis integrandis, ubi poteſtas major eſt lineari, hanc regulam ſequere: accipe logarithmum denominatoris ita multiplicati aut diviſi per protonumerum, ut quantitas ſubſequens ſignum $l$ ſit linearis: ita $\frac{2cx\,dx}{aa+xx}$ habebit integrale in ſyſtemate ſubtangentis $=c$ acceptum $=l\frac{aa+xx}{f}$, exiſtente



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te $f$ protonumero. Reapſe ſi hujus differentiam ſumas, invenies $\frac{2 c x dx}{aa+xx}$. Idem integrale ita etiam poteſt exprimi $2l\sqrt{aa+xx}$. Quod cum ſuperioribus convenit; nam quum ſit $f : \sqrt{aa+xx} :: \sqrt{aa+xx} : \frac{aa+xx}{f}$, patet $l\frac{aa+xx}{f}$ eſſe duplum $l\sqrt{aa+xx}$. Similiter $\frac{c x dx}{aa-xx}$ habebit integrale $\frac{-1}{2} l \frac{aa-xx}{f}$; ſeu $-l\sqrt{aa-xx}$, ſeu $l\frac{ff}{\sqrt{aa-xx}}$. Eadem regula tenenda eſt, quum plures logarithmi in eodem ſyſtemate ſumpti ad unum redigendi ſunt. Ita $lx+ly=l\frac{xy}{f}$, $lx-ly=l\frac{fx}{y}$, $lx+ly+la=l\frac{axy}{f^2}$; atque ita de aliis. Quod ſi logarithmi habeant coefficientes præfixos, coefficientes intra ſignum $l$ transferantur & convertantur in exponentes, tum logarithmi omnes conjungantur, ut traditum eſt. Ita $5lx-3ly+2la=l\frac{x^5}{f^4}+l\frac{f^4}{y^3}+l\frac{a^2}{f}+l\frac{a^2x^5}{f^3y^3}$.

30. Hæc utilitatem afferunt plurimam, quotieſcumque tranſitus faciendus eſt a logarithmis ad numeros, a numeris ad logarithmos. Tranſitus hic fundatur in hoc axiomate: logarithmi æquales habent æquales numeros, & numeri æquales præditi ſunt æqualibus logarithmis: quod axioma clariſſime deſcendit ex geneſi logarithmorum, dummodo logarithmi in eodem ſyſtemate fuerint accepti. Rem facilem uno, aut altero exemplo declaremus. Sit æquatio $2lx-ly=\frac{1}{2}l\sqrt{xx+yy}+la$. Logarithmi in utraque æquationis parte ſiti in unum redigantur, ut fiat $l\frac{x^2}{y}=l\frac{a.\overline{xx+yy}^{\frac{1}{4}}}{f^{\frac{1}{2}}}$; ergo facto tranſitu a logarithmis ad numeros erit $\frac{x^2}{y}=\frac{a.\overline{xx+yy}^{\frac{1}{4}}}{f^{\frac{1}{2}}}$, ex qua provenit $f^2x^8=a^4y^4.\overline{xx+yy}$. Vice verſa ſi ex hac æquatione ad logarithmos tranſeundum ſit, fiat diviſio per $f^9$, ut redigatur ad linearem, & proveniet $\frac{x^8}{f^7}=\frac{a^4y^4.\overline{x^2+y^2}}{f^9}$; ergo $l\frac{x^8}{f^7}=l\frac{a^4y^4.\overline{x^2+y^2}}{f^9}$ ſive $8lx=4ly+l\frac{xx+yy}{f}+4la$, & dividendo per 4, $2lx=ly+\frac{1}{4}l\frac{xx+yy}{f}+la$; ſed $\frac{1}{4}l\frac{xx+yy}{f}=\frac{1}{2}l\sqrt{xx+yy}$; ergo $2lx=ly+\frac{1}{2}l\sqrt{xx+yy}+la$;

quæ

quæ cohæret cum ſuperiore. Similiter data ſit inter logarithmos æquatio $l\overline{x+y} - l\overline{x-y} = 2lx + 2ly - la$, ſive $l\overline{x+y} + l\frac{ff}{x-y} = l\frac{x^2}{f} + l\frac{y^2}{f} + l\frac{f}{a}$, redactiſque in unum logarithmis $l\frac{f.\overline{x+y}}{x-y} = l\frac{x^2 y^2}{f^2 a}$; ergo facto tranſitu ad numeros $f.\frac{x+y}{x-y} = \frac{x^2 y^2}{f^2 a}$, ſeu $f^3 a.\overline{x+y} = x^2 y^2.\overline{x-y}$. Si ab æquatione ad logarithmos velis tranſire, divide per $f^4$, ut ſit $\frac{a.\overline{x+y}}{f} = \frac{x^2 y^2.\overline{x-y}}{f^4}$; ergo $l\frac{a.\overline{x+y}}{f} = l\frac{x^2 y^2.\overline{x-y}}{f^4}$, ſive $la + l\overline{x+y} = 2lx + 2ly + l\overline{x-y}$, quæ convenit cum ſuperiore. In his omnibus ſuppoſui $f$ eſſe protonumerum, cujus logarithmus $=o$. Si logarithmi omnes rejiciantur ad unam æquationis partem, ut fiant $=o$, tunc æquales faciendi ſunt $lf = o$, tum tranſeundum ad numeros. Sit $2lx - ly = o$, fac $2lx + ly = lf$, reliquiſque ut ſupra peractis obtinebis $\frac{x^2}{y} = f$.

## *CAPUT SEPTIMUM.*

### De quadratura circuli, & hyperbolæ æquilateræ ad axes relatæ.

1. SAtis eſt, ſpectare quadraturam ſolius circuli & hyperbolæ æquilateræ, ut formulas inveniamus, quæ conſtrui poſſunt quadratis ellypſi, & hyperbola, quia ellypſium, & ceterarum hyperbolarum quadratura dependet a quadraturà circuli, & hyperbolæ æquilateræ: quod ita demonſtro. Sit ellypſis, & hyperbola AF, (*Fig.* 1, 2.) cujus centrum C, unus ſemiaxis CA, alter CB. Eodem centro C, & ſemiaxibus æqualibus CA, CD deſcribatur circulus, & hyperbola æquilatera AG; tum agantur duæ ordinatæ infinitè proximæ EFG, efg, & ducantur parallelæ abſciſſis Fm, Gn. Ex curvarum proprietate EF: EG:: CB: CD; ſed rectang. EFme: EGne:: EF: EG; ergo EFme: EGne:: CB: CD. Quæ proportio quum conſtans ſit, & valeat de omnibus rectangulis infiniteſimæ latitudinis, quæ inſcribuntur in curvis, conſequitur eorum ſummam eſſe in eadem ratione CB: CD; atqui ſumma rectangulorum adæquant areas, quæ ad eamdem ordinatam EFG terminantur; ergo area AEF: AEG; item in circulo BCEF: DCEG:: CB: CD; igitur datà area circuli, & hyperbolæ æquilateræ, datur area ellypſis, & cujuſcumque hyperbolæ.

2. In circulo AGD (*Fig.* 1.) vocata $AE = x$, radio $CA = r$, erit ordi-

dinata $EG = \sqrt{2x - xx}$; ergo elementum areæ AGE, ſeu rectangulum GEe $= dx\sqrt{2rx - xx}$. Si vero vocetur $CE = x$, & $Ee = -dx$, erit $EG = \sqrt{rr - xx}$, & elementum areæ $AGE = -dx\sqrt{rr - xx}$, & elementum areæ $DCGE = dx\sqrt{rr - xx}$. Si vocetur $GE = y$, erit $CE = \sqrt{rr - yy}$, & ejus differentia $= \frac{-ydy}{\sqrt{rr - yy}}$, quæ ſi multiplicetur per $y$ dabit differentiam areæ $DCEG = \frac{-y^2 dy}{\sqrt{rr - yy}}$, ex qua colligitur differentiam $AEG = \frac{y^2 dy}{\sqrt{rr - yy}}$. Has autem formulas ex ſuperioribus ita licet deducere.

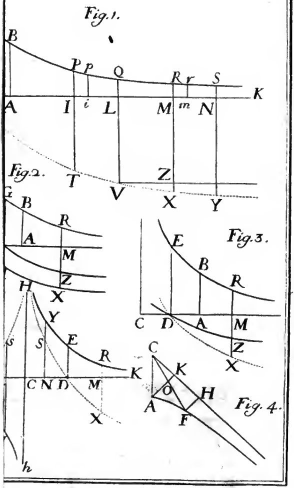
$$Dr\sqrt{rr - yy} = dy\sqrt{rr - yy} - \frac{y^2 dy}{\sqrt{rr - yy}}; \text{ ergo}$$

$$y\sqrt{rr - yy} + S - dy\sqrt{rr - yy} = S\frac{-y^2 dy}{\sqrt{rr - yy}};$$ atqui producta Gn in H rectangulum $GHCE = y\sqrt{rr - yy}$, & area $GHD = S - dy\sqrt{rr - yy}$; ergo

$S\frac{-y^2 dy}{\sqrt{rr - yy}} = GHCE + GHD = GECD$. Spectatis formulis, quæ integrantur per aream circularem, inveniamus illas, quæ poſtulant areas hyperbolicas. Sit hyperbola æquilatera AG; (*Fig.* 2.) vocetur $CA = r$, $AE = x$, erit $EG = \sqrt{2rx + xx}$; ergo elementum areæ AEG, ſive rectangulum GEeg $= dx\sqrt{2rx + xx}$. Quod ſi vocetur $CE = x$, erit $EG = \sqrt{xx - rr}$; ergo elementum areæ $AEG = dx\sqrt{xx - rr}$. Demum ordinata GH, vocetur $CH = x$, erit $HG = \sqrt{xx + rr}$; ergo elementum areæ $CHGA = dx\sqrt{xx + rr}$ Quod ſi velimus exprimere aream AGE per rectam EG; hæc fiat $= y$; ergo $CE = \sqrt{yy + rr}$, & ejus differentia $\frac{ydy}{\sqrt{yy + rr}}$; ergo differentia areæ $AEG = \frac{y^2 dy}{\sqrt{yy + rr}}$. Simili modo ſi optas habere aream CAGH per ordinatam HG, quæ vocetur $y$, erit $CH = \sqrt{yy - rr}$, & ejus differentia $= \frac{ydy}{\sqrt{y^2 - rr}}$; ergo elementum areæ $CAGH = \frac{y^2 dy}{\sqrt{y^2 - rr}}$. Hæ duæ ultimæ formulæ ex ſuperioribus deduci poſſunt, prout ultima formula circularis.

3. Quamquam hujuſmodi formulæ memoria retinendæ ſunt, ut quicumque ad eas pervenerit, cognoſcat, ſe circuli, aut hyperbolæ quadratura indigere; tamen ſatius erit, eas inquirere, quæ exibent ſectores circulares & hyperbolicos, qui diviſi per dimidium ſemiaxis, ſeu ſinus totius præbent arcus circulares, & logarithmos analogos. Sit quadrans circuli ADB, (*Fig.* 3.) cujus centrum C, agatur radius quicumque CD; quæritur formula exprimens ſectorem ACD. Agatur ordinata DE. Si vocato radio $= r$ exprimendus ſit ſector per

ſinum




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ſinum verſum $AE = x$, erit $CE = r - x$, $DE = \sqrt{2rx - xx}$; ergo triangulum $CED = \frac{\overline{r-x}\sqrt{2rx-xx}}{2}$, & ſemiſegmentum $ADE = S\,dx\sqrt{2rx-xx}$; ergo $CAD = \frac{\overline{r-x}\sqrt{2rx-xx}}{2} + S\,dx\sqrt{2rx-xx}$ & differentia ejuſdem ſectoris $= \frac{-dx\sqrt{2rx-xx}}{2} + \frac{\overline{r-x}^2 . dx}{2\sqrt{2rx-xx}} + dx\sqrt{2rx-xx} =$
$\frac{dx\sqrt{2rx-xx}}{2} + \frac{\overline{r^2 - 2rx + xx}. dx}{2\sqrt{2rx-xx}} = \frac{\overline{2rx - xx + rr - 2rx + xx}. dx}{2\sqrt{2rx-xx}} =$
$\frac{r^2 dx}{2\sqrt{2rx-xx}}$; ergo ſector $CAD = S\frac{r^2 dx}{2\sqrt{2rx-xx}}$, quæ diviſa per $\frac{r}{2}$ exhibebit arcum $AD = S\frac{r\,dx}{\sqrt{2rx-xx}}$. Si vero vocemus coſinum $CE = x$ inveniemus ſectorem eumdem $CAD = \frac{x\sqrt{rr-xx}}{2} + S - dx\sqrt{rr-xx} =$
$S\frac{-r^2 dx}{2\sqrt{rr-xx}}$; ergo arcus $AD = S\frac{-r\,dx}{\sqrt{rr-xx}}$. Si vero vocetur ſinus $CE = y$ inveniemus eumdem ſectorem $CAD = \frac{y\sqrt{rr-yy}}{2} + S\frac{y^2 dy}{\sqrt{rr-yy}} =$
$S\frac{r^2 dy}{2\sqrt{rr-yy}}$, & arcum $AF = S\frac{r\,dy}{\sqrt{rr-yy}}$. Si ſectorem, & arcum habere cupis per tangentem $AF = t$, vocatis $CE = x$, $DE = y$ inſtitue calculum $x : y :: r : t$, & quadrando $x^2 : y^2 :: r^2 : t^2$, & componendo
$x^2 + y^2 = r^2 : y^2 :: r^2 + t^2 : t^2$; ergo $y^2 = \frac{r^2 t^2}{rr + tt}$; ergo
$r^2 - y^2 = r^2 - \frac{r^2 t^2}{r^2 + t^2} = \frac{r^4}{r^2 + t^2}$, & $\sqrt{rr-yy} = \frac{r^2}{\sqrt{rr+tt}}$. Præterea
$y = \frac{rt}{\sqrt{rr+tt}}$, & differentiis acceptis $dy = \frac{r\,dt}{\sqrt{rr+tt}} - \frac{rt^2 dt}{\overline{rr+tt}^{\frac{3}{2}}} = \frac{r^3 dt}{\overline{rr+tt}^{\frac{3}{2}}}$.
Quare $\frac{r^2 dy}{2\sqrt{rr-yy}} = \frac{r^3 dt}{2.\overline{rr+tt}}$; ſed ſector $CDA = S\frac{r^2 dy}{2\sqrt{rr-yy}}$; ergo idem ſector $= S\frac{r^3 dt}{2.\overline{rr+tt}}$, ex quo arcus circularis $AD = S\frac{r^2 dt}{rr+tt}$, quæ formu-

mula expers eſt radicalium. Quoniam autem ſecans $= s = \sqrt{rr+tt}$, erit $\sqrt{ss-rr} = t$; ergo $\frac{s\,ds}{\sqrt{ss-rr}} = dt$; ergo $\frac{dt}{rr+tt} = \frac{ds}{s\sqrt{ss-rr}}$; quare ſector CAD $= S\frac{r^3 ds}{2s\sqrt{ss-rr}}$, & arcus AD $= S\frac{r^2 ds}{s\sqrt{ss-rr}}$. Si quæras eaſdem quantitates per cotangentem, & coſecantem, eaſdem formulas invenies, ſed affectas ſigno $-$.

4. Hanc methodum ſequutus ſum, ut pateſiat maxima analogia, quæ eſt inter circulum, & hyperbolam. Ceterum multo elegantiori methodo determinantur in circulo arcus, & ſectores. Agatur $ed$ infinite proxima DE; Dm ſit parallela CA; agatur Cdf, & centro C intervallo CF deſcribatur minimus arcus Fn. Quoniam recti ſunt anguli EDm, CDd, ablato communi CDm remanebit CDE = dDm: ergo erunt ſimilia triangula CDE, dDm: igitur Dd : Dm :: CD : DE. Quare ſi vocetur AE $= x$, erit Dm $= dx$, & DE $= \sqrt{2rx-xx}$; ergo Dd $: dx :: r : \sqrt{2rx-xx}$, & Dd $= \frac{r\,dx}{\sqrt{2rx-xx}}$, & AD $= S\frac{r\,dx}{\sqrt{2rx-xx}}$. Si vero vocetur CE $= x$, erit Dm $= -dx$, & DE $= \sqrt{rr-xx}$; ergo Dd $: -dx :: r : \sqrt{rr-xx}$, & Dd $= \frac{-r\,dx}{\sqrt{rr-xx}}$, & AD $= S\frac{-r\,dx}{\sqrt{rr-xx}}$. Simili modo eſt Dd : dm :: CD : CE; quare vocato DE $= y$ erit dm $= dy$, & CE $= \sqrt{rr-yy}$; ergo Dd $: dy :: r : \sqrt{rr-yy}$, & Dd $= \frac{r\,dy}{\sqrt{rr-yy}}$, & AD $= S\frac{r\,dy}{\sqrt{rr-yy}}$. Præterea eſt

Dd : Fn :: CD = CA : CF

Fn : Ft :: CA : CF; ergo

Dd : Ft :: $CA^2 : CF^2$. Vocata AF $= t$, erit CF $= \sqrt{r^2+t^2}$; ergo

Dd $: dt :: r^2 : r^2+t^2$, Dd $= \frac{r^2 dt}{rr+tt}$. Præterea Fn : fn :: CA : FA, ergo Dd : fn :: $CA^2$ : FA.CF. Vocata CF $= s$, erit fn $= ds$, & FA $= \sqrt{ss-rr}$; ergo Dd $: ds :: r^2 : s\sqrt{ss-aa}$, & Dd $= \frac{r^2 ds}{s\sqrt{ss-aa}}$, & AD $= S\frac{r^2 ds}{s\sqrt{ss-aa}}$. Quæ formulæ ſi multiplicentur per $\frac{r}{2}$ obtinebimus ſectorem CAD.

5. Formulæ hyperbolicæ circularibus prorſus ſunt ſimiles. Sit hyperbola æquilatera AD, (*Fig.* 4.) cujus centrum C, ſemiaxis, ſeu ſinus totus CA $= r$. Juncta CD quæritur formula exprimens ſectorem CAD. Sit ſinus verſus AE $= x$, erit CE $= r+x$, DE $= \sqrt{2rx+xx}$; ergo triangulum CED $= \frac{\overline{r+x}.\sqrt{2rx+xx}}{2}$, & ſemiſegmentum ADE $= S\,dx\sqrt{2rx+xx}$; ergo ſector

CAD

$CAD = CED - ADE = \frac{\overline{r+x}.\sqrt{2rx+xx}}{2} - S\,dx\sqrt{2rx+xx} =$ $S\,\frac{r^2dx}{2\sqrt{2rx+xx}}$; atqui ſector diviſus per $\frac{1}{2}r$ præbet logarithmum analogum; ergo logarithmus analogus expreſſus per ſinum verſum $= S\,\frac{rdx}{\sqrt{2rx+xx}}$. Si coſinus $CE = x$, erit $DE = \sqrt{xx-rr}$, & $AED = S\,dx\sqrt{xx-rr}$. Ergo ſector $CAD = \frac{x\sqrt{xx-rr}}{2} - S\,dx\sqrt{xx-rr} = S\,\frac{r^2dx}{2\sqrt{xx-rr}}$, & logarithmus analogus expreſſus per coſinum $= S\,\frac{rdx}{\sqrt{xx-rr}}$. Fiat ſinus $DE = y$, erit $CE = \sqrt{rr+yy}$, & ſemiſegmentum $ADE = S\,\frac{y^2dy}{\sqrt{rr+yy}}$; ergo ſector $CAD = \frac{y\sqrt{rr+yy}}{2} - S\,\frac{y^2dy}{\sqrt{rr+yy}} = S\,\frac{r^2dy}{2\sqrt{rr+yy}}$, & logarithmus analogus expreſſus per ſinum $= S\,\frac{rdy}{\sqrt{rr+yy}}$. Si ex puncto A agatur AF parallela ſecundo axi CB, vocari poteſt AF tangens hyperbolica. Exprimendus eſt ſector CAD per tangentem. Vocata tangente $AF = t$, coſinu $CE = x$, ſinu $ED = y$ valebit proportio $x:y::r:t$; ergo $x^2:y^2::r^2:t^2$, & dividendo $x^2-y^2 = r^2:y^2::r^2-t^2:t^2$; ergo $y^2 = \frac{r^2t^2}{r^2-t^2}$, & $r^2+y^2 = r^2 + \frac{r^2t^2}{r^2-t^2} = \frac{r^4}{r^2-t^2}$, & $\sqrt{rr+yy} = \frac{r^2}{\sqrt{rr-tt}}$. Præterea $y = \frac{rt}{\sqrt{rr-tt}}$, & facta differentiatione $dy = \frac{rdt}{\sqrt{rr-tt}} + \frac{rt^2dt}{\overline{rr-tt}^{\frac{3}{2}}} = \frac{r^3dt}{\overline{rr-tt}^{\frac{3}{2}}}$. Quare $S\,\frac{r^2dy}{2\sqrt{rr+yy}} =$ $S\,\frac{r^3dt}{2.rr+tt}$ = ſectori CAD, & logarithmus analogus datus per tangentem $= S\,\frac{r^2dt}{rr-tt}$. Ex puncto B extremo ſecundi ſemiaxis agatur BG parallela primo axi, in quam incidat CD producta, ſi opus eſt, in G, BG per analogiam ad circulum vocari poteſt cotangens, quæ vocetur $= z$. Ex triangulorum ſimilitudine eſt $t:r::r:z$; ergo $t = \frac{rr}{z}$, & $dt = \frac{-r^2dz}{zz}$, & $rr-tt = rr - \frac{r^4}{zz} = \frac{rr}{zz}.\overline{zz-rr}$; igitur $\frac{dt}{rr-tt} = \frac{-dz}{zz-rr}$. Quare ſector

ctor $CAD = S\frac{-r^{3}dz}{2 \cdot z\sqrt{zz-rr}}$, & logarithmus analogus $= S\frac{-r^{2}dz}{z\sqrt{zz-rr}}$.

6. Poſtquam formulas exprimentes arcus circulares, & logarithmos analogos expoſuimus, opportunum videtur, eaſdem quantitates per convergentes ſeries exhibere. Formula differentialis arcus circularis expreſſi per ſinum rectum eſt hujuſmodi $\frac{rdy}{\sqrt{rr-yy}}$. Evolvo in ſeriem formulam $\frac{r}{\sqrt{rr-yy}}$, ſive $r \cdot \overline{rr-yy}^{-\frac{1}{2}}$, & invenio

$$1 + \frac{y^{2}}{2r^{2}} + \frac{1 \cdot 3 \cdot y^{4}}{2^{2} \cdot 1 \cdot 2 \cdot r^{4}} + \frac{1 \cdot 3 \cdot 5 \cdot y^{6}}{2^{3} \cdot 1 \cdot 2 \cdot 3 \cdot r^{0}} + \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot y^{8}}{2^{4} \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot r^{8}} + \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdot y^{10}}{2^{5} \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot r^{10}} \text{ \&c.}$$

igitur $\frac{rdy}{\sqrt{rr-yy}}$, ſeu differentia arcus circularis erit $=$

$$dy + \frac{y^{2}dy}{2r^{2}} + \frac{1 \cdot 3 \cdot y^{4}dy}{2^{2} \cdot 1 \cdot 2 \cdot r^{4}} + \frac{1 \cdot 3 \cdot 5 \cdot y^{6}dy}{2^{3} \cdot 1 \cdot 2 \cdot 3 r^{6}} + \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot y^{8}dy}{2^{4} \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot r^{8}} + \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdot y^{10}dy}{2^{5} \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot r^{10}} \text{ \&c.}$$

igitur facta integratione ſine additione conſtantis, quia evaneſcente ſinu $y$ debet avaneſcere arcus, fiet $S\frac{rdy}{\sqrt{rr-yy}}$, ſeu arcus circularis $=$

$$y + \frac{y^{3}}{2 \cdot 3 \cdot r^{2}} + \frac{1 \cdot 3 \cdot y^{5}}{2^{2} \cdot 1 \cdot 2 \cdot 5 \cdot r^{4}} + \frac{1 \cdot 3 \cdot 5 \cdot y^{7}}{2^{3} \cdot 1 \cdot 2 \cdot 3 \cdot 7 \cdot r^{6}} + \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot y^{9}}{2^{4} \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 9 \cdot r^{8}} + \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdot y^{11}}{2^{5} \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 11 \cdot r^{10}} \text{ \&c.}$$

ſeries hæc maxime convergit, & utilis eſt inveniendis arcubus quadrante minoribus. Si ponas $y = r$, habebis quadrantem circumferentiæ $=$

$$r \cdot \overline{1 + \frac{1}{2 \cdot 3} + \frac{1 \cdot 3}{2^{2} \cdot 1 \cdot 2 \cdot 5} + \frac{1 \cdot 3 \cdot 5}{2^{3} \cdot 1 \cdot 2 \cdot 3 \cdot 7} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2^{4} \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 9} + \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9}{2^{5} \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 11} \text{ \&c.}}$$

Si vero ponas $y = \frac{r}{\sqrt{2}}$, obtinebis dimidium quadrantis $=$

$$\frac{r}{\sqrt{2}} \cdot \overline{1 + \frac{1}{2^{2} \cdot 3} + \frac{1 \cdot 3}{2^{4} \cdot 1 \cdot 2 \cdot 5} + \frac{1 \cdot 3 \cdot 5}{2^{6} \cdot 1 \cdot 2 \cdot 3 \cdot 7} + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2^{8} \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 9} + \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9}{2^{10} \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 11} \text{ \&c.}}$$

Si arcum quæras per coſinum, cujus expreſſio eſt hujuſmodi $\frac{-rdx}{\sqrt{rr-xx}}$, exiſtente coſinu $= x$, quæ eadem eſt, ac expreſſio per ſinum ſigno mutato, eadem peragendo, quæ ſupra, invenies $\frac{-rdx}{\sqrt{rr-xx}}$, ſeu differentiam arcus $=$

$$-dx - \frac{x^{2}dx}{2r^{2}} - \frac{1 \cdot 3 \cdot x^{4}dx}{2^{2} \cdot 1 \cdot 2 \cdot r^{4}} - \frac{1 \cdot 3 \cdot 5 \cdot x^{6}dx}{2^{3} \cdot 1 \cdot 2 \cdot 3 \cdot r^{6}} - \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot x^{8}dx}{2^{4} \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot r^{8}} \text{ \&c.}$$

Hæc ita integranda eſt, ut facta $x = r$, arcus nulleſcat. Quapropter proveniet arcus circularis

$+$

$$r + \frac{r}{2 \cdot 3} + \frac{1 \cdot 3 \cdot r}{2^2 \cdot 1 \cdot 2 \cdot 5} + \frac{1 \cdot 3 \cdot 5 \cdot r}{2^3 \cdot 1 \cdot 2 \cdot 3 \cdot 7} + \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot r}{2^4 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 9} \ \&c.$$

$$-x - \frac{x^3}{2 \cdot 3 \cdot r^2} - \frac{1 \cdot 3 \cdot x^5}{2^2 \cdot 1 \cdot 2 \cdot 5 \cdot r^4} - \frac{1 \cdot 3 \cdot 5 \cdot x^7}{2^3 \cdot 1 \cdot 2 \cdot 3 \cdot 7 \cdot r^6} - \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot x^9}{2^4 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 9} \ \&c.$$

7. Formula logarithmi analogi dati per ſinum $\frac{r\,dy}{\sqrt{rr+yy}}$ eadem eſt ac formula arcus, dummodo ſignum mutes quadrato $yy$. Itaque eodem modo tractata eamdem ſeriem producet, ſed cum ſignis alternantibus, ut logarithmus analogus ſit =

$$y - \frac{y^3}{2 \cdot 3 \cdot r^2} + \frac{1 \cdot 3 \cdot y^5}{2^2 \cdot 1 \cdot 2 \cdot 5 \cdot r^4} - \frac{1 \cdot 3 \cdot 5 \cdot y^7}{2^3 \cdot 1 \cdot 2 \cdot 3 \cdot 7 \cdot r^6} + \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot y^9}{2^4 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 9 \cdot r^9} - \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdot y^{11}}{2^5 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 11 \cdot r^{10}} \ \&c.$$

quæ, facta $y = r$, dabit logarithmum analogum, quum ejus ſinus æquat ſinum totum. Si evadat $y > r$, ſeries poſt terminos plures ſpecie convergentes evadit divergens, atque adeo inutilis ad valorem logarithmi inveniendum. Ut hoc vitium vitetur, ſumendæ ſunt ſeries inverſo ordine, ut $r$ numeratorem terminorum conſtituat, $y$ denominatorem. Hoc modo evoluta in ſeriem $\frac{r}{\sqrt{yy+rr}}$, ſive $r \cdot \overline{yy+rr}^{-\frac{1}{2}}$, obtineo

$$\frac{r}{y} - \frac{r^3}{2 \cdot 1 \cdot y^3} + \frac{1 \cdot 3 \cdot r^5}{2^2 \cdot 1 \cdot 2\, y^5} - \frac{1 \cdot 3 \cdot 5 \cdot r^7}{2^3 \cdot 1 \cdot 2 \cdot 3\, y^7} + \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot r^9}{2^4 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot y^9} - \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdot r^{11}}{2^5 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5\, y^{11}} \ \&c.$$

ergo multiplicando per $dy$, proveniet $\frac{r\,dy}{\sqrt{yy+rr}} =$

$$\frac{r\,dy}{y} - \frac{r^3 dy}{2 \cdot y^3} + \frac{1 \cdot 3 \cdot r^5 dy}{2^2 \cdot 1 \cdot 2 \cdot y^5} - \frac{1 \cdot 3 \cdot 5 \cdot r^7 dy}{2^3 \cdot 1 \cdot 2 \cdot 3\, y^7} + \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot r^9 dy}{2^4 \cdot 1 \cdot 2 \cdot 3 \cdot 4\, y^9} - \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9\, r^{11} dy}{2^5 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5\, y^{11}} \ \&c.$$

igitur integrando non ſine additione conſtantis, ſumptiſque logarithmis in ſyſtemate, cujus ſubtangens $= r$, fiet $S \frac{r\,dy}{\sqrt{yy+rr}}$, ſive logarithmus analogus = $ly - lA +$

$$\frac{r^3}{2 \cdot 2 \cdot y^2} - \frac{1 \cdot 3 \cdot r^5}{2^2 \cdot 1 \cdot 2 \cdot 4 \cdot y^4} + \frac{1 \cdot 3 \cdot 5 \cdot r^7}{2^3 \cdot 1 \cdot 2 \cdot 3 \cdot 6 \cdot y^6} - \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot r^9}{2^4 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 8 \cdot y^8} + \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdot r^{11}}{2^5 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 10 \cdot y^{10}} \ \&c.$$

Licet quantitas hæc debeat nulleſcere, facta $y = o$, tamen ſuppoſitio hæc eſſet omnino inutilis, quia redderet omnes terminos infinitos, neque poſſemus determinare conſtantem $lA$. Verum quum per primam ſeriem habeatur logarithmus analogus, exiſtente ſinu $= r$, fiat $y = r$, & hæc formula illi æquetur, ut proveniat $lr - lA =$

$$\frac{r}{2 \cdot 2} - \frac{1 \cdot 3 \cdot r}{2^2 \cdot 1 \cdot 2 \cdot 4} + \frac{1 \cdot 3 \cdot 5 \cdot r}{2^3 \cdot 1 \cdot 2 \cdot 3 \cdot 6} - \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot r}{2^4 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 8} + \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdot r}{2^5 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 10} - \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdot 11 \cdot r}{2^6 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 12} \ \&c.$$

$$-r+\frac{r}{2.3}-\frac{1.3.r}{2^2.1.2.5}+\frac{1.3.5.r}{2^3.1.2.3.7}-\frac{1.3.5.7.r}{2^4.1.2.3.4.9}+\frac{1.3.5.7.9.r}{2^5.1.2.3.4.5.11}-$$

$$\frac{1.3.5.7.9.11.r}{2^6.1.2.3.4.5.6.13}$$ &c. ſive redactis terminis

$$-r+\frac{\overline{2+3}.r}{2.2.3}-\frac{\overline{4+5}.1.3.r}{2^2.1.2.4.5}+\frac{\overline{6+7}.1.3.5.r}{2^3.1.2.3.6.7}-\frac{\overline{8+9}.1.3.5.7.r}{2^4.1.2.3.4.8.9}+$$

$$\frac{\overline{10+11}.1.3.5.7.9.r}{2^5.1.2.3.4.5.10.11}-\frac{\overline{12+13}.1.3.5.7.9.11.r}{2^6.1.2.3.4.5.6.12.13}$$ &c. Quare $S\frac{rdy}{\sqrt{yy+rr}}$ ſive logarithmus analogus $= ly - lr - r$

$$+\frac{\overline{2+3}.r}{2.2.3}-\frac{\overline{4+5}.1.3.r}{2^2.1.2.4.5}+\frac{\overline{6+7}.1.3.5.r}{2^3.1.2.3.6.7}-\frac{\overline{8+9}.1.3.5.7.r}{2^4.1.2.3.4.8.9}+$$

$$\frac{\overline{10+11}.1.3.5.7.9.r}{2^5.1.2.3.4.5.10.11}$$ &c.

$$+\frac{r^3}{2.2.y^2}-\frac{1.3.r^5}{2^2.1.2.4.y^4}+\frac{1.3.5.r^7}{2^3.1.2.3.6.y^6}-\frac{1.3.5.7.r^9}{2^4.1.2.3.4.8.y^8}-$$

$$\frac{1.3.5.7.9.r^{11}}{2^5.1.2.3.4.5.10.y^{10}}$$ &c. Logarithmos $y$, & $r$ potes ſumere in ſyſtemate cujuſcumque protonumeri, dummodo ſubtangens $= r$; ſed ſi ſumas logarithmos hyperbolicos, in quibus protonumerus $= r$, & $lr = o$, formula fit aliquanto ſimplicior. Quomodo autem per ſeriem hi logarithmi inveniendi ſint, traditum eſt capite ſuperiore.

8. Similis, ſed aliquanto expeditior eſt calculus, ſi per coſinum quæratur logarithmus analogus, cujus expreſſio differentialis eſt $\frac{rdx}{\sqrt{xx-rr}}$. Nam ſi evolvatur in ſeriem $\frac{r}{\sqrt{xx-rr}}$, tum multiplicetur per $dx$, demum integretur addita coſtante, invenies $S\frac{rdx}{\sqrt{xx-rr}} = lx - lA$

$$-\frac{r^3}{2.2.x^2}-\frac{1.3.r^5}{2^2.1.2.4.x^4}-\frac{1.3.5.r^7}{2^3.1.2.3.6.x^6}-\frac{1.3.5.7.r^9}{2^4.1.2.3.4.8.x^8}-$$

$$-\frac{1.3.5.7.9.r^{11}}{2^5.1.2.3.4.5.10.x^{10}}$$ &c. Hæc debet nulleſcere, facta $x = r$; ergo fiet

$$S\frac{rdx}{\sqrt{xx-rr}} = lx - lr$$

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$$+\frac{r}{2.2}+\frac{1.3.r}{2^2.1.2.4}+\frac{1.3.5.r}{2^3.1.2.3.6}+\frac{1.3.5.7.r}{2^4.1.2.3.4.8}+\frac{1.3.5.7.9.r}{2^5.1.2.3.4.5.10}\ \&c.$$

$$-\frac{r^3}{2.2.x^2}-\frac{1.3.r^5}{2^2.1.2.4.x^4}-\frac{1.3.5.r^7}{2^3.1.2.3.6.x^6}-\frac{1.3.5.7.r^9}{2^4.1.2.3.4.8.x^8}-$$

$\frac{1.3.5.7.9.r^{11}}{2^5.1.2.3.4.5.10.x^{10}}$ &c. Harum ſerierum ope licet ſemper determinare logarithmum analogum per coſinum. Duo logarithmi $x$, $r$ ſumendi ſunt in ſyſtemate ſubtangentis $=r$. Quod ſi facias protonumerum $=r$, evadet $lr=o$.

9. Quæ formulæ exprimunt arcus circulares, & logarithmos analogos per tangentes, omnium ſunt ſimpliciſſimæ, quia carent radicalibus. Formulas in unam colligam hoc modo $\frac{r^2dt}{rr\pm tt}$; ſignum ſuperius pertinet ad arcum circuli, inferius ad logarithmum analogum. Si formula convertatur in ſeriem fit $=$ $dt\mp\frac{t^2dt}{r^2}+\frac{t^4dt}{r^4}\mp\frac{t^6dt}{r^6}+\frac{t^8dt}{r^8}$ &c., & facta integratione nulla addita conſtante, quia debet nulleſcere poſita $t=o$, habemus $S\frac{r^2dt}{rr\pm tt}=$

$t\mp\frac{t^3}{3r^2}+\frac{t^5}{5r^4}\mp\frac{t^7}{7r^6}+\frac{t^9}{9r^8}$ &c. Si fiat $t=r$, ſeries ſignorum inferiorum, in quibus nulla eſt alternatio, valorem acquirit infinitum; logarithmus enim analogus, ubi $t=r$, infinitus eſt. Series vero ſignorum ſuperiorum, & alternantium fit æqualis dimidio quadrantis. Quare integrum quadrans circuli $=$ $2r.1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}$ &c. Series hæc lente admodum convergit. Quapropter juvabit contrahere terminos, facta vicinorum ſubtractione. Si in unum contrahantur primus & ſecundus, tertius & quartus, atque ita deinceps ſeries fit $4r.\frac{1}{1.3}+\frac{1}{5.7}+\frac{1}{9.11}+\frac{1}{13.15}$ &c. Quare quicumque accipiatur in hac ſerie terminorum numerus, ſemper quadrans circuli major erit valore, qui invenitur. Contra ſi in unum jungantur ſecundus & tertius, quartus & quintus, atque ita deinceps ſeries fiet $2r-4r.\frac{1}{3.5}+\frac{1}{7.9}+\frac{1}{11.13}+\frac{1}{15.17}$ &c. Collige in ſummam quemcumque terminorum numerum, quadrans minor erit valore invento. Quamobrem facto in duabus ſeriebus opportuno computo poteris verum valorem quadrantis conſtituere inter duos numeros, qui inter ſe non differant niſi quantitate quacumque data minore.

10. Alia quoque methodo formulæ propoſitæ in ſeriem convertuntur; methodum paucis aperiam. Perſpicuum eſt



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$$D\,\frac{t^p}{\overline{rr+tt}^{\,q}} = \frac{pt^{p-1}dt}{\overline{rr+tt}^{\,q}} - \frac{2qt^{p+1}dt}{\overline{rr+tt}^{\,q+1}}$$ ; Igitur

$$S\,\frac{t^{p-1}dt}{\overline{rr+tt}^{\,q}} = \frac{t^p}{p.\overline{rr+tt}^{\,q}} + \frac{2q}{p}S\,\frac{t^{p+1}dt}{\overline{rr+tt}^{\,q+1}}$$ . Fiat $p=1$, $q=1$

& omnes termini ducantur in $rr$, & orietur

$$S\,\frac{r^2dt}{rr+tt} = \frac{r^2t}{rr+tt} + 2S\,\frac{rrt^2dt}{\overline{rr+tt}^{\,2}}$$ . Facta $p=3$, $q=2$, erit

$$S\,\frac{r^2t^2}{\overline{rr+tt}^{\,2}} = \frac{rrt^3}{3.\overline{rr+tt}^{\,2}} + \frac{2.2}{3}S\,\frac{r^2t^4dt}{\overline{rr+tt}^{\,3}}$$ . Sit $p=5$, $q=3$

$$S\,\frac{r^2t^4dt}{\overline{rr+tt}^{\,3}} = \frac{r^2t^5}{5.\overline{rr+tt}^{\,3}} + \frac{2.3}{5}S\,\frac{r^2t^6dt}{\overline{rr+tt}^{\,4}}$$ . Sit $p=7$, $q=4$

$$S\,\frac{r^2t^6dt}{\overline{rr+tt}^{\,4}} = \frac{r^2t^7}{7.\overline{rr+tt}^{\,4}} = \frac{2.4}{7}S\,\frac{r^2t^8dt}{\overline{rr+tt}^{\,5}}$$ , atque ita deinceps progrediendo uſque in infinitum. Quare, opportunis peractis ſubſtitutionibus, nanciſcemur

$$S\,\frac{r^2dt}{rr+tt} = \frac{r^2t}{rr+tt} + \frac{2}{3}\,\frac{r^2t^3}{\overline{rr+tt}^{\,2}} + \frac{2^2.2}{3.5}\,\frac{r^2t^5}{\overline{rr+tt}^{\,3}} + \frac{2^3.2.3}{3.5.7}\,\frac{r^2t^7}{\overline{rr+tt}^{\,4}}\ \&c.$$

Itaque ſi ponas $t=r$, invenies circumferentiæ ſemiquadrantem $=$

$$\frac{r}{2} + \frac{r}{2.3} + \frac{2.r}{2.3.5} + \frac{2.3.r}{2.3.5.7} + \frac{2.3.4.r}{2.3.5.7.9}$$ &c., & quadrantem

$$= r + \frac{r}{3} + \frac{2r}{3.5} + \frac{2.3.r}{3.5.7} + \frac{2.3.4.r}{3.5.7.9}$$ &c. quæ convergit aliquanto magis illa, quam ſupra obtinuimus. Eadem methodo invenies

$$S\,\frac{r^2dt}{rr-tt} = \frac{r^2t}{rr-tt} - \frac{2}{3}.\frac{r^2t^2}{\overline{rr-tt}^{\,2}} + \frac{2^2.2}{3.5}.\frac{r^2t^5}{\overline{rr-tt}^{\,3}} - \frac{2^3.2.3}{3.5.7}.\frac{r^2t^7}{\overline{rr-tt}^{\,4}}$$ &c. cum alternatione ſignorum. Per has formulas data tangente invenies arcus circulares, & logaritmhmos analogos.

11. Nunc determinandi ſunt ſinus, coſinus, & tangentes per arcus circulares, & logarithmos analogos; quod problema etiamſi difficilius videatur, tamen non ineleganter ſolvemus adhibito theoremate, de quo loquuti ſumus Cap. 4. num. 20, & ſequentibus. In formula II, & VII ſuppono unicum ineſſe ſignum ſummatorium ita, ut utraque hanc formam accipiat $S \pm \frac{xdx}{z} = z$ ; ſignum

ſignum ſuperius pertinet ad æquationem II, inferius ad VII; igitur facta differentiatione $\pm \frac{x\,dx}{z} = dz$, ſeu $\pm x\,dx = z\,dz$, quam integremus cum additione conſtantis, ut fiat $2b \pm xx = zz$. Si valeat ſignum ſuperius, $b$ eſſe poteſt cum negativa, tum poſitiva, propterea ſi fiat $2b = \pm rr$, erit $xx \pm rr = zz$, ex qua exoriuntur duæ $du = \frac{a\,dx}{\sqrt{xx + rr}}$, $du = \frac{a\,dx}{\sqrt{xx - rr}}$; ex quibus conſtat $u$ eſſe logarithmum analogum datum in prima per ſinum $= x$, in ſecunda per coſinum $= x$. In utraque valet

$$x = \begin{cases} A + \frac{Au^2}{2r^2} + \frac{Au^4}{2.3.4.r^4} + \frac{Au^6}{2.3.4.5.6.r^6} \text{ \&c.} \\[2ex] Bu + \frac{Bu^3}{2.3r^2} + \frac{Bu^5}{2.3.4.5.r^4} + \frac{Bu^7}{2.3.4.5.6.7.r^6} \text{ \&c.} \end{cases}$$

Si valeat ſignum inferius, ut vitentur imaginaria, $b$ poſitive accipienda eſt; quare poſita $2b = rr$, fiet $rr - xx = zz$, & $\pm\sqrt{rr - xx} = z$. Æquatio itaque orietur $du = \frac{\pm r\,dx}{\sqrt{rr - xx}}$, in qua $u$ eſt arcus circularis datus per ſinum $= x$, ſi formula affecta ſit ſigno $+$, datus per coſinum $= x$, ſi ſumatur ſignum $-$. Invenitur autem ex theoremate

$$x = \begin{cases} A - \frac{Au^2}{2.r^2} + \frac{Au^4}{2.3.4.r^4} - \frac{Au^6}{2.3.4.5.6.r^6} \text{ \&c.} \\[2ex] Bu - \frac{Bu^3}{2.3.r^2} + \frac{Bu^5}{2.3.4.5.r^4} - \frac{Bu^7}{2.3.4.5.6.7.r^6} \text{ \&c.} \end{cases}$$

12. Ad determinandos coefficientes $A$, $B$, advertendum eſt, ſi $x$ denotet coſinum, poſita $u = 0$, eſſe $x = r$, $\frac{dx}{du} = 0$, ſi vero $x$ exprimat ſinum, eſſe $x = 0$, $\frac{dx}{du} = 1$; ergo colliges pro coſinu $A = r$, $B = 0$; pro ſinu autem $A = 0$, $B = 1$. Hæ determinationes hujuſmodi ſeries ſuppeditant

$$Cb.u = r + \frac{u^2}{2.r} + \frac{u^4}{2.3.4.r^3} + \frac{u^6}{2.3.4.5.6.r^5} \text{ \&c.}$$

$$Sb.u = u + \frac{u^3}{2.3.r^2} + \frac{u^5}{2.3.4.5.r^4} + \frac{u^7}{2.3.4.5.6.7.r^6} \text{ \&c.}$$

$$Cc.u = r - \frac{u^2}{2.r} + \frac{u^4}{2.3.4.r^3} - \frac{u^6}{2.3.4.5.6.r^5} \text{ \&c.}$$

$$Sc.u = u - \frac{u^3}{2.3.r^2} + \frac{u^5}{2.3.4.5.r^4} - \frac{u^7}{2.3.4.5.6.7.r^6} \text{ \&c.}$$

quæ

quæ formulæ erant inveniendæ.

13. Coſinui hyperbolico addatur, ac detrahatur ſinus hyperbolicus, & fiet

$$Cb.u \pm Sb.u = r \pm u + \frac{u^2}{2r} \pm \frac{u^3}{2.3.r^2} + \frac{u^4}{2.3.4.r^3} \pm \frac{u^5}{2.3.4.5.r^4} \&c.,$$

quæ ſeries, quum indicet numerum, cujus logarithmus hyperbolicus $= u$, conſtat $Cbu \pm Sbu$ æquare numerum, cujus logarithmus hyperbolicus $= u$. Quod ſi, multiplicata expreſſione ſinus circularis per $\sqrt{-1}$, eædem fiant operationes, proveniet

$$Cc.u \pm \sqrt{-1}.Sc.u = r \pm u\sqrt{-1} - \frac{u^2}{2.r} \pm \frac{u^3\sqrt{-1}}{2.3.r^2} + \frac{u^4}{2.3.4.r^3} \pm \frac{u^5\sqrt{-1}}{2.3.4.5.r^4} \&c.$$

quæ ſeries oritur, quotieſcumque in ſuperiore pro $u$ ſubſtituas $u\sqrt{-1}$. Quapropter $Cc.u \pm \sqrt{-1}.Sc.u$ æquat numerum, cujus logarithmus hyperbolicus eſt quantitas imaginaria, nempe $u\sqrt{-1}$.

14. Si poſito $i$ numero infinito, binomium $\overline{1 \pm \frac{u}{ir}}^{\,i}$ evolvatur in ſeriem fiet

$$1 + \frac{iu}{ir} + \frac{i.\overline{i-1}.u^2}{2.i^2r^2} \pm \frac{i.\overline{i-1}.\overline{i-2}.u^3}{2.3.i^3r^3} + \frac{i.\overline{i-1}.\overline{i-2}.\overline{i-3}.u^4}{2.3.4.i^4r^4} \pm \&c$$

atqui quum $i$ infinitus ſit numerus, erit $i-1$, $i-2$, $i-3$ &c. ſemper $= i$ ergo facta ſubſtitutione, & opportuna diviſione, habebimus

$$\overline{1 \pm \frac{u}{ir}}^{\,i} = 1 \pm \frac{u}{r} + \frac{u^2}{2r^2} \pm \frac{u^3}{2.3.r^3} + \frac{u^4}{2.3.4.r^4} \pm \frac{u^5}{2.3.4.5.r^5} \&c.;$$ atqui

hæc ſeries $= \frac{Cb.u \pm Sb.u}{r}$ ex num. ſuperiore; ergo $\frac{Cb.u \pm Sb.u}{r} = \overline{1 \pm \frac{u}{ia}}^{\,i}$.

Quapropter ex combinatione ſignorum

$$\frac{2\,Cb.u}{r} = \overline{1 + \frac{u}{ir}}^{\,i} + \overline{1 - \frac{u}{ir}}^{\,i}, \quad \frac{2\,Sbu}{r} = \overline{1 + \frac{u}{ir}}^{\,i} - (\overline{1 - \frac{u}{ir}}^{\,i}.$$ Simili

ratione ſi evolvamus in ſeriem binomium $\overline{1 \pm \frac{u\sqrt{-1}}{ir}}^{\,i}$, inveniemus

$$1 \pm \frac{u\sqrt{-1}}{r} - \frac{u^2}{2r^2} \mp \frac{u^3}{2.3.r^3} + \frac{u^4}{2.3.4.r^4} \pm \frac{u^5\sqrt{-1}}{2.3.4.5.r^5} \&c.,$$ quæ ſeries

quum æquet $\frac{Cc.u \pm \sqrt{-1}.Sc.u}{r}$, orietur $\frac{Cc.u \pm \sqrt{-1}.Sc.u}{r} = \overline{1 \pm \frac{u\sqrt{-1}}{ir}}^{\,i}$,

ex qua ob ambigua ſigna inveniemus

2 C.u

$$\frac{2Cc.u}{r} = \overline{1 + \frac{u\sqrt{-1}}{ir}}^{\,i} + \overline{1 - \frac{u\sqrt{-1}}{ir}}^{\,i}, \quad \frac{2Sc.u}{r} = \frac{\overline{1 + \frac{u\sqrt{-1}}{ir}}^{\,i} - \overline{1 - \frac{u\sqrt{-1}}{ir}}^{\,i}}{\sqrt{-1}}$$

Quæ expreſſiones, tametſi omnes involvant quantitatem infinitam $i$, & illæ, quæ ad circulum pertinent, etiam quantitates imaginarias; tamen quum inventus fuerit earum valor finitus, & realis, maximam in calculo habent utilitatem. Hæc deſumpſimus ex Vincentij Riccati epiſtola ſecunda ad Joſephum Suzzium, quæ legitur in primo opuſculorum tomo op. ſexto.

15. Quum inventus fuerit tam ſinus, quam coſinus per arcum circuli, aut logarithmum analogum, facile eſt determinare tangentem; eſt enim coſinus ad ſinum, ſicuti ſinus totus ad tangentem; ergo tangens diviſa per ſinum totum æquabit ſinum diviſum per coſinum. Itaque

$$\frac{Tb.u}{r} = \frac{u + \frac{u^3}{2.3.r^2} + \frac{u^5}{2.3.4.5.r^4} + \frac{u^7}{2.3.4.5.6.7.r^6}}{r + \frac{u^2}{2.r} + \frac{u^4}{2.3.4.r^3} + \frac{u^6}{2.3.4.5.6.r^5}} \ \&c.$$

$$\frac{Tc.u}{r} = \frac{u - \frac{u^3}{2.3.r^2} + \frac{u^5}{2.3.4.5.r^4} - \frac{u^7}{2.3.4.5.6.7.r^6}}{r - \frac{u^2}{2.r} + \frac{u^4}{2.3.4.r^3} - \frac{u^6}{2.3.4.5.6.r^5}} \ \&c.$$

Hæc ſufficiant de ſeriebus exprimentibus quantitates circulares, & hyperbolicas.

16. Reliquum eſt, ut declaremus quibus nam curvis uſi ſint ſcriptores ad quadrandum circulum, & hyperbolam, quas proinde quadratrices circuli, & hyperbolæ appellavere. De logarithmica, & quomodo ea deſcripta quadratur hyperbola, dictum eſt ſatis capite ſuperiori. Nunc de ea, quam vocant tractoriam, cujus æquatio $dx = \frac{-dy\sqrt{rr-yy}}{y}$. Ut appareat, quomodo curva hac ſuppoſita hyperbolæ quadratura conſtruatur, fiat $\sqrt{rr-yy} = z$, ex qua $\frac{-dy}{y} = \frac{z\,dz}{rr-zz}$, & formula in hanc convertetur

$$dx = \frac{z^2 dz}{rr-zz} = -dz + \frac{r^2 dz}{rr-zz}$$ ; ergo integrando facta terminorum tranſpoſitione $x + z = S\frac{r^2 dz}{rr-zz}$. Summatoria hæc exprimit logarithmum analogum, cujus tangens $= z$, exiſtente ſinu toto $= r$. Quapropter ſemiaxe CA $= r$, (*Fig.* 5.) deſcribatur hyperbola æquilatera AD, cui ducatur tangens AF, & ducatur quælibet CD; erit logarithmus analogus ſeu ſector CAD diviſus

per $\frac{1}{2}$ CA $= S \frac{r^2 dz}{rr - zz}$, vocata AF $= z$. Itaque eodem radio CA deſcribe quadrantem circuli ALM. Abſcinde CI $=$ AF $= z$, & duc ordinatam IL $= \sqrt{rr - zz} = y$. Duc radio CM parallelam LH æqualem logarithmo analogo. Punctum H erit in curva quæſita. Etenim ducta HK parallela CA vocetur CK $= x$. Habebimus LH $= x + z = S \frac{r^2 dz}{rr - zz}$, & differentiando

$$dx + dz = \frac{r^2 dz}{rr - zz}, \text{ ſive } dx = \frac{z^2 dz}{rr - zz} = \frac{-dy}{y}\sqrt{rr - yy}.$$ Curva in puncto A incipiens accedit ultra quemcumque limitem ad rectam CO, quæ eſt ejus aſymptotum; habet præterea ad alteram partem rectæ AC alium ramum prorſus æqualem, & ſimilem ramo AHO.

17. Perſpicuum jam eſt, quomodo hyperbola æquilatera quadratur deſcripta tractoria. Sit hyperbola AD, cujus ſemiaxis CA. Ad eamdem CA deſcribatur tractoria AHO, cujus aſymptotum CO eſt normale CA. Radio eodem CA deſcribe quadrantem ALM. Abſcinde in radio CM rectam CI æqualem tangenti AF, age ordinatam IL, & ex puncto L rectam LH parallelam CO. Eſt LH logarithmus analogus; ergo rectangulum ex LH in $\frac{1}{2}$ CA æquabit ſectorem CAD. Ex punctis A, D in aſymptotum hyperbolæ ducantur normales AP, DQ. Sector CAD æquat quadrilaterum APQD; ergo hoc quadrilaterum mixtilineum eſt æquale rectangulo ex LH in $\frac{1}{2}$ CA. Ducatur circuli radius CL, cui ſit parallela HN, quam ſuo loco demonſtrabimus eſſe curvæ tangentem, erit CN $=$ Lm; ergo prædicta ſpatia hyperbolica æquant CN in $\frac{1}{2}$ CA. Quo pacto per inſtrumentum organice delineentur logarithmica, & tractoria alias declarabimus.

18. Super circulo ADB (*Fig. 6.*) intelligatur conſtitutus cilinder rectus, quem per diametrum ACB ita ſeca, ut planum ſecans faciat cum baſi cilindri angulum ſemirectum; deſcribetur in ſuperficie cilindri ellypſis AEB. Perpendiculares diametro ACB agantur CD, HF; & ex punctis D, F normales baſi agantur DE, FG, quæ incidant in ellypſim. Propter angulum ſemirectum, quem planum ſecans facit cum baſi, conſtat rectas DE, GF æquare lineas DC, FH, quæ ſunt ſinus arcuum AF, AD. His poſitis extendatur in lineam rectam ſemicircumferentia ADB, & ſuperficies ungulæ complanetur, orietur curva AGEBF (*Fig. 7.*) cujus abſciſſæ AF æquant arcus circulares, & ordinatæ FG æquant eorum ſinus. Curva hæc ab aliquibus vocata eſt ungula cylindrica, ſed plerumque nominatur curva ſinuum. Hæc eadem eſt curva coſinuum circularium, in qua ſcilicet abſciſſæ ſunt arcus, ordinatæ coſinus. Nam diviſa AB bifariam in D, ſi FG eſt ſinus arcus AF, conſtat eamdem FG eſſe coſinum arcus FD. Deſcripta hac curva, quum habeatur recta æqualis arcui circulari, perſpicuum eſt, quomodo per hanc quadratura circuli obtineatur. Curvæ æquatio invenitur nullo negotio; nam vocata AF $= x$, FG $= y$; erit $dx = \frac{r\,dy}{\sqrt{rr - yy}}$, ſi vero fiat DF $= x$ erit $dx = \frac{-r\,dy}{\sqrt{rr - yy}}$. Ne putes cur-

curvam hanc conſtare ſola parte A E B, ſed ad utramque partem procedit in M, N, & infra A B format partem ſimilem A E B, tum aſcendit ſupra A B ſemper efformans partes ſimiles, & æquales, atque ita in infinitum. Si planum ſecans cylindrum faceret cum ejus baſi quemlibet angulum orirεtur curva cujus ordinatæ non eſſent quidem ſinus, & coſinus circulares, ſed ad hos datam haberent proportionem.

19. Quemadmodum ſpectantur curvæ ſinuum, & coſinuum circularium, ita ſpectari poſſunt curvæ ſinuum, & coſinuum hyperbolicorum, in quibus ſcilicet abſciſſæ æquales ſint logarithmis analogis, ordinatæ vero eorum ſinibus aut coſinibus. Earum æquationes erunt, vocatis abſciſſis $= x$, ordinatis $= y$, $dx = \frac{r\,dy}{\sqrt{yy + rr}}$, $dx = \frac{r\,dy}{\sqrt{yy - rr}}$. Progreſſus curvarum ſit A G, B H, (*Fig.* 8) in quibus A B æquat ſinum totum, A F logarithmos analogos, F G eorum ſinus, F H coſinus. Verum harum curvarum non ita facilis habetur deſcriptio; qua de re ad quadrandam hyperbolam ſatius erit uti logarithmica, aut tractoria. Sed quadrata hyperbola, curvæ ſinuum, & coſinuum hyperbolicorum, determinatis punctis quotquot volueris, conſtruuntur. Utraque autem curva prædita eſt quatuor ramis ſimilibus, & æqualibus in infinitum recedentibus.

20. Venio ad cycloidem, quæ curva propter inſignes ejus proprietates a geometris eſt nobilitata. Ea ita deſcribitur. Tangat circulus G H K lineam D E, (*Fig.* 9.) & punctum H in ejus circumferentia fixum primum jaceat in D. Super rectam D E rotetur circulus, ut ſingula circumferentiæ puncta ſucceſſive ſe accommodent punctis rectæ D E. In hoc motu punctum H delineabit curvam D H B E. Ut natura curvæ patefiat, concipe circulum veniſſe in locum G H K, dum punctum deſcribens eſt in H. Hujuſce circuli duc diametrum G K perpendicularem D E. Progrediatur circulus, donec punctum H veniat in B, quod maxime remotum eſt a recta D E, ejuſque poſitio ſit A F B. Ex puncto H duc H I parallelam D E, quæ ſecet G K in L, & circumferentiam A F B in F. Agantur chordæ H G, F A. Quoniam arcus H G, F A circulorum æqualium intercipiuntur inter parallelas H I, D A, æquales erunt; ergo tum chordæ H G, F A, tum anguli H G K, F A B æquales erunt; ergo chordæ H G, F A æquales, & parallelæ; igitur quadrilaterum H G A F erit parallelogrammum. Jam vero, quoniam ſemicircumferentia A F B ſeſe applicavit ſucceſſive lineæ D A, eidem erit æqualis; ſed eadem de cauſſa arcus G H, ſeu illi æqualis A F æquat rectam D G; ergo deducendo, arcus B F erit æqualis rectæ G A; atqui G A æquat H F; ergo F B æquat H F. Quæ eſt princeps cycloidis proprietas; nimirum ſi ſuper axe B A deſcriptus intelligatur circulus A F B, ducta quacumque H I parallela baſi D E, intercepta H F inter cycloidem & circuli circumferentiam æquat arcum F B. Deſcripta itaque cycloide circuli quadratura obtinetur nullo negotio.

21. Ex inventa proprietate æquationem cycloidis exhibeamus. Vocetur B I $= x$, I H $= y$, erit ex natura circuli F I $= \sqrt{2rx - xx}$, & arcus B F $= S \frac{r\,dx}{\sqrt{2rx - xx}}$; ergo quum H F $=$ B F, ſive H I $=$ B F $+$ F I erit $y = S \frac{r\,dx}{\sqrt{2rx - xx}} + \sqrt{2rx - xx}$, ſeu differentiis acceptis



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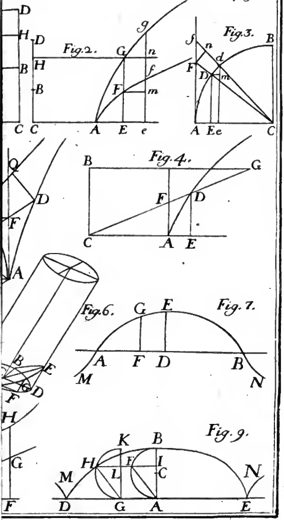
$dy = \frac{r\,dx}{\sqrt{2rx-xx}} + \frac{r\,dx - x\,dx}{\sqrt{2rx-xx}} = \frac{\overline{2r-x}.dx}{\sqrt{2rx-xx}} = \frac{dx\sqrt{2r-x}}{\sqrt{x}}$, quæ eſt æquatio quæſita. Quod ſi voces C I $= x$, exiſtente C centro circuli, erit $dy = \frac{-dx\sqrt{r+x}}{\sqrt{r-x}}$. Demum ſi A I $= x$, fiet $dy = \frac{-dx\sqrt{x}}{\sqrt{2r-x}}$. Curva hæc progreditur ad utramque partem lineæ D E per M, N, & efformat infinitas partes æquales, & ſimiles D B E, omnes ſitas ſupra lineam D E.

## CAPUT OCTAVUM.

### De integratione fractionum rationalium.

1. FRactiones rationales eas dicimus, in quibus differentia variabilis multiplicatur, & dividitur per functionem rationalem ejuſdem variabilis, cujuſmodi eſſet $\frac{x^3 + ax^2 - aa^2x - a^3.dx}{x^4 + 2ax^3 - a^2x^2 + 2a^3x + a^4}$. Hujuſmodi formulas perſpicuum eſt dividi poſſe in plures, in quibus poteſtas variabilis ducta in differentiam dividitur per functionem rationalem; quare qui has noverit integrare, omnes fractiones rationales pariter integrabit. Quotieſcumque diviſor reſolvi poteſt in factores reales aut primi, aut ſecundi gradus, affirmo formulas integrari aut abſolute, aut ſaltem ſuppoſita circuli, & hyperbolæ quadratura. Quomodo autem plures functiones rationales reſolvantur in factores reales primi, aut ſecundi gradus, docuimus in primo tomo, & diximus non temere ſuſpicari nos, omnes functiones rationales eodem modo reſolvi poſſe. Ut propoſitionem demonſtrem gradatim progredior. Ex cap. *6.* conſtat $\frac{dx}{x+a}$ integrari per logarithmos, ſeu ſuppoſita hyperbolæ quadratura, ejuſque ſummatoriam $= \frac{1}{c}\, l\,\overline{x+a}$ exiſtente $c$ ſyſtematis ſubtangente; ſimiliter S$\frac{\pm dx}{x \pm a} = \pm \frac{1}{c}\, l\,\overline{x \pm a}$, quæ formula a ſuperiore non eſt diverſa, una enim in aliam mutatur ſi aut $x$, aut $a$, aut utraque negative accipiatur. Verum ad evitandum logarithmum quantitatis negativæ formula $\frac{\pm dx}{x-a}$, quum $x < a$, ita ſcribatur $\frac{\mp dx}{a-x}$, cujus integralis erit $\pm\, l\,\overline{a-x}$. Ad ſuperiorem reducitur formula $\frac{x^n dx}{x+a}$. Nam ſi fiat continua diviſio, donec exponens numeratoris ſit $=0$, mutabitur in ſequentem $x^{n-1}dx - ax^{n-2}dx + a^2x^{n-3}dx \ldots\ldots \frac{\pm a^n dx}{x+a}$, ſignum $+$ valet ſi $n$ ſit numerus par, ſignum $-$ ſi ſit impar. Formulæ inventæ termini omnes ſunt algebraice integrabiles, excepto ultimo, qui dependet a logarithmis. Similis methodus valet in formula $\frac{dx}{x^n.\overline{x+a}}$. Nam ita diſpo-

ua-


Fig. 2.
Fig. 3.
Fig. 4.
Fig. 6.
Fig. 7.
Fig. 9.

natur $\frac{dx}{ax^n + x^{n+1}}$, tum fiat diviſio continua, donec exponens $n$ in denominatore $= 0$ & formula in hanc mutabitur $\frac{dx}{ax^n} - \frac{dx}{a^2 x^{n-1}} + \frac{dx}{a^3 x^{n-2}} \ldots\ldots \pm \frac{dx}{a^n x} \mp \frac{dx}{a^n \cdot (x+a)}$; ſignum ſuperius valet ſi $n$ ſit impar, inferius ſi ſit par. In hac omnes termini ſunt algebraice integrabiles, ſi duos ultimos excipias, qui dependent a logarithmis.

2. Tranſeo ad formulam $\frac{x^n dx}{(x+a)^m}$, in qua $m$, $n$ ſunt numeri integri, & poſitivi. Ut formulam ad integrationem perducam, utor methodo ſubſtitutionis, ponens $x + a = y$, quæ convertit formulam in hanc $\frac{dy \cdot (y-a)^n}{y^m} = \frac{dy}{y^{m-n}} - \frac{n a dy}{y^{m-n+1}} + \frac{n \cdot n-1}{2} \cdot \frac{a^2 dy}{y^{m-n+2}} \ldots \pm \frac{a^n dy}{y^m}$; valet $+$ ſi $n$ ſit par, valet $-$ ſi $n$ ſit impar. Si $m = n + 1$ primus terminus integratur per logarithmos, reliqui omnes ſunt abſolute integrabiles. Si $m = 1$, ultimus terminus a logarithmis dependet, reliqui algebraicam integrationem recipiunt. Si valor $m$ medius ſit inter $n + 1$, $1$, aliquis terminus medius inter primum, & ultimum dependebit a logarithmis; ſi vero $m > n + 1$ omnes integrabiles ſunt. Quapropter propoſita formula integrabilis eſt algebraice ſi $m > n + 1$, ſi vero ſit $m =$, aut $< n + 1$ dependet ab hyperbolæ quadratura. Quod ſi data ſit formula $\frac{dx}{x^n \cdot (x+a)^m}$, utere ſubſtitutione $x = \frac{a^2}{y}$, & formula in hanc convertetur $\frac{-y^{m+n-2} dy}{a^{m+2n-2} \cdot (a+y)^m}$, quæ redacta eſt ad formam ſuperiorem. Quum autem $m$ non poſſit eſſe major $m + n - 1$, ſed aut ſit æqualis, aut minor, perſpicuum eſt formulam ad ſui integrationem indigere hyperbolæ quadratura.

3. Ad has formulas integrandas juvat addere aliam methodum mihi familiarem, quæ ſubſtitutionum non eget. Accipio differentiam formulæ $\frac{x^q}{(x+a)^p}$ hoc modo $D. \frac{x^q}{(x+a)^p} = \frac{q x^{q-1} dx}{(x+a)^p} - \frac{p x^q dx}{(x+a)^{p+1}}$; ergo

$$S \frac{x^q dx}{(x+a)^{p+1}} = \frac{-1}{p} \cdot \frac{x^q}{(x+a)^p} + \frac{q}{p} \cdot S \frac{x^{q-1} dx}{(x+a)^p}.$$

Hac habita formula pono primum $q = n$, $p + 1 = m$, ut habeam



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$$\mathrm{S}\frac{x^n dx}{\overline{x+a}^{\,m}} = \frac{-1}{m-1}\cdot\frac{x^n}{\overline{x+a}^{\,m-1}} + \frac{n}{m-1}\cdot\mathrm{S}\frac{x^{n-1}dx}{\overline{x+a}^{\,m-1}}.$$ Fiat $q=n-1$, $p+1=m-1$

$$\mathrm{S}\frac{x^{n-1}dx}{\overline{x+a}^{\,m-1}} = \frac{-1}{m-2}\cdot\frac{x^{n-1}}{\overline{x+a}^{\,m-2}} + \frac{n-1}{m-2}\cdot\mathrm{S}\frac{x^{n-2}dx}{\overline{x+a}^{\,m-2}}.$$ Fac $q=n-2$, $p+1=m-2$

$$\mathrm{S}\frac{x^{n-2}dx}{\overline{x+a}^{\,m-2}} = \frac{-1}{m-3}\cdot\frac{x^{n-2}}{\overline{x+a}^{\,m-3}} + \frac{n-2}{m-3}\cdot\mathrm{S}\frac{x^{n-3}dx}{\overline{x+a}^{\,m-3}}.$$ Duplicem caſum diſtinguamus neceſſe eſt. Vel enim $n < m$, vel ſecus. In primo caſu operatio toties iteranda eſt, donec in ultima ſummatoria exponens numeratoris fiat $=0$, exiſtente ejus coefficiente $= \frac{1}{m-n}$. Jam vero ſi cujuslibet ſummatoriæ valorem opportune ſubſtituas, invenies

$$\mathrm{S}\frac{x^n dx}{\overline{x+a}^{\,m}} = \frac{-1}{m-1}\cdot\frac{x^n}{\overline{x+a}^{\,m-1}} - \frac{n}{\overline{m-1}.\overline{m-2}}\cdot\frac{x^{n-1}}{\overline{x+a}^{\,m-2}} - \frac{n.\overline{n-1}}{\overline{m-1}.\overline{m-2}.\overline{m-3}}\cdot\frac{x^{n-2}}{\overline{x+a}^{\,m-3}}\ldots\ldots + \frac{n.\overline{n-1}.\overline{n-2}\ldots.1}{\overline{m-1}.\overline{m-2}.\overline{m-3}\ldots\overline{m-n}}\cdot\mathrm{S}\frac{dx}{\overline{x+a}^{\,m-n}},$$

jam vero ſi $m-n=1$, ultima ſummatoria habetur per logarithmos. Si vero $m-n>1$ abſolute integrabilis eſt; fit enim

$$\frac{n.\overline{n-1}.\overline{n-2}\ldots1}{\overline{m-1}.\overline{m-2}.\overline{m-3}\ldots.\overline{m-n-1}}\cdot\frac{-1}{\overline{x+a}^{\,m-n-1}}.$$

4. In caſu altero, ubi $n$ eſt major, aut æqualis $m$, analyſis numeri ſuperioris producenda eſt, donec in ultima ſummatoria exponens denominatoris $=1$. Atque hoc pacto factis ſubſtitutionibus ultimus terminus invenietur eſſe

$$\frac{n.\overline{n-1}.\overline{n-2}\ldots\ldots\overline{n-m+1}}{\overline{m-1}.\overline{m-2}.\overline{m-3}\ldots\ldots1}\cdot\mathrm{S}\frac{x^{n-m+1}dx}{x+a},$$

quæ, ut conſtat ex num. 1., peracta diviſione dat plures terminos integrabiles, addito termino, qui dependet a logarithmis. Quare ſi diviſor conſtet factoribus omnibus primi gradus & æqualibus, noſtræ formulæ ſummatoria erit algebraica, ſi exponens numeratoris ſit minor ſaltem duabus unitatibus numero factorum, ſive exponente denominatoris, ſecus includit terminum, qui per logarithmos integratur. Quod ſi ſupponatur integranda formula $\frac{dx}{x^n.\overline{x+a}^{\,m}}$, non eſt opus novam inſtituere analyſim, ſed ſatis eſt in formulis numeri ſuperioris mutare ſignum ſpeciei $n$, & operationem producere, donec binomium $x+a$ præditum ſit exponente $=1$.

Pro-

Proveniet autem $S\frac{dx}{x^n\cdot\overline{x+a}^m}=\frac{-1}{m-1}\cdot\frac{1}{x^n\cdot\overline{x+a}^{m-1}}+\frac{n}{\overline{m-1}\cdot\overline{m-2}}\cdot$

$\frac{1}{x^{n+1}\cdot\overline{x+a}^{m-2}}-\frac{n\cdot\overline{n+1}}{\overline{m-1}\cdot\overline{m-2}\cdot\overline{m-3}}\cdot\frac{1}{x^{n+2}\cdot\overline{x+a}^{m-3}}\ldots\pm$

$\frac{n\cdot\overline{n+1}\cdot\overline{n+2}\ldots\overline{n+m+1}}{\overline{m-1}\cdot\overline{m-2}\cdot\overline{m-3}\ldots 1}S\frac{dx}{x^{m+n-1}\cdot\overline{x+a}}$. In ultima ſummatoria ſignum $+$ accipiendum eſt, ſi $m$ ſit impar, ſignum $-$, ſi $m$ ſit par. Eam autem a logarithmis dependere probatum eſt num. 1. Loquutus ſum de formulis $\frac{dx}{x^n\cdot\overline{x+a}}$, $\frac{dx}{x^n\cdot\overline{x+a}^m}$ quia methodus ipſa me vocavit. Cæterum in his $x^n$ ſpectari poteſt, ut factor ejus denominatoris, atque illæ methodi locum habebunt, quas deinceps trademus.

5. Revocandum eſt, quod docuimus cap. 7. num. 3 & 5, nimirum $\frac{a^2dx}{x^2+a^2}$ eſſe elementum arcus circularis, cujus ſinus totus $=a$, tangens $=x$; & $\frac{-a^2dx}{x^2+a^2}$ eſſe elementum arcus circuli poſita cotangente $=x$; ſimiliter $\frac{a^2dx}{a^2-x^2}$, ſive $\frac{-a^2dx}{x^2-a^2}$, eſſe elementum logarithmi analogi, cujus tangens $=x$, & $\frac{-a^2dx}{a^2-x^2}$, ſive $\frac{a^2dx}{x^2-a^2}$ eſſe elementum logarithmi analogi poſita cotangente $=x$. Ex his collige, ex duabus formulis $\pm\frac{dx}{x^2+a^2}$, $\pm\frac{dx}{x^2-a^2}$ primam a quadratura circuli dependere, ſecundam a quadratura hyperbolæ; ergo $\frac{dx}{x^2+ga}$ dependebit a quadratura circuli ſi $g$ ſit $=+a$, ab hyperbolæ ſi $g=-a$, quam ſpeciem $g$ ita introducimus, ut eundem calculum utrique formulæ applicemus. Memento etiam $\frac{xdx}{x^2+ga}$ habere pro integrale $\frac{1}{c}l\sqrt{x^2+ga}$, exiſtente $c$ ſyſtematis ſubtangente. His poſitis ſit formula $\frac{x^ndx}{x^2+ga}$; hæc per continuam diviſionem ad ſequentem reducitur $x^{n-2}dx-gax^{n-4}dx+g^2a^2x^{n-6}dx-g^3a^3x^{n-8}dx$ &c. Duplex diſtinguendus eſt caſus $n$ numeri imparis, & $n$ numeri paris. Si $n$ ſit impar, ultimus terminus proveniet $\pm\frac{g^{\frac{n-1}{2}}a^{\frac{n-1}{2}}xdx}{x^2+ga}$; ſignum ſuperius valebit

ſi n

ſi $n$ ſit numerus ex ſerie 1, 5, 9, 13 &c.; inferius ſi $n$ contineatur in ſerie 3, 7, 11, 15, &c.. Ultimus hic terminus dependet a logarithmis, reliqui omnes ſunt algebraice integrabiles; quare in hoc caſu propoſita formula ſemper dependet ab hyperbolæ quadratura. Si $n$ ſit par, ultimus terminus fiet $= \frac{\pm g^{\frac{n}{2}} a^{\frac{n}{2}} dx}{xx+ga}$; ſignum $+$ præfigendum ſi $n$ ſit pariter par, ſcilicet ex ſerie 0, 4, 8, 12 &c.; ſi vero ſit impariter par, hoc eſt ex ſerie 2, 6, 10, 14 &c. apponendum eſt ſignum $-$. Quapropter formula reſolvitur in plures terminos omnes algebraice integrabiles excepto ultimo, qui poſcit circuli quadraturam, ſi $g$ ſit poſitiva, quadraturam hyperbolæ, ſi $g$ ſit negativa.

6. Progredior ad formulam $\frac{dx}{x^n\,.\,\overline{x^2+ga}}$, quæ in hanc mutatur $\frac{dx}{gax^n+x^{n+2}}$. Hæc per continuam diviſionem provenit

$\frac{dx}{gax^n} - \frac{dx}{g^2a^2x^{n-2}} + \frac{dx}{g^3a^3x^{n-4}}$ &c. Si $n$ ſit numerus par, ultimus terminus addendus, fiet $\pm \frac{dx}{g^{\frac{n}{2}} a^{\frac{n}{2}}\,.\,\overline{xx+ga}}$, qui integratur poſita circuli, aut hyperbolæ quadratura. Quoad ſigna, eadem valent, quæ antea. Si $n$ ſit impar producenda eſt operatio donec deveniamus ad duos hoſce ultimos terminos $\pm \frac{dx}{g^{\frac{n-1}{2}} a^{\frac{n-1}{2}} x} \mp \frac{x\,dx}{g^{\frac{n+1}{2}} a^{\frac{n+1}{2}}\,.\,\overline{x^2+ga}}$ quorum uterque eſt quantitas logarithmica, & dependet a quadratura hyperbolæ. Signis ſuperioribus utere ſi $n$ contineatur in ſerie 1, 5, 9, 13 &c. inferioribus, ſi $n$ ſit ex ſerie 3, 7, 14, 15 &c.

7. Tractanda jam eſt formula univerſalior $\frac{x^n dx}{\overline{xx+ga}^{\,m}}$. Utens mea methodo ſumo differentiam formulæ $\frac{x^q}{\overline{xx+ga}^{\,p}}$ hoc modo

$$D\,\frac{x^q}{\overline{xx+ga}^{\,p}} = \frac{qx^{q-1}dx}{\overline{xx+ga}^{\,p}} - \frac{2px^{q+1}dx}{\overline{xx+ga}^{\,p+1}}; \text{ ergo}$$

$$S\,\frac{x^{q+1}dx}{\overline{xx+ga}^{\,p+1}} = -\frac{1}{2p}\,.\,\frac{x^q}{\overline{xx+ga}^{\,p}} + \frac{q}{2p}\,.\,S\,\frac{x^{q-1}dx}{\overline{xx+ga}^{\,p}}.$$ Hoc theoremate

inven

invento, pone primum $q+1=n$, $p+1=m$, erit

$$S\frac{x^{n}dx}{\overline{xx+ga}^{m}}=\frac{-1}{2.\overline{m-1}}.\frac{x^{n-1}}{\overline{xx+ga}^{m-1}}+\frac{n-1}{2.\overline{m-1}}.S\frac{x^{n-2}dx}{\overline{xx+ga}^{m-1}}.$$ Fiat $q+1=n-2$, $p+1=m-1$

$$S\frac{x^{n-2}dx}{\overline{xx+ga}^{m-1}}=\frac{-1}{2.\overline{m-2}}.\frac{x^{n-3}}{\overline{xx+ga}^{m-2}}+\frac{n-3}{2.\overline{m-2}}.S\frac{x^{n-4}dx}{\overline{xx+ga}^{m-2}}.$$ Pone $q+1=n-4$, $p+1=m-2$

$$S\frac{x^{n-4}dx}{\overline{xx+ga}^{m-2}}=\frac{-1}{2.\overline{m-3}}.\frac{x^{n-5}}{\overline{xx+ga}^{m-3}}+\frac{n-5}{2.\overline{m-3}}.S\frac{x^{n-6}}{\overline{xx+ga}^{m-3}}.$$ Hac methodo procedendum eſt, donec opus fuerit. Si $n$ ſit numerus par, producatur analyſis, neceſſe eſt, donec exponens diviſoris ultimi termini $=1$, atque ultima ſummatoria fiat $S\frac{x^{n-2m+2}dx}{xx+ga}$, in qua ſi $n$ ſit aut major, aut æqualis $2m-2$, habemus integrationem ex num. 5; ſi vero $n<2m-2$ ultimus terminus reducitur ad formulam num. 6, ad quam pariter reducitur, ſi $n$ ſit negativus vel par, vel impar, ſi analyſim protrahas uſque dum binomii $xx+ga$ exponens ſit $=1$.

8. Si $n$ ſit numerus impar & poſitivus. Vel $n$ eſt major, aut æqualis $2m-1$, & producatur analyſis, donec exponens denominatoris $=1$, ut proveniat formula $\frac{x^{n-2m+2}dx}{xx+ga}$, quæ integratur ex num. 5. Vel eſt $n<2m-1$. In hoc caſu protrahenda eſt analyſis uſque ad terminum, cujus numerator $=x\,dx$, atque in hoc ultimo termino habebitur differentialis $\frac{x\,dx}{\overline{xx+ga}^{\frac{2m-n-1}{2}}}$, quæ ſemper integrabilis eſt, excepto caſu $n=2m-3$, in quo pertinet ad logarithmicam. Pro hypotheſi vero numeri imparis, fortaſſe expeditior fiet methodus ſubſtitutionis. Nam ita diſpone formulam $\frac{x^{n-1}.x\,dx}{\overline{xx+ga}^{m}}$. Fiat $xx=az$, & $x\,dx=\frac{a\,dz}{2}$, & $x^{n-1}=a^{\frac{n-1}{2}}.z^{\frac{n-1}{2}}$; ergo formula in hanc mutatur

$$\frac{1}{2}.a^{\frac{n+1-2m}{2}}.\frac{z^{\frac{n-1}{2}}dz}{\overline{z+g}^{m}},$$ quæ integratur per methodos num. 2, 3, 4.

9. Quoniam $\frac{x^{n}dx}{\overline{xx+ga}^{m}}$ exiſtentibus $n$, $m$ numeris integris vel poſitivis

vel negativis, ſemper integrabilis eſt vel algebraice, vel ſaltem ſuppoſitis circuli, & hyperbolæ quadraturis; nullo negotio id ipſum probabitur de formula $\frac{x^n dx}{\overline{xx+2bx+fa}^{\,m}}$. Fiat enim $x+b=z$; ergo $xx+2bx=zz-bb$, & $xx+2bx+fa=zz-bb+fa$; poſitaque $fa-bb=ga$, reliquiſque peractis ſubſtitutionibus, formula in hanc mutabitur $\frac{dz.\overline{z-b}^{\,n}}{\overline{zz+ga}^{\,m}}$, quæ elevato binomio $z-b$ ad poteſtatem $n$ integram, & affirmativam, in plures convertitur $\frac{z^n dz}{\overline{zz+ga}^{\,m}} - \frac{nbz^{n-1}dz}{\overline{zz+ga}^{\,m}} + \frac{n.n-1}{2}.\frac{b^2 z^{n-2}dz}{\overline{zz+ga}^{\,m}} \ldots \ldots \pm \frac{b^n dz}{\overline{zz+ga}^{\,m}}$, qui termini omnes conceſſis quadraturis circuli & hyperbolæ integrantur.

10. Ad ſuperiorem reducitur formula $\frac{dx}{x^n.\overline{xx+2bx+fa}^{\,m}}$. Fiat enim $x=\frac{fa}{z}$; ergo $xx+2bx+fa=\frac{f^2a^2}{z^2}+\frac{2bfa}{z}+fa=\frac{fa}{z^2}.\overline{zz+2bz+fa}$, quare, omnibus peractis ſubſtitutionibus, formula in hanc mutabitur $\frac{-z^{2m+n-2}.dz}{f^{m+n-1}.a^{m+n-1}.\overline{zz+2bz+fa}^{\,m}}$, quæ habet formam num. 9, atque adeo, conceſſis quadraturis ſectionum conicarum, integratur. Ex his, quæ hactenus demonſtrata ſunt, hoc colligimus, formulam $x^n dx$, in qua $n$ eſt numerus integer vel poſitivus, vel negativus, diviſam per unum factorem vel ſimplicem vel ſecundi gradus, ſeu per plures omnes æquales, conceſſis circuli & hyperbolæ quadraturis numquam non integrari. Nunc ad factores inæquales animum convertamus oportet.

11. Formula $\frac{dx}{x+a.x+b}$, in qua ſuppono $a>b$, in duas transformatur hoc modo $\frac{dx}{x+b.a-b}$ $\frac{-dx}{x+a.a-b}$. Hoc tibi conſtabit, ſi fractiones duas ad eandem denominationem deducas, facto enim calculo propoſita formula ſeſe offert. Verum ſi hoc per calculum analyticum optas invenire, pone $\frac{dx}{x+a.x+b} = \frac{Adx}{x+a} + \frac{Bdx}{x+b}$. Tum ut coefficientes determines, ad eumdem denominatorem deductis fractionibus invenies $\frac{Ax+Ab}{Bx+Ba}.\frac{dx}{x+a.x+b}$. Hujus numeratorem compara cum numeratore propoſitæ, & habebis $A+B=0$,

$Ab$

$Ab+Ba=1$, ex quibus $A=\frac{-1}{a-b}$, $B=\frac{1}{a-b}$; ergo

$$\frac{dx}{\overline{x+a}\cdot\overline{x+b}}=\frac{dx}{\overline{a-b}\cdot\overline{x+b}}-\frac{dx}{\overline{a-b}\cdot\overline{x+a}}.\ \text{Q. E. I.}$$

Quum autem duæ inventæ ſint formulæ logarithmicæ, patet propoſitam per logarithmos integrari. Si ultima æquatio multiplicetur per $x$, patet formulam $\frac{x^n dx}{\overline{x+a}\cdot\overline{x+b}}$ in duas reſolvi, quæ tractatæ ſunt num. 1. Methodus hæc deficit ſi $a=b$, quia diviſores multiplicantur per $o$. Verum hæc hypotheſis dat duos factores ſimplices æquales, quæ ſupra tractata eſt.

12. Quamquam hæc reſolutio ſufficiat ad ea demonſtranda, quæ ſuperſunt de pluribus factoribus ſimplicibus inæqualibus; tamen, ut elegantius & facilius omnia deinceps determinentur, præſtabit oſtendere formulam $\frac{x^n dx}{\overline{x+a}\cdot\overline{x+b}}$ in duas reſolvi, quæ unicum habent factorem in diviſore, & exponentem numeratoris unitate minorem. Etenim ex numero ſuperiore habemus

$$\frac{dx}{\overline{x+a}\cdot\overline{x+b}}=\frac{1}{a-b}\cdot\overline{\frac{dx}{x+b}-\frac{dx}{x+a}};$$

igitur multiplicando per $x^n$ fiet

$$\frac{x^n dx}{\overline{x+a}\cdot\overline{x+b}}=\frac{1}{a-b}\cdot\overline{\frac{x^n dx}{x+b}-\frac{x^n dx}{x+a}}.$$

In utraque fractione fiat diviſio per denominatorem, ut oriatur

$$\frac{x^n dx}{\overline{x+a}\cdot\overline{x+b}}=\frac{1}{a-b}\cdot\begin{cases}-x^{n-1}dx+\dfrac{ax^{n-1}dx}{x+a}\\[2ex]+x^{n-1}dx-\dfrac{dx^{n-1}dx}{x+b}\end{cases}=\frac{1}{a-b}:$$

$$\overline{\frac{ax^{n-1}dx}{x+a}-\frac{bx^{n-1}dx}{x+b}},$$

in quibus numeratoris exponens eſt unitate minor.

13. Formula $\frac{dx}{\overline{x+a}\cdot\overline{xx+2bx+ge}}$, cujus diviſor duos habet factores alterum primi, alterum ſecundi gradus, transformatur in duas ſequentes

$$\frac{dx}{\overline{aa-2ab+ge}\cdot\overline{x+a}}+\frac{\overline{a-2b}\cdot dx-xdx}{\overline{aa-2ab+ge}\cdot\overline{xx+2bx+ge}},$$

quod tibi manifeſtum fiet, ſi haſce ad eundem denominatorem perducas. Si malis analytica methodo uti, fac

$$\frac{dx}{\overline{x+a}\cdot\overline{xx+2bx+ge}}=\frac{Adx}{x+a}+\frac{Bdx+Cxdx}{xx+2bx+ge}=$$

$$\frac{\begin{matrix}Ax^2+2Abx+Age\\ Cx^2+Bx+Ba\\ +Cax\end{matrix}}{}\ \ \frac{dx}{\overline{x+a}\cdot\overline{xx+2bx+ge}}.$$

Hanc compara cum data,

& invenies æquationes tres, $A + C = 0$, $2Ab + B + Ca = 0$, $Age + Ba = 1$, ex quibus proveniunt valores $A = \frac{1}{aa - 2ab + ge}$, $B = \frac{a - 2b}{aa - 2ab + ge}$, $C = \frac{-1}{aa - 2ab + ge}$, quibus ſubſtitutis habes formulas, in quas propoſita diſtribuitur. Inventæ autem habent unicum factorem pro diviſore, vel primi vel ſecundi gradus. Methodus deficit, ſi foret $aa + ge = 2ab$. Verum ſi pro $ge$ ponas in factore ſecundi gradus valorem $2ab - aa$, factor fiet $xx + 2bx + 2ab - aa = x + a . \overline{x - a + 2b}$; adeoque propopoſita fiet $\frac{dx}{\overline{x+a}^2 . \overline{x - a + 2b}}$, cujus diviſor conſtat duobus factoribus æqualibus $x + a$, & uno inæquali, niſi $b = a$, quo in caſu tres ſunt factores æquales. De caſu trium factorum æqualium jam diximus, de caſu duorum factorum æqualium, & unius inæqualis paulo infra dicemus.

14. Hic quoque utile erit formulam $\frac{x^n dx}{\overline{x+a} . \overline{xx + 2bx + ge}}$ in duas reſolvere ita, ut exponens numeratoris ejus, cui eſt factor ſimplex $= n - 2$, exponentes autem duorum terminorum, qui conſtituunt alterius numeratorem, ſint $n - 1$, $n - 2$. Ex numero ſuperiore

$$\frac{dx}{\overline{x+a} . \overline{xx + 2bx + ge}} = \frac{1}{aa - 2ab + ge} . \overline{\frac{dx}{x+a} - \frac{x dx + \overline{a - 2b} . dx}{xx + 2bx + ge}}.$$

Multiplica æquationem per $x^n$, ut fiat

$$\frac{x^n dx}{\overline{x+a} . \overline{xx + 2bx + ge}} = \frac{1}{aa - 2ab + ge} . \overline{\frac{x^n dx}{x+a} - \frac{x^{n+1} dx + \overline{a - 2b} . x^n dx}{xx + 2bx + ge}}.$$

Utramque fractionem per ſuum denominatorem divide, ut fiat

$$\frac{x^n dx}{\overline{x+a} . \overline{xx + 2bx + ge}} = \frac{1}{aa - 2ba + ge} .$$

$$\overline{x^{n-1} dx - a x^{n-2} dx + \frac{a^2 x^{n-2} dx}{x+a}}$$

$$- x^{n-1} dx + a x^{n-2} dx + \frac{\overline{ge - 2ab} . x^{n-1} dx - agex^{n-2} dx}{xx + 2bx + ge} =$$

$$\frac{1}{aa - 2ab + ge} . \overline{\frac{a^2 x^{n-2} dx}{x+a} + \frac{\overline{ge - 2ab} . x^{n-1} dx - agex^{n-2} dx}{xx + 2bx + ge}}. \text{ Q.E.I.}$$

Hæc reſolutio inſtituenda eſt, ſi $n$ aut ſit $=$, aut $> 2$. Verum ſi eſſet $n = 1$, non duo, ſed unus tantum terminus in utraque fractione eliciendus eſſet per diviſionem, qui propter ſignorum contrarietatem elideretur; atque hoc modo formulæ propoſita ita reſolveretur, $\frac{1}{aa - 2ab + ge} . \overline{\frac{-a dx}{x+a} + \frac{axdx + gedx}{xx + 2bx + ge}}$.

15. For-

15. Formula continens duos factores reales ſecundi gradus, nempe $\frac{dx}{\overline{xx+2ax+fe}\,.\,\overline{xx+2bx+ge}}$ diſtribuitur in duas ſequentes

$$\frac{-fe+ge+2a.\overline{2a-2b}\,.\,dx+\overline{2a-2b}\,.\,xdx}{xx+2ax+fe}+$$

$$\frac{\overline{fe-ge-2b.\overline{2a-2b}}\,.\,dx-(\overline{2a+2b}\,.\,xdx}{xx+2bx+ge}$$, ſi ambæ dividantur per $\overline{fe+ge}^2+\overline{2age-2bfe}\,.\,\overline{2a-2b}$. Etenim ſi formulas ad eandem denominationem deducas, termini omnes ex contrarietate ſignorum elidentur excepto ultimo, cujus coefficiens $=\overline{fe-ge}^2+\overline{2age-2bfe}\,.\,\overline{2a-2b}$. Hoc theorema analytica methodo inveniri poteſt, ſed induſtria opus eſt, ne nimis longus, & moleſtus evadat calculus. Suppone formulam propoſitam in duas ſequentes eſſe diſtributam $\frac{Adx+Bxdx}{xx+2ax+fe}+\frac{Cdx+Dxdx}{xx+2bx+ge}$. Has ad eandem denominationem fac perducas, ut evadant

$$\frac{\begin{array}{l} Bx^3+2Bbx^3+\ \ Bgex+Age \\ Dx^3+\ \ Ax^2+2Abx\ +Cfe \\ \quad +2Dax^2+\ \ Dfex \\ \quad +\ \ Cx^2+2Cax \end{array}\ .\,dx}{\overline{xx+2ax+fe}\,.\,\overline{xx+2bx+ge}}$$

Ex comparatione cum propoſita quatuor naſcuntur æquationes.

I. $B+D=0$, II. $2Bb+A+2Da+C=0$. III. $Bge+2Ab+Dfe+2Ca=0$, IV. $Age+Cfe=1$. Primam ductam prius in $2b$, poſt in $2a$ detrahe ex ſecunda, ut habeas $-2Db+A+2Da+C=0$, $2Bb+A-2Ba+C=0$. Ex his habes $D=\frac{-A-C}{2a-2b}$, $B=\frac{A+C}{2a-2b}$. Hi valores $D$, $B$ ponendi ſunt in æquatione tertia, ut fiat $\frac{ge.\overline{A+C}}{2a-2b}+2Ab-\frac{fe.\overline{A+C}}{2a-2b}+2Ca=0$, ex qua fit $-C.\overline{fe-ge-2a.\overline{2a-2b}}=A.\overline{fe-ge-2b.\overline{2a-2b}}$. Ut brevitati conſulamus, ſupponamus $fe-ge-2a.\overline{2a-2b}=K$, $fe-ge-2b.\overline{2a-2b}=H$, ut fiat $-CK=AH$. Ex hac & quarta inveniemus $A=\frac{-K}{feH-geK}$, $C=\frac{H}{feH-geK}$, ex his collige $A+C=\frac{H-K}{feH-geK}$; atqui $H-K=\overline{2a-2b}^2$; ergo $A+C=\frac{\overline{2a-2b}^2}{feH-geK}$; ergo $D=\frac{-(2a-2b}{feH-geK}$, $B=\frac{2a-2b}{feH-geK}$; ergo formula propoſita dividitur in haſce

$-Kdx$

$$\frac{-Kdx+\overline{2a-2b}.xdx}{\overline{feH-geK}.\overline{xx+2ax+fe}}+\frac{Hdx-(\overline{2a-2b}.xdx}{\overline{feH-geK}.\overline{xx+2bx+ge}}.$$ Quum autem $feH-geK=\overline{fe-ge}^2+\overline{2age-2bfe}.\overline{2a-2b}$, ſubſtitutis valoribus $H$, $K$, & $feH-geK$ ex reſultant formulæ, quas ſupra invenimus.

16. Inutilis evadit methodus, quotieſcumque diviſor $\overline{fe-ge}^2+\overline{2age-2bfe}.\overline{2a-2b}=0$. Videamus, quænam conditio ex hac æqualitate in factores trinomios transferatur. Ea æqualitas in hanc mutatur $\overline{fe-ge}^2-4ab.\overline{ge+fe}=-4a^2ge-4b^2fe$. Utrique parti addatur $4fge^2$, & fiet $\overline{fe+ge}^2-4ab.\overline{ge+fe}=-4a^2ge-4b^2fe+4fge^2$. Addatur pariter $4a^2b^2$, & proveniet $\overline{fe+ge}^2-4ab.\overline{fe+ge}+4a^2b^2=4a^2b^2-4a^2ge-4b^2fe+4fge^2$. Extrahatur radix quadrata, fiet $fe+ge-2ab=\pm 2.\sqrt{a^2b^2-a^2ge-b^2fe+fge^2}=\pm 2\sqrt{a^2-fe}.\sqrt{b^2-ge}$. Tranſlato $fe+ge$, additoque utrobique $a^2+b^2$, naſcitur $a^2-2ab+b^2=a^2-fe+b^2-ge\pm 2\sqrt{a^2-fe}.\sqrt{b^2-ge}$, extracta radice quadrata $a-b=\pm\sqrt{a^2-fe}\pm\sqrt{b^2-ge}$; ergo $a\mp\sqrt{a^2-fe}=b\pm\sqrt{b^2-ge}$; atqui $a\mp\sqrt{a^2-fe}$ eſt radix unius factoris, $b\pm\sqrt{b^2-ge}$ eſt radix alterius; ergo duo factores debent habere unam ſaltem radicem æqualem. Hoc autem contingere poteſt, ubi radices fuerint reales, quo in caſu formula propoſita poterit in hanc formam diſponi $\frac{dx}{\overline{x+a}^2.\overline{x+b}.\overline{x+e}}$, de qua paullo infra. Si vero radices fuerint imaginariæ, non poteſt una radix unius trinomii eſſe æqualis uni radici alterius trinomii, niſi altera alteri æqualis fuerit. Nam ſit $b+\sqrt{b^2-ge}=a-\sqrt{a^2-fe}$; ergo quum realia realibus, imaginaria imaginariis æqualia ſint oporteat, erit $a=b$, $-\sqrt{a^2-fe}=\sqrt{b^2-ge}$; ergo etiam $a+\sqrt{a^2-fe}=b-\sqrt{b^2-ge}$, quæ formula oſtendit alias quoque radices æquales eſſe. Quum autem radices æquantur, trinomia ipſa æquentur neceſſe eſt, adeoque formula in hunc modum poteſt exprimi $\frac{dx}{\overline{xx+2ax+fe}^2}$, de qua antea diximus.

17. Quod num. 12, 14 obſervatum eſt in fractionibus, quarum denominator continet aut duos factores primi gradus, aut unum primi, & alterum ſecundi, idem applicandum fractionibus, quarum denominator duos factores ſecundi gradus complectitur, & demonſtrandum $\frac{x^n dx}{\overline{xx+2ax+fe}.\overline{xx+2bx+ge}}$ in duas

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reſolvi poſſe, in quibus numeratoris termini duos habeant exponentes $n-2$ $n-3$. Si formulas num. 15 multiplices per $x^n$, habebis formulam propoſitam reſolutam in duas, nempe

$$\frac{\overline{2a-2b}\,.\,x^{n+1}dx+\overline{ge-fe+2a\,.\,\overline{2a-2b}}\,.\,x^n dx}{xx+2ax+fe}+$$

$$\frac{\overline{2b-2a}\,.\,x^{n+1}dx+\overline{fe-ge-2b\,.\,\overline{2a-2b}}\,.\,x^n dx}{xx+2bx+ge}$$, dummodo utraque dividatur per quantitatem, quæ ibi ponitur. In utraque fiat diviſio, ut oriatur

$$\overline{2a-2b}\,.\,x^{n-1}dx+\overline{ge-fe}\,.\,x^{n-2}dx+\overline{2bfe-2age}\,.\,x^{n-3}dx$$

$$+\frac{2a\,.\,\overline{2age-2bfe}\,.\,x^{n-2}dx+fe\,.\,\overline{2age-2bfe}\,.\,x^{n-3}dx}{xx+2ax+fe}$$

$$+\overline{2b-2a}\,.\,x^{n-1}dx+\overline{fe-ge}\,.\,x^{n-2}dx+\overline{2age-2bfe}\,.\,x^{n-3}dx$$

$$+\frac{2b\,.\,\overline{2bfe-2age}\,.\,x^{n-2}dx+ge\,.\,\overline{2bfe-2age}\,.\,x^{n-3}dx}{xx+2bx+ge}$$

Termini omnes integri ex ſignorum contrarietate eliduntur, & reliquæ ſunt duæ fractiones, quæ quærebantur. Ita protrahenda eſt diviſio, ut in ſingulis fractionibus tres termini integri detegantur, ſi ſit $n=$, aut $>3$. Verum ſi $n=2$ duo tantum termini integri inveniendi ſunt per diviſionem, & ſi $n=1$ unus integer terminus ſolummodo; atque in utroque caſu numerator fractionum duobus terminis conſtabit hujuſce formæ $Ax\,dx+B\,dx$.

18. His demonſtratis detecta ſunt principia, per quæ oſtendamus omnes fractiones rationales aut abſolute, aut ſaltem per quadraturas circuli, & hyperbolæ integrabiles eſſe. Etenim ponamus primo diviſorem conſtare ex ſolis factoribus ſimplicibus. Si iſti forent omnes æquales, formula ita exponi poſſet $\dfrac{x^n dx}{\overline{x+a}^m}$, quæ in ſuperioribus numeris tractata eſt. Si vero factores adſint inæquales, ſumptis inæqualibus factoribus duobus, hanc formam induit fractio $\dfrac{x^n dx}{\overline{x+a}\,.\,\overline{x+b}\,.\,M}$. Quoniam ex num. 11 $\dfrac{dx}{\overline{x+a}\,.\,\overline{x+b}}=\dfrac{dx}{\overline{a-b}\,.\,\overline{x+b}}-\dfrac{dx}{\overline{a-b}\,.\,\overline{x+a}}$, facta multiplicatione per $\dfrac{x^n}{M}$ invenies $\dfrac{x^n dx}{\overline{x+a}\,.\,\overline{x+b}\,.\,M}=\dfrac{x^n dx}{\overline{a-b}\,.\,\overline{x+b}\,.\,M}-\dfrac{x^n}{\overline{a-b}\,.\,\overline{x+a}\,.\,M}$. Itaque formula diviſa eſt in duas, in quibus unus ex duobus inæqualibus ejectus eſt. Singulas eadem methodo pertracta, donec in denominatore factores inæquales non inveniantur, ſed vel omnes æquales ſint, vel unus tantum remaneat.

19. Eadem methodus tenenda eſt, ſi omnes factores reales fuerint ſecundi gradus. Etenim, ſi omnes fuerint æquales, conſtat ex ſuperioribus, eam poſitis qua-

quadraturis circuli & hyperbolæ integrari. Si vero inæquales ſint, aſſumantur duo factores inæquales, & fractio in hanc formam perducatur

$$\frac{x^n dx}{xx+2ax+fe\,.\,xx+2bx+ge\,.\,M}$$. Ex numero 15 habes

$$\frac{dx}{xx+2ax+fe\,.\,xx+2bx+ge}$$ diviſa in duas

$$\frac{Adx+Bxdx}{xx+2ax+fe}+\frac{Cdx+Dxdx}{xx+2bx+ge}$$. Valores conſtantium $A, B, C, D$ ibi expoſiti ſunt; ergo multiplicando per $\frac{x^n}{M}$ formula propoſita in has convertetur

$$\frac{Ax^n dx+Bx^{n+1}dx}{xx+2ax+fe\,.\,M}+\frac{Cx^n dx+Dx^{n+1}dx}{xx+2bx+ge\,.\,M}$$, in quibus unus ex diviſoribus inæqualibus abeſt. Hæ eadem methodo ſingulæ tractentur, donec aut factores omnes æquales ſint, aut unius ſuperſit factor, & formulæ omnes ex præcedentibus integrationem accipient ſaltem per quadraturas circuli, & hyperbolæ.

20. Demum diviſor fractionis contineat factores tum primi tum ſecundi gradus. Per methodos expoſitas formula diſtribui poteſt in plures, in quibus factores primi gradus, atque ſecundi æquales ſint, quæ fractiones ita ſcribuntur

$$\frac{x^n dx}{x+a\,.\,xx+2bx+ge\,.\,M}$$. Quoniam ex num. 13

$$\frac{dx}{x+a\,.\,xx+2bx+ge}=\frac{Adx}{x+a}+\frac{Bdx+Cxdx}{xx+2bx+ge}$$; $A, B, C$ determinatæ reperientur. Itaque, facta multiplicatione per $\frac{x^n}{M}$, fiet

$$\frac{x^n dx}{x+a\,.\,xx+2bx+ge\,.\,M}=\frac{Ax^n dx}{x+a\,.\,M}+\frac{Bx^n dx+Cx^{n+1}dx}{xx+2bx+ge\,.\,M}$$, in quarum prima deeſt unus factor ſecundi gradus, in altera primi. Singulæ eadem methodo tractentur, donec aut ſolos contineant factores primi, aut ſolos ſecundi gradus, in quibus demonſtratum eſt ſupra ſufficere ad integrationem obtinendam quadraturas circuli, & hyperbolæ.

21. Hæc ſatis ſunt ad demonſtrandum fractiones omnes rationales, quarum denominatores reſolubiles ſunt in factores rationales aut primi aut ſecundi gradus ad ſuimet integrationem ſolis circuli, & hyperbolæ quadraturis indigere. Verum ſi ſequaris methodum hactenus indicatam, longos moleſtoſque calculos opus eſſet inſtituere, ut fractiones ſingulæ reducerentur. Quare aliqua præmiſi ad demonſtrationem non neceſſaria, quæ tamen calculos poſſunt breviores efficere. Hanc ob rem oportet diſtinguere puras fractiones a ſpuriis. Puræ fractiones illæ dicuntur, in quibus exponens numeratoris minor eſt exponente denominatoris; eæ, in quibus exponens numeratoris aut æqualis, aut major eſt exponente denominatoris dicuntur ſpuriæ. Spuriæ nullo negotio ad puras reducun-

ſuntur, ſi diviſio inſtituatur, donec exponens numeratoris minor ſit exponente denominatoris. Ita fractio ſpuria convertitur in unum aut plures terminos integros, & in fractionem puram. Quæ dicemus deinceps, ad ſolas fractiones veras pertinebunt, nam ſpurias redactas fuiſſe ad puras ſupponemus.

22. Si denominator fractionis puræ conſtet ex pluribus factoribus ſimplicibus omnibus inæqualibus, quotus eſt factorum numerus, in tot fractiones dividetur, quarum ſingulæ hanc formam habebunt $\frac{A\,dx}{x+a}$ exiſtente $A$ quantitate conſtante. Sit factorum numerus $=m$; ergo, quum fractio pura ſit, maximum exponens numeratoris erit ad ſummum $m-1$. Ergo ex num. 12 dividetur fractio in duas, in quibus numerus factorum $=m-1$, numeratoris autem exponens ad ſummum erit $m-2$. Hæ in alias dividentur, quæ numerum factorum continebunt $m-2$, exiſtente numeratoris exponente ad ſummum $m-3$. Producatur eodem modo analyſis, donec ad unum factorem deveniamus, quo in caſu exponens numeratoris $=0$. Ergo evidens eſt fractionem dividi in plures ſimplices, in quibus quantitas multiplicans differentiale eſt conſtans. Quod ſi exponens numeratoris fractionis puræ eſſet $<m-1$, citius ad numeratorem conſtantem pervenires, & ſolutio per num. 11. deinceps abſolveretur.

23. Quod ſi factor aliquis ſimplex habet alios ſibi æquales, & æqualium factorum numerus $=m$, ajo formulam reſolvi in factorem hujus formæ

$$\frac{Ax^{m-1}+Bx^{m-2}\ldots\ldots H}{\overline{x+a}^{m}}.\,dx.$$

Hujus rei demonſtratio, licet difficilis ſit, tamen ex ſuperioribus ita perficitur. Pono primum unicum factorem, continere plures æquales $x+a$, reliquos omnes inæquales. Ex methodo numeri ſuperioris fractio in plures reſolvetur formæ $\frac{Ax^{m}+Bx^{m-1}\ldots H}{\overline{x+b}\,.\,\overline{x+a}^{m}}.\,dx$ in quibus unus eſt tantum factor inæqualis in diviſore, qui per æquales multiplicatur. Quare, qui has in plures reſolvet, etiam datam reſolvet. Specta factores inæquales $x+b$, $x+a$, veluti prætermiſſo $\overline{x+a}^{m-1}$. Fractionem ita acceptam in duas divide ex num. 12, quibus deinde reſtitue factorem $\overline{x+a}^{m-1}$. Formulæ, quæ provenient habebunt maximum exponentem numeratoris $m-1$; diviſor autem erit in una $\overline{x+b}\,.\,\overline{x+a}^{m-1}$, in altera $\overline{x+a}^{m}$. Hæc ſecunda habet formam, quæ probanda eſt. Prima eadem methodo tractetur, & duæ novæ orientur, in quibus numeratoris exponens $=m-2$, diviſores autem erunt $\overline{x+b}\,.\,\overline{x+a}^{m-2}$, $\overline{x+a}^{m-1}$. Hæc ſecunda ſi multiplicetur & dividatur per $x+a$, acquiret formam, quæ probanda eſt. Prima eadem methodo tractetur, & formulæ, quibus eſt diviſor $x+a$ poteſtas aliqua, multiplicentur, & dividantur per congruam ejuſdem poteſtatem ut in diviſore exiſtat $\overline{x+a}^{m}$, & tandem propoſitam formulam in duas reſolutas habebis, quarum una habet numeratorem conſtantem, diviſorem $x+b$, altera propoſitam formam induet: nam quæ eundem diviſorem habent in unam fractionem redigi debent.

24. Si plures exiſtant factores, habentes expoſitam conditionem, licet demonſtratio ſit difficilior, tamen ex principiis iiſdem peragitur. Sit enim primo fractio $\frac{x^{m+n-1}\,dx}{\overline{x+a}^m \, . \, \overline{x+b}^n}$. Spectentur duo factores inæquales $x+a$, $x+b$, & ex num. 12. in duas divide, quarum una habebit diviſorem $\overline{x+a}^{m-1} . \overline{x+b}^n$, altera $\overline{x+a}^m . \overline{x+b}^{n-1}$. Ita eadem methodo tractentur, donec deveniamus ad factorem, qui non habet æqualem: quod ubi contingit, ex num. ſuperiore demonſtratio conficietur. Quod ſi tres fuerint factores habentes æquales, methodus ad duas deducet. Quapropter apertum eſt ſingulis factoribus habentibus æquales, fractionem reſpondere ejus formæ, quam num. ſuperiore propoſui.

25. Quamquam ſuperior demonſtratio ſit genuina; tamen ſi per methodum uſurpatam numeratores fractionum inveniendi eſſent, analyſis longiſſima evaderet, ac plena faſtidii. Quare facilior methodus explicanda, per quam quæſiti numeratores inveniantur. Ac primo de peculiari fractione, quæ oritur a factore ſimplici, cui nullus eſt æqualis. Sit itaque fractio pura $\frac{M\,dx}{\overline{x+a}.N}$, in qua $M$, $N$ ſunt functiones integræ variabilis $x$; quæritur numerator conſtans fractionis $\frac{A\,dx}{x+a}$, qui provenit a factore $x+a$, qui careat æquali. Sit alia fractio huic adjungenda $\frac{S\,dx}{N}$, ut ſit $\frac{M\,dx}{\overline{x+a}.N} = \frac{A\,dx}{x+a} + \frac{S\,dx}{N}$. Manifeſtum eſt $S$ eſſe functionem integram ipſius $x$, quia ſi eſſet fracta, redactis duabus ad eandem denominationem, haberetur alius diviſor præter $x+a$, & $N$, quod eſt contra hypotheſim. Redigantur ad eandem denominationem, ut fiat $\frac{M\,dx}{\overline{x+a}.N} = \frac{AN\,dx + \overline{x+a}.S\,dx}{\overline{x+a}.N}$; ergo $M = AN + \overline{x+a}.S$, ſive $\frac{M-AN}{x+a} = S$. Quoniam $S$ eſt functio rationalis, & integra, manifeſtum eſt $M-AN$ fore diviſibilem per $x+a$; ergo poſita $x=-a$ fiet $M-AN=0$, ſive $A=\frac{M}{N}$, ſi in $M$, $N$ pro $x$ ſcribatur $-a$.

26. Methodum exemplo uno aut altero illuſtremus. Sit fractio pura $\frac{ax\,dx}{\overline{x-2a}.\overline{xx-aa}}$, & invenienda ſit fractio ſimplex conveniens factori $x-2a$. Sit ea $\frac{A\,dx}{x-2a}$, & quæ convenit diviſori reſiduo $xx-aa$, ſit $\frac{S\,dx}{xx-aa}$, ut fiat $\frac{ax\,dx}{\overline{x-2a}.\overline{xx-aa}} = \frac{A\,dx}{x-2a} + \frac{S\,dx}{xx-aa}$, ut ſit $M=ax$, $N=xx-aa$; ergo quum debeat $A=\frac{M}{N}$ poſita pro $x$ in his $2a$, erit $A=\frac{2aa}{3aa}=\frac{2}{3}$; ergo fractio requiſita erit $\frac{2}{3}.\frac{dx}{x-2a}$. Simili modo reperietur fractio conve-

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veniens factori $x+a$, quæ ſit $\frac{A\,dx}{x+a}$, alia fractione exiſtente $\frac{S\,dx}{xx-3ax+2aa}$; ergo $M=ax$, $N=xx-3ax+2aa$. In his ponatur $x=-a$, & fiet $M=-aa$, $N=6aa$; ergo $A=\frac{M}{N}=\frac{-aa}{6aa}=\frac{-1}{6}$; adeoque fractio conveniens factori $x+a$ eſt hæc $\frac{-dx}{6.\overline{x+a}}$. Si optas fractionem convenientem tertio factori $x-a$, ea ſit $=\frac{A\,dx}{x-a}$, alia fractione exiſtente $\frac{S\,dx}{xx-ax-2aa}$; ergo $M=ax$, $N=xx-ax-2aa$. Fac $x=a$, ut ſit $M=aa$, $N=aa-aa-2aa=-2aa$; ergo $A=\frac{-1}{2}$; ergo fractio $\frac{-dx}{2.\overline{x-a}}$. Reapſe tres fractiones inventæ, ſi ad eandem denominationem reducantur propoſitam reſtituunt.

27. Exemplum alterum ſufficiat fractio $\frac{aa\,dx}{\overline{x-a}.\overline{xx+aa}}$. Fractio ſimplex conveniens factori $x-a$ ſit $\frac{A\,dx}{x-a}$, reliqua exiſtente $\frac{S\,dx}{xx+aa}$; ergo $M=aa$, $N=xx+aa$, in quibus fiat $x=a$, & erit $N=2aa$; ergo $A=\frac{M}{N}=\frac{1}{2}$; ergo fractio ſimplex $=\frac{dx}{2.\overline{x-a}}$. Reliqua autem fractio erit $\frac{-dx.\overline{x+a}}{2.\overline{xx+aa}}$.

28. Hæc methodus docet, quomodo per ſimplicem calculum inveniatur fractio ſimplex conveniens factori ſimplici, cui nullus æqualis ſit. Sed ſi factor ſimplex habeat æquales, methodus aliquanto ſit difficilior. Hanc ut clare explicem, gradatim progredior, & incipio a caſu ubi æquales factores ſint duo. Sit igitur fractio pura $\frac{M\,dx}{N.\overline{x+a}^2}$, in qua $N$ non cotineat factorem $x+a$. Fractio conveniens factori $\overline{x+a}^2$ ſit $\frac{Ax\,dx+B\,dx}{\overline{x+a}^2}$, alia exiſtente $\frac{S\,dx}{N}$; ergo $\frac{M}{\overline{x+a}^2.N}=\frac{Ax+B}{\overline{x+a}^2}+\frac{S}{N}$; ergo $\frac{M-N.\overline{Ax+B}}{\overline{x+a}^2}=S$. Quoniam $S$ debet eſſe functio integra, neceſſario quantitas $M-N.\overline{Ax+B}$ bis poterit dividi per $x+a$. Quare poſita in ea $x=-a$ fiet $=o$, & determinabitur $B$, cujus valor erit $\frac{M}{N}-Ax=B$ exiſtente $x=-a$. Eadem quantitas, ſubſtituto valore $B$, dividatur primum per $x+a$. Quoniam autem quotiens iſtius diviſionis remanet adhuc diviſibilis per $x+a$, poſita iterum $x=-a$, fiet $=o$; ergo nova orietur æquatio, quæ juncta ſuperiori utrumque valorem $A$, $B$ determinabit; quod erat faciendum.

29. Exempla methodum illuſtrabunt. Propoſita fractione vera $\frac{ax - 2xx \,.\, dx}{\overline{x+a}^2 \,.\, \overline{xx - 2aa}}$, quæritur fractio reſpondens factori quadrato $\overline{x+a}^2$. Primo intuitu colliges $M = ax - 2xx$, $N = xx - 2aa$. Quantitas igitur $M - N \,.\, \overline{Ax + B}$, per quam valores $A$, $B$ ſunt definiendi, evadit hujuſmodi $ax - 2xx + 2aa - xx \,.\, \overline{Ax + B}$. Pone in hac $x = -a$, & fiet $-3aa + aa \,.\, \overline{B - Aa} = 0$, ergo $B = 3 + Aa$. Hunc valorem in eadem quantitate ſtatue, ut fiat $ax - 2xx + 2aa - xx \,.\, \overline{Ax + 3 + Aa}$, quæ item debet eſſe diviſibilis per $x + a$. Eam ita diſpone $ax - 5xx + 6aa + 2aa - xx \,.\, A \,.\, \overline{x+a}$, ſive $6a - 5x \,.\, \overline{x+a} + 2aa - xx \,.\, A \,.\, \overline{x+a}$. Hanc divide per $x + a$, ut oriatur $6a - 5x + \overline{2aa - xx} \,.\, A$. Facta iterum $x = -a$, erit $11a + aaA = 0$; igitur $A = -\frac{11}{a}$, & $B = -8$; quibus inventis fractio quæſita provenit $= \frac{-11xdx - 8adx}{a \,.\, \overline{x+a}^2}$.

30. Alterum exemplum ſufficiat fractio $\frac{\overline{x^4 + a^4} \,.\, dx}{\overline{x - 2a}^2 \,.\, \overline{x^3 + 2a^3}}$, ex qua quæratur fractio conveniens factori duplici $\overline{x - 2a}^2$, cujus hæc eſt forma $\frac{Axdx + Bdx}{\overline{x - 2a}^2}$. Statim vides $M = x^4 + a^4$, $N = x^3 + 2a^3$; ergo quantitas determinans valores $A$, $B$ ſcilicet $M - N \,.\, \overline{Ax + B}$, erit $x^4 + a^4 - (\overline{x^3 + 2a^3} \,.\, \overline{Ax + B}$. Pone $x = 2a$, ut habeas $17a^4 - 10a^3 \,.\, \overline{2Aa + B} = 0$; ergo $B = \frac{17}{10}a - 2Aa$. Hunc valorem ſubſtitue in ſuperiore quantitate, ut oriatur $x^4 + a^4 - (x^3 + 2a^3) \,.\, Ax + \frac{17a}{10} - 2Aa$. Hanc ita diſpone $\overline{x^4 - \frac{17ax^3}{10} - \frac{12a^4}{5}} - (x^3 + 2a^3) \,.\, A \,.\, \overline{x - 2a}$, ſive

$$x^3 + \frac{3ax^2}{10} + \frac{3a^2x + 6a^3}{5} \,.\, \overline{x - 2a} - (x^3 + 2a^3) \,.\, A \,.\, \overline{x - 2a},$$

quæ diviſa per $x - 2a$ ſufficit $x^3 + \frac{3ax^2}{10} + \frac{3a^2x + 6a^3}{5} - (x^3 + 2a^3) \,.\, A$. Pone $x = 2a$, ut habeas $8 + \frac{18}{5} - 10 \,.\, A = 0$, ſive $A = \frac{29}{25}$, & $B = \frac{-31}{50}a$; ergo fractio quæſita erit $\frac{\frac{29}{25}xdx - \frac{31}{50}adx}{\overline{x - 2a}^2}$.

31. Methodus per ſeſe oſtendit, quo pacto inſtituendus ſit calculus, ſi factor fuerit $\overline{x+a}^3$, in quo caſu fractio ſimplex habebit hanc formam $\frac{Ax^2+Bx+C.dx}{\overline{x+a}^3}$. Nam ſi numerator multiplicans $dx$ in propoſita $=M$, & $N$ ſit quantitas, quæ ducta in $\overline{x+a}^3$ det diviſorem, invenies, & probabis $M-N.Ax^2+Bx+C$ eſſe diviſibilem per $\overline{x+a}^3$. Pone in illa $x=-a$, & ſtatue $=o$. Hæc æquatio definiet valorem $C$ datum per $A$, $B$. Hanc pone in inventa quantitate, eamdemque divide per $x+a$, factaque $x=-a$ pone $=o$. Per hanc æquationem determinabis valorem $B$ datum per $A$, quem ſubſtitue, & iterum inſtitue diviſionem per $x+a$. Quotientem, facta $x=-a$, fac $=o$, & invenies valorem $A$. Demum regrediendo cognoſces valores $B$, $C$. Methodus eſt eadem ſi factores æquales fuerint quatuor, quinque &c. Eædem enim faciendæ ſunt operationes, ſed ſæpius iterandæ.

32. Ad methodum oſtendendam propono exemplum $\frac{a^6dx}{\overline{x-a}^5.x}$, ut ſit $M=a^6$, $N=x$, & quantitas, quæ probatur diviſibilis per $\overline{x-a}^5$, ſive $M-N.Ax^4+Bx^3+Cx^2+Dx+E$ erit hujuſmodi

I. $-Ax^5-Bx^4-Cx^3-Dx^2-Ex+a^6$. Facta $x=a$, invenies

$E=-Aa^4-Ba^3-Ca^2-Da+a^5$. Pone in formula prima hunc valorem, ut fiat

$$-Ax^5-Bx^4-Cx^3-Dx^2+\left.\begin{matrix}+Aa^4\\+Ba^3\\+Ca^2\\+Da\\-a^5\end{matrix}\right\}.x+a^6$$

Hanc divide per $x-a$, ut oriatur

$$\text{II.}\;-Ax^4\;\begin{matrix}-Aa\\-B\end{matrix}\;.x^3\;\begin{matrix}-Aa^2\\-Ba\\-C\end{matrix}\;.x^2\;\begin{matrix}-Aa^3\\-Ba^2\\-Ca\\-D\end{matrix}\;.x\;-a^5$$

Pone $x=a$, & invenies $-4Aa^4-3Ba^3-2Ca^2-Da-a^5=o$; ergo $D=-4Aa^3-3Ba^2-2Ca-a^4$; quem valorem colloca in II. ut habeas

$$-Ax^4\;\begin{matrix}-Aa\\-B\end{matrix}\;.x^3\;\begin{matrix}-Aa^2\\-Ba\\-C\end{matrix}\;.x^2\;\begin{matrix}+3Aa^3\\+2Ba^2\\+Ca\\+a^4\end{matrix}\;.x\;-a^5$$

Hanc

Hanc divide per $x - a$, ut proveniat

III. $\begin{array}{l} -Ax^3 - 2Aa \\ \quad\;\; - B \end{array} \cdot x^2 \begin{array}{l} -3Aa^2 \\ -2Ba \\ -C \end{array} \cdot x + a^4$. In hac pone $x = a$, ut oriatur æquatio $-6Aa^3 - 3Ba^2 - Ca + a^4 = 0$; ergo erit $C = -6Aa^2 - 3Ba + a^3$, quem colloca in III. ut oriatur

$$\begin{array}{l} -Ax^3 - 2Aa \\ \quad\;\; - B \end{array} \cdot x^2 \begin{array}{l} +3Aa^2 \\ +Ba \\ -a^3 \end{array} \cdot x + a^4,$$ quam divide per $x - a$, ut fiat

IV.) $\begin{array}{l} -Ax^2 - 3Aa \\ \quad\;\; - B \end{array} \cdot x - a^3$. Pone $x = a$, ut reſultet æquatio $-4Aa^2 - Ba - a^3 = 0$; ergo $B = -4Aa - a^2$, quem colloca in IV. ut fiat $\begin{array}{l} -Ax^2 + Aa \\ \quad\;\; + aa \end{array} \cdot x - a^3$, quam divide per $x - a$, ut reſultet

V. $-Ax + a^2$. Pone $x = a$, & invenies $A = a$. Igitur regrediendo, $B = -5a^2$, $C = 10a^3$, $D = -10a^4$, $E = 5a^5$; ergo formula quæſita erit $\frac{ax^4 - 5a^2x^3 + 10a^3x^2 - 10a^4x + 5a^5 \cdot dx}{\overline{x-a}^5}$. Quod ſi per methodum num. 27 requiras fractionem convenientem factori $x$, invenies $-\frac{adx}{x}$. Reapſe iſtæ fractiones ad eundem denominatorem redactæ, propoſitam reſtituunt.

33. Si fractio pura habeat diviſorem ſecundi gradus $xx + 2ax + ge$, cui nullus æqualis ſit, ex num. 14, & 17 poſſem demonſtrare eam ita poſse reſolvi, ut fractio conveniens diviſori ſecundi gradus habeat hanc formam $\frac{Axdx + Bdx}{x^2 + 2ax + ge}$ exiſtentibus $A$, $B$ quantitatibus datis, quæ independent ab $x$. Quod ſi factor habeat æquales, ut exprimi poſsit per $\overline{x^2 + 2ax + ge}^m$; fractio eidem conveniens erit $\frac{Ax^{2m-1} + Bx^{2m-2} \ldots H \cdot dx}{\overline{x^2 + 2ax + ge}^m}$. Quantitates conſtantes $A$, $B$ &c. determinari poſſent eadem methodo, qua in fractionibus ſimplicibus uſi ſumus. Verum quoniam ex una parte calculus evadit longus & difficilis, ex altera non vitantur quantitates imaginariæ, quæ oriuntur ex reſolutione factoris quadratici; idcirco utilius judico eum reducere ad duos factores ſimplices, & ex numeris ſuperioribus invenire fractiones illis convenientes, tametſi imaginaria contineant, quia, fractiones inventas ſi ad eandem denominationem perducas, & in unam jungas, imaginaria abibunt, & fractio conveniens factori ſecundi gradus determinabitur. Si factor ſecundi gradus reſolvi poſſit in duos

duos ſimplices reales; res difficultate caret. Quare ſpecto factorem, qui reſolvatur in duos factores imaginarios $x+a+b\sqrt{-1}$, $x+a-b\sqrt{-1}$, ſeu facta $x+a=z$ in duos $z+b\sqrt{-1}$, $z-b\sqrt{-1}$. Demonſtrandum eſt fractiones duas, quæ hiſce factoribus conveniunt, vel habeant vel non habeant æquales, ſimul junctas realem fractionem producere. Ut hoc præſtem, aliquot theoremata ſpectantia ad quantitates imaginarias præmitto.

34. Per terminos imaginarios intelligo reales illos terminos, qui ducuntur in $\sqrt{-1}$, quæ imaginaria eſt. $P$, $\overset{I}{P}$ denotant formulas duas in quibus termini reales habent eadem ſigna, imaginarii diverſa. Si $P$, $\overset{I}{P}$ multiplicentur ambæ per $\sqrt{-1}$, orientur formulæ $P.\sqrt{-1}$, $\overset{I}{P}.\sqrt{-1}$ in quibus termini reales præditi ſunt ſignis diverſis, imaginarii iiſdem. Nam termini reales per hanc multiplicationem evadunt imaginarii, & retinent eadem ſigna, & termini imaginarii evadunt reales, & ſigna aut in utraque formula retinent eadem, aut in utraque mutant. Quapropter ſi alterius tantum ſigna mutentur, ut fiant $P.\sqrt{-1}$, $-\overset{I}{P}\sqrt{-1}$, formulæ exurgent, in quibus eadem ſigna afficiunt terminos reales, diverſa imaginarios. Idem dicendum eſſe conſtat, ſi multiplicatio fiat per $a\sqrt{-1}$. Quamobrem ſi ex formulis altera ducatur in $a\sqrt{-1}$, altera in $-a\sqrt{-1}$, eandem proprietatem conſervabunt, ut identitas ſignorum ſit in terminis realibus, contrarietas in imaginariis.

35. Sint formulæ $P$, $\overset{I}{P}$ functiones ejuſdem $x$, atque in una pro $x$ ſubſtitue $-a\sqrt{-1}$, in altera $+a\sqrt{-1}$, provenient novæ formulæ habentes eadem ſigna in terminis realibus, diverſa in imaginariis. Nam ſigna terminorum non continentium $x$ nihil mutantur. Qui continent $x$ vel reales ſunt, vel imaginarii. Sint primo reales. Si exponens $x$ eſt numerus par, conſtat tam $-a\sqrt{-1}$, quam $+a\sqrt{-1}$ elatas ad parem poteſtatem, evadere reales & eodem affici ſigno: ergo termini iſti remanent reales, & eodem ſigno afficientur. Si exponens eſt numerus impar, perſpicuum eſt $-a\sqrt{-1}$, $+a\sqrt{-1}$ ad hanc poteſtatem elatas remanere imaginarias, & diverſis ſignis affici: igitur peracta ſubſtitutione termini & imaginarii evadunt, & diverſis ſignis affecti. Sint deinde termini continentes $x$ imaginarii. Si exponens eſt par, $-a\sqrt{-1}$, $+a\sqrt{-1}$ fiunt reales, eiſdemque ſignis affectæ; ergo facta ſubſtitutione termini remanent imaginarii, & quum antea diverſa haberent ſigna, nunc quoque habebunt; ſi vero exponens $x$ eſt impar, $-a\sqrt{-1}$, $+a\sqrt{-1}$ ad hanc poteſtatem elatæ remanent imaginariæ, & diverſa ſigna recipiunt; ergo cum termini imaginarii haberent antea ſigna diverſa, convertuntur in reales affectas iiſdem.

36. Si $P$, $\overset{I}{P}$ invicem multiplicentur, productum exurgit reale. Nam in multiplicatione, qui termini producuntur ex ductu realium in reales, imaginariorum in imaginarios, omnes ſunt reales, qui vero oriuntur ex ductu realium in imaginarios, remanent quidem imaginarii, ſed ob contrarietatem ſignorum eliduntur. Idem dicas ſi multiplicentur $P$, $-\overset{I}{P}$.

37. Sint $Q$, $\overset{I}{Q}$ aliæ quantitates eadem affectæ conditione ac $P$, $\overset{I}{P}$: ajo $PQ$,

$\overset{I}{P}\overset{I}{Q}$

$P'Q'$ habere eadem ſigna in realibus terminis, diverſa in imaginariis. Nam, qui termini naſcuntur ex ductu realis in realem, & reales ſunt, & iiſdem ſignis affecti; qui vero oriuntur ex ductu imaginarii in imaginarium, fiunt reales, & cum ambo utrobique ſigna diverſa habeant, per multiplicationem eadem ſigna apparebunt. Demum, qui oriuntur ex ductu realis, qui utrobique idem habet ſignum, in imaginarium, qui ſignum habet contrarium, contrarietas ſignorum conſervabitur, ſicuti conſervatur imaginarium. Si conditio requiſita exiſteret in formulis $P$, $P'$, at non in $Q$, $Q'$, in quibus licet iidem habeantur termini, tamen aut aliquis ex realibus diverſo ſigno, aut aliquis ex imaginariis eodem ſigno afficeretur, $PQ'$, $P'Q$ evidens eſt non poſſe habere conditionem, ut omnes termini reales idem ſignum, imaginarij diverſum obtineant.

38. Si $P$, $P'$ dividi poſſint per $Q$, $Q'$ reſpective, & in utriſque conditio habeatur, in quotientibus eadem conditio remanebit, ut reales idem ſignum recipiant, imaginarii diverſum. Nam quum $P$ dividi poſſit per $Q$, ita exprimatur $SQ$: ſimiliter fiat $P' = S'Q'$. Si $S$, $S'$ conditionem non haberent, facta multiplicatione per $Q$, $Q'$ prædicta conditione carerent: ſed non carent: ergo neque carent $S$, $S'$: ſed facta diviſione quotientes, ſunt $S$, $S'$: patet ergo quod erat demonſtrandum.

39. Si $P$, $P'$ ejiſdem terminis componantur, ſed ita, ut in realibus ſigna ſint eadem, in imaginariis diverſa, erit $P + P'$ quantitas realis, quia reales termini duplicantur, imaginarii ex contrarietate ſignorum eliduntur. Item $P.\sqrt{-1} - P'.\sqrt{-1}$ eſt quantitas realis, quia ex dictis $P.\sqrt{-1}$, $-P'.\sqrt{-1}$ habent reales terminos affectos iiſdem ſignis, imaginarios diverſis. At $P - P'$, aut $P\sqrt{-1} + P'.\sqrt{-1}$ conſtant ſolis terminis imaginariis, ac proinde reales fierent, ſi aut multiplicarentur, aut dividerentur per $\sqrt{-1}$.

40. Idem dictum volo de fractionibus, quarum numeratores ac denominatores eadem proprietate donantur. Ita $\frac{P}{Q} + \frac{P'}{Q'}$ eſt realis; reducantur enim ad eandem denominationem, ut fiat $\frac{PQ' + P'Q}{QQ'}$. Jam vero $QQ'$ eſt productum reale; $PQ'$, $P'Q$ ſunt formulæ, in quibus reales termini iiſdem ſignis afficiuntur, imaginarii diverſis; igitur tum numerator, tum denominator realis effectus eſt. Realis pariter eſt $\frac{P.\sqrt{-1}}{Q} - \frac{P'.\sqrt{-1}}{Q'}$ quia redactis terminis ad eandem denominationem fit $\frac{PQ'.\sqrt{-1} - P'Q\sqrt{-1}}{QQ'}$, quam conſtat eſſe realem.

At

At $\frac{P}{Q} - \frac{\overset{1}{P}}{\overset{1}{Q}}$, aut $\frac{P.\sqrt{-1}}{Q} + \frac{\overset{1}{P}\sqrt{-1}}{\overset{1}{Q}}$ ita imaginariæ ſunt, ut facta multiplicatione ac diviſione per $\sqrt{-1}$, evadant reales.

41. Venio ad demonſtrationem. Sit fractio pura $\frac{M\,dz}{\overline{z+b.\sqrt{-1}}.\overline{z-1.\sqrt{-1}}.N}$; demonſtrandum mihi propono, fractiones ſimplices reſpondentes factoribus $z+b.\sqrt{-1}$, $z-b\sqrt{-1}$ ita eſſe affectas, ut ſi ſimul accipiantur, & ad eandem denominationem reducantur, exhibeant fractionem, realem carentem ſignis imaginariis. Fractiones iſtæ ſint $\frac{A\,dz}{z+b.\sqrt{-1}}$, $\frac{\overset{1}{A}\,dz}{z-b.\sqrt{-1}}$. Ex num. 25 conſtat formulam, ex qua colligi debet valor $A$ eſſe $\overline{M-AN}.\overline{z-b\sqrt{-1}}$. Pro $z$ ſcribe $-b\sqrt{-1}$, & hac peracta ſubſtitutione quantitates $M$, $N$ evadant $P$, $Q$, & habebis

$$P + 2AQ.b\sqrt{-1} = 0:$$

eam vero, ex qua colligitur valor $\overset{1}{A}$, eſſe $\overline{\overset{1}{M}-\overset{1}{A}\overset{1}{N}}.\overline{z+b\sqrt{-1}}$. Pro $z$ ſcribe $+b\sqrt{-1}$, & quantitates $\overset{1}{M}$, $\overset{1}{N}$ fiant $\overset{1}{P}$, $\overset{1}{Q}$, ut habeas

$$\overset{1}{P} - 2\overset{1}{A}\overset{1}{Q}\,b\sqrt{-1} = 0.$$

Palam eſt per hoc tantum diferre quantitates $P$, $\overset{1}{P}$, item $Q$, $\overset{1}{Q}$, quod in ipſis termini reales præditi ſint iiſdem ſignis, imaginarii autem, ſeu qui multiplicantur per $\sqrt{-1}$, habere præfixa ſigna oppoſita. Ex duabus æquationibus inventis determina valores $A$, $\overset{1}{A}$ hoc modo

$$A = \frac{-P}{2bQ.\sqrt{-1}} = \frac{P.\sqrt{-1}}{2bQ} \quad\Big|\quad \overset{1}{A} = \frac{\overset{1}{P}}{2b\overset{1}{Q}.\sqrt{-1}} = \frac{-\overset{1}{P}.\sqrt{-1}}{2b\overset{1}{Q}}.$$

Quoniam $P$, $\overset{1}{P}$ habent terminos reales iiſdem ſignis affectos, imaginarios diverſis; $P$, $-\overset{1}{P}$ habebunt terminos reales affectos diverſis ſignis, imaginarios iiſdem; ergo $P\sqrt{-1}$, $-\overset{1}{P}\sqrt{-1}$ habebunt eadem ſigna in terminis realibus, diverſa in imaginariis: igitur etiam $A$, $\overset{1}{A}$ continebunt terminos reales iiſdem ſignis affectos, imaginarios diverſis. Quapropter fractiones $\frac{A\,dz}{z+b\sqrt{-1}}$, $\frac{\overset{1}{A}\,dz}{z-b\sqrt{-1}}$ in unam ſummam collectæ coaleſcent in fractionem realem.

42. Progredior ad fractionem puram $\frac{M\,dz}{\overline{z+b\sqrt{-1}}^{2}.\overline{z-b\sqrt{-1}}^{2}.N}$, in qua ſinguli factores ſimplices imaginarii ad ſecundam poteſtatem evehuntur. Fractiones illis reſpondentes ſint $\frac{\overline{Az+B}.dz}{\overline{z+b.\sqrt{-1}}^{2}}$ $\frac{\overline{\overset{1}{A}z+\overset{1}{B}}.dz}{\overline{z-b.\sqrt{-1}}^{2}}$.

Formulæ, per quas ex numero 28 debent determinari valores conſtantium, ſunt

$M - N\,.\,\overline{z - b\,.\sqrt{-1}}^{2}\,.\,\overline{Az + B}$

Pone $z = -b\sqrt{-1}$, & $M$, $N$ evadant $P$, $Q$, ut fiat æquatio

$P + 4bbQ\,.\,\overline{(-Ab\sqrt{-1} + B} = 0$,

ſive $B = \frac{-P}{4bbQ} + Ab\sqrt{-1}$.

$\overset{1}{M} - \overset{1}{N}\,.\,\overline{z + b\sqrt{-1}}^{2}\,.\,\overline{\overset{1}{A}z + \overset{1}{B}}$

Fac $z = b\sqrt{-1}$, & $\overset{1}{M}$, $\overset{1}{N}$ evadant $\overset{1}{P}\,.\,\overset{1}{Q}$, ut oriatur æquatio

$\overset{1}{P} + 4bb\overset{1}{Q}\,.\,\overline{\overset{1}{A}b\sqrt{-1} + \overset{1}{B}} = 0$,

ſive $\overset{1}{B} = \frac{-\overset{1}{P}}{4bb\overset{1}{Q}} - \overset{1}{A}b\sqrt{-1}$.

Ex duabus æquationibus perſpicuum eſt, tum quantitates, quæ multiplicant $A$, $\overset{1}{A}$, tum eas, quæ non multiplicant, in terminis realibus præditas eſſe ſignis iiſdem, ſecus in imaginariis. Valores $B$, $\overset{1}{B}$ ſubſtituantur in primis formulis ut evadant

$$M - N\,.\,\overline{z - b\sqrt{-1}}^{2}\,.\,\overline{Az + Ab\sqrt{-1} \frac{-P}{4bbQ}}\,;\quad \overset{1}{M} - \overset{1}{N}\,.\,\overline{z + b\sqrt{-1}}^{2}\,.\,\overline{\overset{1}{A}z - \overset{1}{A}b\sqrt{-1}\ \frac{-\overset{1}{P}}{4bb\overset{1}{Q}}}$$

ſive

$$M + \frac{NQ}{4bbQ}\,.\,\overline{z - b\sqrt{-1}}^{2} - NA\,.\,\overline{z - b\sqrt{-1}}^{2}\,.\,\overline{z + b\sqrt{-1}}$$

$$\overset{1}{M} + \frac{\overset{1}{N}\overset{1}{P}}{4bb\overset{1}{Q}}\,.\,\overline{z + b\sqrt{-1}}^{2} - \overset{1}{N}\overset{1}{A}\,.\,\overline{z + b\sqrt{-1}}^{2}\,.\,\overline{z - b\sqrt{-1}}$$

Prima ex his debet eſſe diviſibilis per $\overline{z + b\sqrt{-1}}$, ſecunda per $\overline{z - b\sqrt{-1}}$.

Quare ponatur $M + \frac{NP}{4bbQ}\,.\,\overline{z - b\sqrt{-1}}^{2} = R\,.\,\overline{z + b\sqrt{-1}}$,

$\overset{1}{M} + \frac{\overset{1}{N}\overset{1}{P}}{4bbQ}\,.\,\overline{z + b\sqrt{-1}}^{2} = \overset{1}{R}\,.\,\overline{z - b\sqrt{-1}}$. Quoniam harum æquationum quantitates omnes habent terminos reales iiſdem ſignis affectos, imaginarios diverſis, etiam $R$, $\overset{1}{R}$ eadem erunt conditione donatæ. Itaque factis diviſionibus prodibunt

$R - NA\,.\,\overline{z - b\sqrt{-1}}^{2}$

In hac fiat $z = -b\sqrt{-1}$, & $R$ evadat $S$, ut ſit $S + 4bbQA = 0$

$\overset{1}{R} - \overset{1}{N}\overset{1}{A}\,.\,\overline{z + b\sqrt{-1}}^{2}$

in qua pone $z = b\sqrt{-1}$, & $\overset{1}{R}$ evadat $\overset{1}{S}$, ut ſit $\overset{1}{S} + 4bb\overset{1}{Q}\overset{1}{A} = 0$

Species $S$, $\overset{1}{S}$ exprimunt formulas habentes terminos reales iiſdem ſignis affectos, imaginarios diverſis. Igitur

$$A = \frac{-S}{4bbQ}\,,\quad \overset{1}{A} = \frac{-\overset{1}{S}}{4bbQ}\,;$$

Itaque conſtat in quantitatibus $A$, $\overset{1}{A}$ terminos reales iiſdem eſſe ſignis dona-

tos

tos, diverſis imaginarios; igitur idem eveniet in quantitatibus $B$, $\overset{1}{B}$. Quapropter ſumma duarum fractionum

$$\frac{\overline{Az+B}.dz}{\overline{z+b\sqrt{-1}}^2},\quad \frac{\overline{\overset{1}{A}z+\overset{1}{B}}.dz}{\overline{z-b\sqrt{-1}}^2}$$

evadet realis facta reductione ad eumdem denominatorem.

43. Si factores $z+b\sqrt{-1}$, $z-b\sqrt{-1}$, elati eſſent ad poteſtatem tertiam, quartam &c. eadem methodus perficeret demonſtrationem, tametſi calculus eo longior, atque implicatior evaderet, quo altior eſſet factorum poteſtas.

44. Ad exemplum proponatur reſolvenda formula $\frac{\overline{xx+2ax+aa-bb}.dx}{\overline{x+a}.\overline{xx+2ax+aa+bb}}$. Ut factores ſimpliciores ſint, pone $x+a=z$ ut formula in hanc mutetur $\frac{\overline{z^2-b^2}.dz}{z.\overline{zz+bb}}$, ſive $\frac{\overline{zz-bb}.dz}{z.\overline{z+b\sqrt{-1}}.\overline{z-b\sqrt{-1}}}$.

Fractiones inventæ ſint

$\frac{Adz}{z+b\sqrt{-1}}$. Quantitas per quam eſt determinanda $A$ ex num. 41. eſt $zz-bb-Az.\overline{z-b\sqrt{-1}}$. Pro $z$ ſcribe $-b\sqrt{-1}$ ut habeas $-2bb+2Abb=0$; ergo $A=1$. Quare fractio quæſita $\frac{dz}{z+b\sqrt{-1}}$.

$\frac{\overset{1}{A}dz}{z-b\sqrt{-1}}$. Conſtans $\overset{1}{A}$ ex num. 41. definienda eſt per $z^2-b^2-\overset{1}{A}z.\overline{z+b\sqrt{-1}}$. Pro $z$ ſcribe $+b\sqrt{-1}$, ut fiat $-2bb+2\overset{1}{A}bb=0$; igitur $\overset{1}{A}=1$. Itaque fractio quæſita $\frac{dz}{z+b\sqrt{-1}}$.

Duæ iſtæ fractiones ad eandem denominationem redactæ, & in ſummam collectæ dant $\frac{2zdz}{z^2+b^2}$, quæ realis eſt. Reapſe ſi hæc conjungatur cum $-\frac{dz}{z}$, quæ oritur ex alio factore $z$, reſtituit $\frac{\overline{zz-bb}.dz}{z.\overline{zz+bb}}$. In hiſce ſi pro $z$ ponas $x+a$, cognoſces quid in propoſita eveniat.

45. Exemplum alterum propono in formula $\frac{b^6dz}{zz.\overline{z+b\sqrt{-1}}^2.\overline{z-b\sqrt{-1}}^4}$, in qua inveniendæ ſint fractiones peculiares reſpondentes factoribus $\overline{z+b\sqrt{-1}}^2$, $\overline{z-b\sqrt{-1}}^2$. Hæ autem ſint



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$\frac{\overline{Az+B}\,.\,dz}{\overline{z+b\sqrt{-1}}^{\,2}}$. Formula, ex qua determinantur $A, B$, eſt $b^6 - z^2 . \overline{z-b\sqrt{-1}}^{\,2} . \overline{Az+B}$. Pone $z = -b\sqrt{-1}$ ut fiat $b^6 + 4Ab^5\sqrt{-1} + 4b^4B = 0$; ergo

$$B = \frac{bb}{4} + Ab\sqrt{-1}$$

$\frac{\overline{\overset{1}{A}z+\overset{1}{B}}\,.\,dz}{\overline{z-b\sqrt{-1}}^{\,2}}$. Conſtantes $\overset{1}{A}, \overset{1}{B}$ determinandæ ſunt per $b^6 - z^2 . \overline{z+b\sqrt{-1}}^{\,2} . \overline{\overset{1}{A}z+\overset{1}{B}}$. Pone $z = b\sqrt{-1}$, ut oriatur $b^6 - 4\overset{1}{A}b^5\sqrt{-1} - 4b^4\overset{1}{B} = 0$; ergo $\overset{1}{B} = \frac{bb}{4} - \overset{1}{A}b\sqrt{-1}$.

Valores hujuſmodi $B, \overset{1}{B}$ ponantur in formulis primis, ut habeas

$$b^6 - z^2 . \overline{z-b\sqrt{-1}}^{\,2} . \frac{bb}{4} - zz . \overline{z-b\sqrt{-1}}^{\,2} . A . \overline{z+b\sqrt{-1}}.$$

Si hanc factis opportunis calculis dividas per $z+b\sqrt{-1}$ invenies

$$\frac{-b^2z^3}{4} + \frac{3b^3z^2\sqrt{-1}}{4} + b^4z - b^5\sqrt{-1} - A . \overline{z^4 - 2bz^3\sqrt{-1} - b^2z^2}.$$

Pro $z$ ſcribe $-b\sqrt{-1}$, & habebis $-3b^5\sqrt{-1} - A . 4b^4 = 0$; ergo $A = \frac{-3b\sqrt{-1}}{4}$; igitur $B = \frac{bb}{4} + \frac{3bb}{4} = bb$. Itaque fractio quæſita erit

$$\frac{\frac{-3b\sqrt{-1}}{4} . z + bb}{\overline{z+b\sqrt{-1}}^{\,2}} . dz$$

$$b^6 - zz . \overline{z+b\sqrt{-1}}^{\,2} . \frac{bb}{4} - zz . \overline{z+b\sqrt{-1}}^{\,2} . \overset{1}{A} . \overline{z-b\sqrt{-1}}.$$

Factis neceſſariis calculis hanc divide per $z-b\sqrt{-1}$, ut habeas

$$\frac{-b^2z^3}{4} + \frac{3b^3z^2\sqrt{-1}}{4} + b^4z + b^5\sqrt{-1} - \overset{1}{A} . \overline{z^4 + 2bz^3\sqrt{-1} - b^2z^2}.$$

Pro $z$ pone $b\sqrt{-1}$, & invenies $3b^5\sqrt{-1} - \overset{1}{A}4b^4 = 0$; ergo $\overset{1}{A} = \frac{3}{4}b\sqrt{-1}$; igitur $\overset{1}{B} = \frac{bb}{4} + \frac{3bb}{4} = bb$. Fractio igitur quæſita erit

$$\frac{\frac{3}{4}b\sqrt{-1} . z + bb}{\overline{z-b\sqrt{-1}}^{\,2}} . dz$$

quarum ſumma dat fractionem realem, nempe $\frac{-b^2z^2 - 2b^4}{\overline{z^2+b^2}^{\,2}} . dz$. Reapſe hæc conjuncta cum fractione $\frac{b^2}{z^2} . dz$, quæ eſt fractio conveniens factori $zz$, reſtituet fractionem propoſitam $\frac{b^6}{z^2 . \overline{z^2+b^2}^{\,2}} . dz$.

46. Quo-

46. Quoniam methodus hæc tandem nos perducit ad duas formulas logarithmicas imaginarias, nimirum vel ad $\frac{a\,dz}{z+b\sqrt{-1}}+\frac{a\,dz}{z-b\sqrt{-1}}$, quæ integrata dat $\frac{a}{c}\,l\,\overline{z+b\sqrt{-1}}+\frac{a}{c}\,l\,\overline{z-b\sqrt{-1}}$; vel ad $\frac{a\sqrt{-1}.dz}{z+b\sqrt{-1}}-\frac{a\sqrt{-1}.dz}{z-b\sqrt{-1}}$, quæ integrata præbet $\frac{a\sqrt{-1}}{c}\,l\,\overline{z+b\sqrt{-1}}-\frac{a\sqrt{-1}}{c}\,l\,\overline{z-b\sqrt{-1}}$, exiſtente $=c$ ſubtangente logarithmica, memoria retinendum eſt, quænam ſit quantitas realis ab hiſce logarithmis imaginariis exhibita. Quoad primam $\frac{a}{c}\,l\,\overline{z+b\sqrt{-1}}+\frac{a}{c}\,l\,\overline{z-b\sqrt{-1}}$; ſume differentias $\frac{a\,dz}{z+b\sqrt{-1}}+\frac{a\,dz}{z-b\sqrt{-1}}$; has ad eandem denominationem perduc, ut fit $\frac{2azdz}{zz+bb}$, quam ſi integres habebis $\frac{2a}{c}\,l\sqrt{zz+bb}$; ergo $l\,\overline{z+b\sqrt{-1}}+l\,\overline{z-b\sqrt{-1}}=2\,l\sqrt{zz+bb}$; igitur ſumma duorum logarithmorum imaginariorum $l\,\overline{z+b\sqrt{-1}}$, $l\,\overline{z-b\sqrt{-1}}$ eſt logarithmus $\sqrt{zz+bb}$ bis ſumptus. Quod ſi ſeſe offerret $l\,\overline{b\sqrt{-1}+z}+l\,\overline{b\sqrt{-1}-z}$, facta differentiatione haberemus $\frac{c\,dz}{b\sqrt{-1}+z}-\frac{c\,dz}{b\sqrt{-1}-z}=\frac{-2czdz}{-bb-zz}=\frac{2czdz}{bb+zz}$ & facta integratione $l\,\overline{b\sqrt{-1}+z}+l\,\overline{b\sqrt{-1}-z}=2\,l\sqrt{b^2+z^2}$ prorſus ut antea. At formulæ ſequentes imaginaria continentes imaginariæ ſunt $l\,\overline{z+b\sqrt{-1}}-l\,\overline{z-b\sqrt{-1}}$, $l\,\overline{b\sqrt{-1}+z}-l\,\overline{b\sqrt{-1}-z}$. Oſtendamus in prima. Differentietur; $\frac{c\,dz}{z+b\sqrt{-1}}-\frac{c\,dz}{z-b\sqrt{-1}}$; reducatur ad eandem denominationem, & fiet $\frac{-2cb\sqrt{-1}.dz}{zz+bb}$, quæ profecto imaginaria eſt.

47. Venio ad formulam alteram $\sqrt{-1}.l\,\overline{z+b\sqrt{-1}}-\sqrt{-1}\,l\,\overline{z-b\sqrt{-1}}$. ſume differentiam $\frac{c\sqrt{-1}\,dz}{z+b\sqrt{-1}}-\frac{c\sqrt{-1}\,dz}{z-b\sqrt{-1}}=\frac{2cbdz}{zz+bb}=\frac{2c}{b}.\frac{bbdz}{zz+bb}$, quæ æquat arcum circularem, cujus radius $=b$, tangens $=z$ ductum in $\frac{2c}{b}$; igitur $\sqrt{-1}\,l\,\overline{z+b\sqrt{-1}}-\sqrt{-1}\,l\,\overline{z-b\sqrt{-1}}$ eſt ad arcum circularem radii $b$, tangentis $z$, ut duplex tangens ſyſtematis logarithmi ad $b$. Idem dicas volo de formula $\sqrt{-1}.l\,\overline{b\sqrt{-1}+z}-\sqrt{-1}.l\,\overline{b\sqrt{-1}-z}$, ut inſtituenti calculum palam fiet. Verum formulæ $\sqrt{-1}.l\,\overline{z+b\sqrt{-1}}+\sqrt{-1}\,l\,\overline{z-b\sqrt{-1}}$, $\sqrt{-1}.l\,\overline{b\sqrt{-1}+z}+\sqrt{-1}\,l\,\overline{b\sqrt{-1}-z}$ imaginariæ ſunt.

48. Ve-

48. Verum, quoniam ex reſolutione factorum ſecundi gradus, ſæpe factores ſimplices reales oriuntur, quæ ad reales formulas deducunt, hæ quoque licet faciliores non videntur prætermittendæ. Formulæ reales $l\,\overline{z+b}+l\,\overline{z-b}$, $l\,\overline{b+z}+l\,\overline{b-z}$ æquales ſunt $2l\sqrt{zz-bb}$, $2l\sqrt{bb-zz}$. Hoc quamquam conſtat ex theoria logarithmorum, tamen differentiatis formulis eruamus. Differentietur prima $\frac{c\,dz}{z+b}+\frac{c\,dz}{z-b}=\frac{2cz\,dz}{zz-bb}$; ſed hujus integrale eſt $2l\sqrt{z^2-b^2}$. ergo; eadem methodus valebit in ſecunda.

49. Formula $l\,\overline{z+b}-l\,\overline{z-b}$ ex logarithmica theoria $=l.\frac{f.\overline{z+b}}{z-b}$, exiſtente $f$ protonumero ſyſtematis. Sed videamus quid per methodum differentiandi proveniat. Fiet $\frac{c\,dz}{z+b}-\frac{c\,dz}{z-b}=\frac{2c}{b}.\frac{bb\,dz}{zz-bb}$; atqui $S\,\frac{b^2dz}{zz-bb}$ eſt logarithmus analogus, cujus ſinus totus $=b$, cotangens $=z$; ergo $l\,\overline{z+b}-l\,\overline{z-b}$ eſt ad hunc logarithmum, ut $2c:b$. Similiter $l\,\overline{b+z}-l\,\overline{b-z}$, quam eſſe conſtat $=lf.\frac{\overline{b+z}}{b-z}$, differentietur ut fiat $\frac{dz}{b+z}+\frac{dz}{b-z}=\frac{2c}{b}.\frac{b^2dz}{bb-zz}$; ſed $S\,\frac{b^2dz}{bb-zz}$ eſt logarithmus analogus, cujus ſinus totus $=b$, tangens $=z$; ergo $l\,\overline{b+z}-l\,\overline{b-z}$ eſt ad hunc logarithmum, ut $2c:b$.

50. Si in omnibus formulis, de quibus egimus, pro $z$ ponamus aut $x+a$, aut $x-a$, eadem omnino valebunt, dummodo ubique hanc ſubſtitutionem adhibeas.

## CAPUT NONUM.

### De calculo logarithmico, & exponentiali.

1. IN univerſo hoc capite ſpecies $c$ deſignabit ſyſtematis logarithmici ſubtangentem, $f$ protonumerum, $e$ baſim logarithmicam, ſeu numerum illum, cujus logarithmus æquat protonumerum, quarum quantitatum duæ quælibet tertiam determinant. Docuimus cap. 6. num. 27. differentialem $lx$ eſſe $\frac{c\,dx}{x}$, & contra integralem $\frac{c\,dx}{x}$, eſſe $lx$. Hinc diſcimus $(lx)^m$ differentiam $=m(lx)^{m-1}.\frac{c\,dx}{x}$, & verſa vice $(lx)^m.\frac{c\,dx}{x}$ integrale $=\frac{(lx)^{m+1}}{m+1}$. Imo formula quælibet, quæ non contineat niſi $lx$, nullo negotio differentiabitur per regulas traditas, ſi $\frac{c\,dx}{x}$ ſpectetur tamquam differentia $lx$.

Ita

Ita $\sqrt{aa-(lx)^2}$ habebit differentiam $= \frac{-c\,dx\,lx}{x\sqrt{aa-(lx)^2}}$. E contrario formula, quæ non includit niſi $lx$, & ejus differentiam $\frac{c\,dx}{x}$, ſi prædita ſit conditionibus, quas ſupra adnotavimus, integrabitur per logarithmos, atque adeo per quadraturam hyperbolæ; niſi poſcat præterea circuli quadraturam. Criterium cognoſcendi, utrum illis conditionibus donata ſit, facile eſt, & expeditum; pone enim $lx=y$, & $\frac{c\,dx}{x}=dy$, factaque ſubſtitutione orietur formula expers logarithmorum, quæ utrum ſit integrabilis algebraice, utrum dependeat ab hyperbolæ, circulive quadratura, ex regulis traditis ſtatim cognoſces. Sed hæc facilia ſunt.

2. Difficilior eſt formula $x^m.(lx)^n dx$, quæ tamen per logarithmos integratur, quoties $n$ fuerit numerus integer; & affirmativus. Ut eam integrem aſſumo formulam $x^p(lx)^q$, ejuſque differentiam accipio hoc modo
$D x^p(lx)^q = p x^{p-1}dx.(lx)^q + q x^p (lx)^{q-1}.\frac{c\,dx}{x}$. Quæ æquatio in hunc modum diſtribuenda eſt
$S x^{p-1}(lx)^q dx = \frac{1}{p}x^p(lx)^q - \frac{qc}{p} S x^{p-1}(lx)^{q-1}dx$. Fiat $p-1=m$, $q=n$, & exurget $S x^m(lx)^n dx = \frac{1}{m+1}x^{m+1}(lx)^n - \frac{nc}{m+1} S x^m (lx)^{n-1}dx$.
Non mutato valore $p$, pone $q=n-1$, ut habeas
$S x^m(lx)^{n-1}dx = \frac{1}{m+1}x^{m+1}(lx)^{n-1} - (\frac{n-1}{m+1} c S x^m(lx)^{n-2}dx$.
Pone $q=n-2$, eritque
$S x^m(lx)^{n-2}dx = \frac{1}{m+1}x^{m+1}(lx)^{n-2} - (\frac{n-2}{m+1} c S x(lx)^{n-3}dx$.
Fiat $q=n-3$, ut ſit
$S x^m(lx)^{n-3}dx = \frac{1}{m+1}x^{m+1}(lx)^{n-3} - (\frac{n-3}{m+1}. S x^m(lx)^{n-4}dx$.
Atque hoc modo progredere, donec exponens $(lx)$ fiat $=o$, quo in caſu ultima formula proveniet abſolute integrabilis. Deinde regredere, factiſque ſubſtitutionibus invenies

$$S x^m(lx)^n dx = \frac{1}{m+1}x^{m+1}(lx)^n - \frac{n}{\overline{m+1}^{\,2}} c x^{m+1}(lx)^{n-1} +$$

$$\frac{n.\overline{n-1}}{\overline{m+1}^{\,3}} c^2 x^{m+1}(lx)^{n-2} - \frac{n.\overline{n-1}.\overline{n-2}}{\overline{m+1}^{\,4}} c^3 x^{m+1}(lx)^{n-3} \ldots\ldots$$

$\pm \frac{n.\overline{n-1}\ldots\ldots 1}{\overline{m+1}^{\,n+1}} c^n x^{m+1}$. Signum ſuperius in ultimo termino valet, ſi $n$ ſit

par,

par, inferius ſi ſit impar. Hæc formula inutilis prorſus evaderet, ſi $m=-1$; ſed in hoc caſu propoſita in hanc mutaretur $(lx)^n . \frac{dx}{x}$, cujus ſummatoria $=\frac{(lx)^{n+1}}{n+1 . c}$.

3. Si $n$ foret negativa, ea in poſitivam mutatatur, ut formula integranda evadat $\frac{x^m dx}{(lx)^n}$; tunc aſſumenda eſt $q$ negativa, & ita diſponenda formula, quæ in auxilium vocatur

$$S\frac{x^{p-1}dx}{(lx)^{q+1}} = \frac{-1}{qc} . \frac{x^p}{(lx)^q} + \frac{p}{qc} . S\frac{x^{p-1}dx}{(lx)^q}.$$ Fiat perpetuo $p-1=m$, tum $q+1=n$, ut proveniat

$$S\frac{x^m dx}{(lx)^n} = \frac{-1}{n-1 . c} . \frac{x^{m+1}}{(lx)^{n-1}} + \frac{m+1}{n-1 . c} . S\frac{x^m dx}{(lx)^{n-1}}.$$ Pone $q+1=n-1$

$$S\frac{x^m dx}{(lx)^{n-1}} = \frac{-1}{n-2 . c} . \frac{x^{m+1}}{(lx)^{n-2}} + \frac{m+1}{n-2 . c} . S\frac{x^m dx}{(lx)^{n-2}}.$$ Fiat $q+1=n-2$

$$S\frac{x^m dx}{(lx)^{n-2}} = \frac{-1}{n-3 . c} . \frac{x^{m+1}}{(lx)^{n-3}} + \frac{m+1}{n-3 . c} S\frac{x^m dx}{(lx)^{n-3}}.$$ Analyſim produc, donec exponens $lx$ æquet unitatem; ſi enim ulterius progredereris, incideres in quantitates infinitas. Deinde facto regreſſu invenies

$$S\frac{x^m dx}{(lx)^n} = \frac{-1}{n-1 . c} . \frac{x^{m+1}}{(lx)^{n-1}} - \frac{(m+1)}{\overline{n-1} . \overline{n-2} . cc} . \frac{x^{m+1}}{(lx)^{n-2}} - \frac{\overline{m+1}^2}{\overline{n-1} . \overline{n-2} . \overline{n-3} . c^3} . \frac{x^{m+1}}{(lx)^{n-3}} \dots + \frac{\overline{m+1}^{n-1}}{\overline{n-1} . \overline{n-2} \dots 1} + S\frac{x^m dx}{lx}.$$ Quæ formula oſtendit $S\frac{x^m dx}{(lx)^n}$ dependere a $S\frac{x^m dx}{lx}$. Si $n$ eſſet numerus fractus, vel poſitivus, vel negativus, analyſis tam numeri ſuperioris, quam præſentis nos deduceret ad ſeriem infinitam: ex duabus autem ſeriebus illam uſurpa—, quæ provenit convergens.

4. Si differentiandus ſit logarithmus logarithmi ſcilicet $l . lx$, qui ita brevitatis cauſa ſcribi poteſt $l^2x$; ponatur $l^2x = ly$; ergo $lx=y$, & $\frac{cdx}{x}=dy$. Præterea $D . l^2x = \frac{cdy}{y} = \frac{c^2dx}{xlx}$. Quapropter integrale $\frac{c^2dx}{xlx}$ erit $l^2x$. Hinc

S

$S(l^2x)^m \cdot \frac{c^2 dx}{x\,lx} = \frac{(l^2x)^{m+1}}{m+1}$, & quælibet formula, quæ non includat niſi $l^2x$, & ejus differentiale $\frac{c^2 dx}{x\,lx}$ per regulas expoſitas tractabitur. Præterea $(lx)^m \cdot (l^2x)^n \cdot \frac{c\,dx}{x}$, ſi $n$ ſit numerus integer & poſitivus per logarithmos integrabitur. Pone enim $lx = y$; ergo formula in hanc mutabitur $y^m . (ly)^n dy$, quæ poſito $n$ numero integro, & poſitivo ex num. 2. integratur. Simili modo invenies $l^3x$ habere differentiam $= \frac{c^3 dx}{x . lx . l^2x}$, & generatim $l^nx$ habebit differentiam $= \frac{c^n dx}{x . lx . l^2x \ldots l^{n-1}x}$; & viciſſim hujus integrale $= l^nx$. Hæc ſatis ſint de quantitatibus logarithmicis.

5. Tranſeamus ad exponentiales, quæ oriuntur, dum a logarithmis ſit tranſitus ad numeros. Sit primum æquatio $m\,lx = ly$, ex qua $l\frac{x^m}{f^{m-1}} = ly$; ergo facto tranſitu ad numeros fiet $\frac{x^m}{f^{m-1}} = y$, ſive $x^m = f^{m-1}y$. Si $m$ ſit numerus datus, & rationalis, æquatio inventa eſt algebraica, ejuſque conſtructio fit per regulas calculi carteſiani. Verum ſi $m$ ſit datus quidem, ſed irrationalis, æquatio complectitur quantitates exponentiales, quæ exhiberi non poſſunt niſi per logarithmicam, aut per curvam aliam tranſcendentem. Atque hæc eſt prima, & infima claſſis quantitatum exponentialium, quarum exponentes ſunt numeri dati, ſed irrationales. Idem dicas velim de generaliori æquatione $m\,lx + n\,la = ly$; facto enim tranſitu ad numeros fiet $\frac{a^n x^m}{f^{m+n-1}} = y$, in qua ſi $n = -m+1$, proveniet $x^m = a^{m-1}y$, ubi locum non habet protonumerus ſyſtematis, in quo logarithmi accipiuntur.

6. Ut appareat qua methodo hujuſmodi æquationes ad conſtructionem perduci poſſint, aſſumo conſtruendam æquationem $x^{\sqrt{2}} = f^{\sqrt{2}-1}y$. Tranſi a numeris ad logarithmos, ut habeas $\sqrt{2}.lx = ly$. Deſcribe quamlibet logarithmicam PBM; (*Fig.* 4.) ſubtangentem, quæ arbitraria eſt, prout tibi commodum fuerit, accipias. In ea ſume protonumerum $AB = f$. Accipe quamlibet $AC = x$, ordinata CD, aut ei æqualis $AE = lx$. Fac ut $1 : \sqrt{2} :: AE : AF$, erit $AF = \sqrt{2}.lx = ly$; ergo $FG = y$. Huic pone CH æqualem, curva quæſita tranſibit per punctum H. Si fiat $x = AB = f$, erit $lx = o$; ergo etiam $\sqrt{2}\,lx = ly = o$; ergo $y = f$; igitur BI ordinata curvæ quæſitæ in puncto B erit æqualis protonumero. Si vero $x = o$, ejus logarithmus erit infinitus, & negativus; ergo etiam $ly$, ergo $y = o$, quare curva tranſibit per punctum A.

Conſtructio itaque docet, curvam incipere in puncto A, progredi per I in H, & recedere in infinitum. Quoniam vero tam $x$, quam $y$ accipi poteſt tum poſitive, tum negative curva conſtabit quatuor ramis AH, A2H, A3H, A4H ſimilibus, & æqualibus.

7. Nihil eſt difficile invenire methodum differentiandi hujuſmodi quantitates exponentiales. Sit differentianda quantitas $x^m$. Pone $x^m = f^{m-1} y$, tranſi ad logarithmos $m l x = l y$; ſume differentias $\frac{m c d x}{x} = \frac{c d y}{y}$. Subtangens ſyſtematis $= c$ per diviſionem ab equatione expellitur, quod indicat, logarithmos ſumi poſſe in ſyſtemate cujuſcumque ſubtangentis. Habes itaque $\frac{m d x}{x} = \frac{d y}{y}$;

ergo $\frac{m y d x}{x} = d y$; ſed $y = \frac{x^m}{f^{m-1}}$; ergo $\frac{m x^{m-1} d x}{f^{m-1}} = d y$, ſive

$m x^{m-1} d x = f^{m-1} d y$, quæ eſt differentialis quæſita. Quapropter regula ſumendi differentias in poteſtatibus rationalibus, valet etiam in radicalibus, nempe differentia $x^m$ æquat $m x^{m-1} d x$, vice verſa ſummatoria $x^m d x$ æquat $\frac{x^{m+1}}{m+1}$, quicumque ſit numerus $m$ vel rationalis vel irrationalis, dummodo conſtans. Sed ad alias exponentium claſſes gradum faciamus.

8. Sit æquatio logarithmicæ $l x = y$. In hac nunquam fiet tranſitus a logarithmis ad numeros, niſi logarithmi in utraque æquationis parte reperiantur, quare introducendi ſunt in eam partem, ubi non inveniuntur. Quando $l e = f$, logarithmus enim baſis logarithmicæ æqualis eſt protonumero, erit $\frac{l e}{f} = 1$; ergo $y = \frac{y}{f} l e$; ergo æquatio aſſumpta hanc formam accipit $l x = \frac{y}{f} l e$. Hoc artificio, quod tum protonumerum, tum baſim logarithmicam cognitam requirit, introductis in utramque partem logarithmis, tranſeundum eſt ad numeros, ut

proveniat æquatio exponentialis $x = \frac{e^{\frac{y}{f}}}{f^{\frac{y}{f} - 1}} = f . \left( \frac{e}{f} \right)^{\frac{y}{f}}$, in qua ratio conſtans $\frac{e}{f}$ elevatur ad poteſtatem variabilem $\frac{y}{f}$; atque hæc eſt altera claſſis quantitatum exponentialium. Habemus itaque æquationem logarithmicæ per quantitates exponentiales expreſſam. Sed æquationem univerſaliorem tractemus. Sit æquatio $m l x + n l a = y = \frac{y}{f} l e$; igitur facto tranſitu ad numeros

$$\frac{a^n x^m}{f^{m+n-1}} = \frac{e^{\frac{y}{f}}}{f^{\frac{y}{f} - 1}} = f . \left( \frac{e}{f} \right)^{\frac{y}{f}}, \text{ ſive } x^m = \frac{f^{m+n}}{a^n} . \left( \frac{e}{f} \right)^{\frac{y}{f}}.$$

9. Quo

9. Quo pacto harum exponentialium differentia ſit accipienda, accurate tradendum eſt. Sit differentianda $(\frac{e}{f})^{\frac{y}{f}}$, in qua $e$ eſt baſis, $f$ protonumerus ſyſtematis logarithmici. Pone eam æqualem $\frac{x}{f}$, ut fiat $\frac{e^{\frac{y}{f}}}{f^{\frac{y}{f}-1}} = x$, a numeris gradum fac ad logarithmos $\frac{y}{f} le = y = lx$, differentia $dy = \frac{c\,dx}{x}$; ergo $\frac{x\,dy}{c} = dx$, & dividendo per $f$, $\frac{x}{f} \cdot \frac{dy}{c} = \frac{dx}{f}$; atqui $\frac{x}{f} = (\frac{e}{f})^{\frac{y}{f}}$; ergo $(\frac{e}{f})^{\frac{y}{f}} \cdot \frac{dy}{c} = \frac{dx}{f}$, quæ eſt differentia quæſita. Habebis itaque differentiam quantitatis exponentialis $(\frac{e}{f})^{\frac{y}{f}}$, ſi eam multiplicatam per $dy$ dividas per ſyſtematis logarithmici ſubtangentem. Quare per operationem contrariam ſummatoria $(\frac{e}{f})^{\frac{y}{f}} \cdot dy$ erit $c \cdot (\frac{e}{f})^{\frac{y}{f}}$, hoc eſt eadem quantitas exponentialis ducta in ſubtangentem. Subtangens autem ſyſtematis data eſt, datis baſi logarithmica & protonumero. Quod ſi denominator exponentis non ſit protonumerus $f$, ut in exponentiali $(\frac{e}{f})^{\frac{y}{g}}$, tunc exponens mutetur in fractionem, cujus denominator $= f$, facto $\frac{y}{g} = \frac{z}{f}$, ut fiat $(\frac{e}{f})^{\frac{z}{f}}$. Hujus differentia erit $= (\frac{e}{f})^{\frac{z}{f}} \cdot \frac{dz}{c}$; atqui $dz = \frac{f\,dy}{g}$; ergo differentia $(\frac{e}{f})^{\frac{y}{g}}$ erit $(\frac{e}{f})^{\frac{y}{g}} \cdot \frac{f\,dy}{cg}$; ſcilicet quantitas exponentialis ducta in differentiam exponentis multiplicanda eſt per protonumerum, & dividenda per ſubtangentem ſyſtematis. Contra ſummatoria $(\frac{e}{f})^{\frac{y}{g}} \cdot dy$ erit $\frac{cg}{f} \cdot (\frac{e}{f})^{\frac{y}{g}}$.

10. Quod ſi quantitas elevanda ad exponentem variabilem non ſit ratio $e:f$, baſis logarithmicæ ad protonumerum, ſed alia quælibet $a:b$, hac methodo ejus differentia determinatur. Pone $(\frac{a}{b})^{\frac{y}{g}} = \frac{x}{f}$; ergo facto tranſitu ad

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logarithmos $\frac{y}{g}.\overline{la-lb}=lx$; & differentiis acceptis $\frac{dy}{g}.\overline{la-lb}=\frac{c\,dx}{x}$, ſive $\frac{x\,dy.\overline{la-lb}}{fcg}=\frac{dx}{f}$; ſed $\frac{x}{f}=(\frac{a}{b})^{\frac{y}{g}}$; ergo $\frac{(\frac{a}{b})^{\frac{y}{g}}dy.\overline{la-lb}}{cg}=\frac{dx}{f}$, quæ eſt differentia quæſita. In hac quamquam baſis logarithmica, & protonumerus non videntur locum habere; tamen nunquam invenies $la$, $lb$, niſi illis determinatis. Hinc diſcis, quomodo integranda ſit formula $(\frac{a}{b})^{\frac{y}{g}}dy$; ejus enim ſummatoria $\frac{cg}{la-lb}.(\frac{a}{b})^{\frac{y}{g}}$.

11. Quæ hactenus tradita ſunt, ſufficiunt expeditam regulam ad differentiandas quantitates exponentiales hujus claſſis, in qua quantitas elevanda ad poteſtatem ſit conſtans. Ejus enim differentia eſt ipſa quantitas exponentialis ducta in differentiam exponentis, & in fractionem conſtantem, cujus numerator eſt differentia logarithmorum numeratoris & denominatoris, diviſor eſt ſubtangens ſyſtematis logarithmici. Ex oppoſito quantitatis exponentialis ductæ in differentiam exponentis ſummatoriam habebis, ſi eamdem exponentialem multiplices per ſubtangentem ſyſtematis logarithmici, & dividas per differentiam logarithmorum numeratoris, & denominatoris. Hæ autem regulæ valent, tametſi quantitatis exponens ſit & ipſe quantitas exponentialis. Etenim quemadmodum ex tranſitu a logarithmis ad numeros oriuntur quantitates, in quibus exponens eſt variabilis; ita ex duplici tranſitu a logarithmorum logarithmis ad numeros naſcuntur quantitates, quarum exponens eſt quantitas exponentialis. Hæc, quando quantitas elevanda ad poteſtatem variabilem eſt conſtans, in eadem claſſe conſtituitur, ſed ad ſecundum gradum pertinere cenſenda eſt. Si exponens ſit quantitas exponentialis gradus ſecundi, ipſa ad tertium gradum pertinebit, atque ita deinceps. Hæ omnes per eamdem regulam differentiantur. Namque exponens ponatur $=\frac{z}{g}$, deinde differentia ſumatur; demum pro $\frac{dz}{g}$ ſubſtitue differentiam exponentis, & habebis differentiam quæſitam. Quoniam autem hæc differentia expreſſa per $z$ eſt æqualis $(\frac{a}{b})^{\frac{z}{g}}.\frac{c\,dz}{g}.\frac{la-lb}{c}$; ſi ſubſtituas valorem $\frac{dz}{g}$, invenies differentiam æquare quantitatem exponentialem ductam in differentiam exponentis, & in fractionem $\frac{la-lb}{c}$ ſicut regula præcipit. Eodem modo probabis, ſummatoriam quantitatis exponentialis cujuſcumque gradus ductæ in differentiam exponentis $=\frac{c}{la-lb}$ in quantitatem ipſam exponentialem.

12. Antequam progredior, advertendum eſſe judico, quantitatem exponentialem, cujus exponens eſt logarithmus quantitatis variabilis, eſſe algebraicam, aut exponentialem primæ claſſis, ac proinde ſimplicius exprimi poſſe. Ajo itaque

que primum $\left(\frac{e}{f}\right)^{\frac{nly}{f}} = \frac{y^n}{f^n}$. Ponatur $\left(\frac{e}{f}\right)^{\frac{nly}{f}} = \frac{z}{f}$; ergo facto tranſitu a numeris ad logarithmos $nly = lz$. Iterum tranſi ad numeros, ut habeas $\frac{y^n}{f^{n-1}} = z$; ergo $\left(\frac{e}{f}\right)^{\frac{nly}{f}} = \frac{y^n}{f^n}$. Q. E. D. Ajo deinde $\left(\frac{e}{f}\right)^{\frac{nly}{g}} = \frac{y^{\frac{nf}{g}}}{f^{\frac{nf}{g}}}$. Fac ut ſupra $\left(\frac{e}{f}\right)^{\frac{nly}{g}} = \frac{z}{f}$; ergo $\frac{nly}{g} \cdot le = lz$, ſive propter $le = f$, $\frac{nf}{g} ly = lz$, & redeundo ad numeros $\frac{y^{\frac{nf}{g}}}{f^{\frac{nf}{g}-1}} = z$; igitur $\left(\frac{e}{f}\right)^{\frac{nly}{g}} = \frac{y^{\frac{nf}{g}}}{f^{\frac{nf}{g}}}$.

Q. E. D. Demum ajo $\left(\frac{a}{b}\right)^{\frac{nly}{g}} = \frac{y^{\frac{n.\overline{la-lb}}{f}}}{f^{\frac{n.\overline{la-lb}}{f}}}$. Pone ut antea $\left(\frac{a}{b}\right)^{\frac{nly}{g}} = \frac{z}{f}$, & facto tranſitu ad logarithmos a numeris $\frac{nly}{g} \cdot \overline{la-lb} = lz$, & ad numeros redeundo $\frac{y^{\frac{n.\overline{la-lb}}{g}}}{f^{\frac{n.\overline{la-lb}}{g}-1}} = z$; igitur $\left(\frac{a}{b}\right)^{\frac{nly}{g}} = \frac{y^{\frac{n.\overline{la-lb}}{g}}}{f^{\frac{n.\overline{la-lb}}{g}}}$. Q. E. D.

23 Claſſis tertia quantitatum exponentialium habetur, quum non ſolum exponens, ſed etiam quantitas eſt variabilis. Hæc quoque naſcitur, dum fit tranſitus a logarithmis ad numeros. Sit æquatio $ylx = flz$, ſeu $\frac{y}{f}lx = lz$; ergo facto tranſitu ad numeros $\frac{x^{\frac{y}{f}}}{f^{\frac{y}{f}-1}} = z$, ſeu $f.\left(\frac{x}{f}\right)^{\frac{y}{f}} = z$; en quantitas variabilis elevata ad poteſtatem variabilem. Videamus primo, quomodo differentiandæ ſit $\left(\frac{x}{f}\right)^{\frac{y}{f}}$. Fac eam æqualem $\frac{z}{f}$, ut ſiat $f.\left(\frac{x}{f}\right)^{\frac{y}{f}} = z$, ergo tranſitu facto ad logarithmos $\frac{y}{f}lx = lz$, & differentiando $\frac{dylx}{f} + \frac{ydx}{fx} = \frac{dz}{z}$,

ſive

ſive $\frac{z}{f} \cdot \frac{xdylx+cydx}{cx} = ds$. Pro $\frac{z}{f}$ ſubſtitue ejus valorem, & habebis $(\frac{x}{f})^{\frac{y}{f}} \cdot \frac{xdylx+cydx}{cx} = dz$, quæ eſt differentia quæſita. Quum autem $\frac{xdylx+cydx}{x} = Dylx$, differentia quantitatis $(\frac{x}{f})^{\frac{y}{f}}$ ita brevius exprimi poterit $(\frac{x}{f})^{\frac{y}{f}} \cdot \frac{Dylx}{c}$. Itaque e contrario ſummatoria differentialis $(\frac{x}{f})^{\frac{y}{f}}$. $Dylx$ fiet $= c \cdot (\frac{x}{f})^{\frac{y}{f}}$.

14. Quod ſi diviſores fractionum non eſſent protonumerus, ſed alia quæcumque quantitas vel data, vel variabilis, ut accidit in $(\frac{x}{p})^{\frac{y}{q}}$, tunc pone $\frac{x}{p} = \frac{z}{f}$, & $\frac{y}{q} = \frac{u}{f}$, ut formula in hanc vertatur $(\frac{z}{f})^{\frac{u}{f}}$, quæ differentiata dat $(\frac{z}{f})^{\frac{u}{f}} \cdot \frac{Dulz}{c} = (\frac{x}{p})^{\frac{y}{q}} \cdot \frac{Dulz}{c}$; atqui $u = \frac{fy}{q}$, & $lz = lx - lp$; ergo differentialis quæſita invenietur eſſe $= \frac{f}{c} \cdot (\frac{x}{p})^{\frac{y}{q}} \cdot D\frac{y}{q} \cdot \overline{lx - lp}$. Quare differentialis quantitatis exponentialis habetur, ſi eadem multiplicetur primum per fractionem $\frac{f}{c}$, deinde per differentiale exponentis ducti in logarithmum numeratoris dempto logarithmo denominatoris quantitatis elevandæ. Contra $(\frac{x}{p})^{\frac{y}{q}} \cdot D\frac{y}{q} \cdot \overline{lx - lp}$ habet ſummatoriam $= \frac{f}{c} \cdot (\frac{x}{p})^{\frac{y}{q}}$. Quæ regulæ ut manifeſtum eſt locum habent, tametſi exponens ſit & ipſe quantitas exponentialis cujuſcumque gradus.

15. Si autem conſtruenda ſit curva, cujus æquatio ſit exponentialis, oportet, æquationem transferre a numeris ad logarithmos, & auxilio logarithmicæ curvam delineare. Exemplum præbeat æquatio $(\frac{x}{f})^{\frac{y}{f}} = \frac{e}{f}$. Facto tranſitu ad logarithmos fit $\frac{y}{f} lx = le = f$; ergo $ylx = ff$. Indefinitæ Mm, Na (*Fig.* 2.) ſeſe

ſeſe ad angulos rectos ſecent, & in his accipiantur CA, Ca $= f$, & formetur quadrata CABD, cabd; tum deſcribantur hyperbolæ oppoſitæ MBN, mbn. Exiſtentibus abſciſſis CE $= lx$, erunt ordinatæ EF $= y$. Præterea ad aſymptotum Cn deſcribantur logiſticæ nDG, ndg tranſeuntes per puncta D, d. Manifeſtum eſt, ordinatas EG $= x$, exiſtentibus CE $= lx$: igitur ordinatæ ad hyperbolam præbent $y$, ordinatæ ad logyſticam exhibent $x$, quæ duæ ſunt coordinatæ curvæ deſcribendæ. Ex his apparet $y$ nullam eſſe, ſi $x$ ſit infinita, decreſcentibus $x$ creſcunt $y$, ut tandem infinitæ fiant, ſi $x = f$. Exiſtente $x < f$, $y$ fiunt negativæ, & imminutis $x$, ita $y$ creſcunt, ut facta $x = o$, ipſæ infinitæ evadant. Quod dictum eſt de $x$ poſitivis idem prorſus dicendum de negativis. Quapropter hic invenietur progreſſus curvæ quæſitæ. In K (*Fig.* 3.) ſumatur abſciſſarum initium, ad partes P erunt poſitivæ, negativæ ad partes p. Abſcinde KH, Kh $= f$, & per H, h age QR, qr normales Pp. Curva in duobus angulis PHQ, phq prædita erit duobus ramis ſimilibus, & æqualibus PQ, pq, quorum HP, HQ, hp, hq ſunt aſymptota. Præterea duos alios ramos habet ſimiles, & æquales KR, Kr ad partes ordinatarum negativarum, qui pariter accedunt ad aſymptota HR, hr.

*16.* Secundum exemplum deſumo a formula $\left(\frac{x}{f}\right)^{\frac{f}{x}} = \frac{y}{f}$. Fiat tranſitus a numeris ad logarithmos, ut oriatur $\frac{f}{x}\, lx = ly$, ſeu $x : f :: lx : ly$. Ex hac analogia curvam ita conſtrues. Ad aſymptotum MAC (*Fig.* 4.) deſcribe logarithmicam BFL, in qua accipe BA $= f$, qui eſt protonumerus. Accipe AI $= x$, ordina IL $= lx$, tum duc LM parallelam BA. Junge IM, cui age parallelam BN; erit AN $= ly$; ergo NP $= y$. Huic æqualem IQ colloca ad angulos rectos ad AI, punctum Q erit in curva quæſita. Ut curvæ progreſſum obtineas, abſcinde AC $=$ AB, & excita CH normalem CA. Tum abſcinde AD æqualem ſubtangenti logyſticæ. Ductis DF, FE, inveni per conſtructionem effectam $y$ convenientem abſciſſæ AE. Ea autem ſit EG, Curva prædita eſt ramis duobus, nempe ramo GQH, qui magis magiſque accedit ad CH ejus aſymptotum, & ramo GBK tranſeunte per punctum B, & accedente ad aſymptotum AK. Sub logyſticæ aſymptoto AK, alia curva prorſus ſimilis, & æqualis priori oritur ex inferiori logarithmica.

17. Antequam calculum exponentialem deſeram, opportunum eſt aliquot theoremata demonſtrare, quæ eumdem calculum conjungunt cum calculo ſinuum, & coſinuum tam hyperbolicorum quam circularium. Hæc theoremata hiſce quatuor formulis continentur, in quibus $f$ eſt ſinus totus

$$\frac{1}{2} f . \left(\left(\frac{e}{f}\right)^{\frac{z}{f}} - \left(\frac{e}{f}\right)^{\frac{-z}{f}}\right) = sb . \frac{fz}{c}$$

$$\frac{1}{2} f . \left(\left(\frac{e}{f}\right)^{\frac{z}{f}} + \left(\frac{e}{f}\right)^{\frac{-z}{f}}\right) = cb . \frac{fz}{c}$$

$$\frac{1}{2} f . \left(\left(\frac{e}{f}\right)^{\frac{z\sqrt{-1}}{f}} - \left(\frac{e}{f}\right)^{\frac{-z\sqrt{-1}}{f}}\right) = sc . \frac{fz}{c} \qquad \frac{1}{2}$$

$\frac{1}{2} f.\overline{(\frac{e}{f})^{\frac{z\sqrt{-1}}{f}} + (\frac{e}{f})^{\frac{-z\sqrt{-1}}{f}}} = Cc.\frac{fz}{c}$. Singula demonſtranda ſunt, incipiamus a primo.

18. Fiat $\frac{f.(\frac{e}{f})^{\frac{z}{f}} - f(\frac{e}{f})^{\frac{-z}{f}}}{2} = x$, quæ ita ſcribatur

$(\frac{e}{f})^{\frac{z}{f}} - \frac{1}{(\frac{e}{f})^{\frac{z}{f}}} = \frac{2x}{f}$, ſive multiplicando per $(\frac{e}{f})^{\frac{z}{f}}$, & tranſponendo terminos $(\frac{e}{f})^{\frac{2z}{f}} - \frac{2x}{f}.(\frac{e}{f})^{\frac{z}{f}} = 1$, quæ tractata ad modum æquationis quadraticæ, & reſoluta dat $(\frac{e}{f})^{\frac{z}{f}} = \frac{x+\sqrt{ff+xx}}{f}$. Fiat tranſitus a numeris ad logarithmos $z = l\overline{x+\sqrt{ff+xx}}$, & ſumptis differentiis

$$dz = c.\frac{dx+\frac{xdx}{\sqrt{ff+xx}}}{x+\sqrt{ff+xx}} = c.\frac{dx\sqrt{ff+xx}+xdx}{\sqrt{ff+xx}.\overline{x+\sqrt{ff+xx}}} = \frac{c}{f}.\frac{fdx}{\sqrt{ff+xx}},$$

ergo integrando $\frac{fz}{c} = S\frac{fdx}{\sqrt{ff+xx}}$; atqui $S\frac{fdx}{\sqrt{ff+xx}}$ eſt logarithmus analogus, cujus ſinus totus $=f$, ſinus $=x$; ergo etiam $\frac{fz}{c}$; igitur $x$, ſeu $\frac{f}{2}.\overline{(\frac{e}{f})^{\frac{z}{f}} - (\frac{e}{f})^{\frac{-z}{f}}}$ eſt ſinus logarithmi analogi $\frac{fz}{c}$, exiſtente $f$ ſinu toto. Q. E. D.

19. Simili methodo demonſtrabis ſecundam formulam quia eodem facto calculo pervenies ad hanc $z=l\overline{x+\sqrt{xx-ff}}$; ergo differentiando

$$dz = c\frac{dx+\frac{xdx}{\sqrt{xx-ff}}}{x+\sqrt{xx-ff}} = \frac{c}{f}.\frac{fdx}{\sqrt{xx-ff}},$$ & integrando $\frac{fz}{c} = S\frac{fdx}{\sqrt{xx-ff}}$, quæ exhibet logarithmum analogum expreſſum per coſinum $=x$; igitur

Cb.

ob. $\frac{fz}{c} = x = f.(\frac{c}{f})^{\frac{z}{f}} - (\frac{c}{f})^{\frac{-z}{f}}$. Q. E. D.

20. Ad tertiam formulam accedo, quæ quantitates circulares requirit. Fiat $\frac{f}{2\sqrt{-1}}.(\frac{c}{f})^{\frac{z\sqrt{-1}}{f}} - (\frac{c}{f})^{\frac{-z\sqrt{-1}}{f}} = x$; ergo

$(\frac{c}{f})^{\frac{z\sqrt{-1}}{f}} - \frac{1}{(\frac{c}{f})^{\frac{z\sqrt{-1}}{f}}} = \frac{2x\sqrt{-1}}{f}$; igitur

$(\frac{c}{f})^{\frac{2z\sqrt{-1}}{f}} - \frac{2x\sqrt{-1}}{f}.(\frac{c}{f})^{\frac{z\sqrt{-1}}{f}} = 1$. Addatur utrique parti $-\frac{xx}{ff}$, & fiet $(\frac{c}{f})^{\frac{2z\sqrt{-1}}{f}} - \frac{2x\sqrt{-1}}{f}.(\frac{c}{f})^{\frac{z\sqrt{-1}}{f}} - \frac{xx}{ff} = \frac{ff-xx}{ff}$, & extracta radice, translatoque termino $(\frac{c}{f})^{\frac{z\sqrt{-1}}{f}} = \frac{x\sqrt{-1}+\sqrt{ff-xx}}{f}$. Tranſeuntes a numeris ad logarithmos invenimus $\sqrt{-1}.z = l\,x\sqrt{-1}+\sqrt{ff-xx}$, & differentiis acceptis,

$$dz\sqrt{-1} = c.\frac{dx\sqrt{-1} - \frac{xdx}{\sqrt{ff-xx}}}{x\sqrt{-1}+\sqrt{ff-xx}} = c.\frac{dx\sqrt{-1}.\sqrt{ff-xx} - xdx}{\sqrt{ff-xx}.x\sqrt{-1}+\sqrt{ff-xx}},$$

& facta diviſione per $\sqrt{-1}$ oritur $dz = c.\frac{dx\sqrt{ff-xx} + \frac{1}{\sqrt{-1}}xdx}{\sqrt{ff-xx}.x\sqrt{-1}+\sqrt{ff-xx}}$;

atqui $\frac{-1}{\sqrt{-1}} = \sqrt{-1}$, ergo $dz = c.\frac{dx\sqrt{ff-xx}+xdx\sqrt{-1}}{\sqrt{ff-xx}.x\sqrt{-1}+\sqrt{ff-xx}} = \frac{c}{f}.\frac{fdx}{\sqrt{ff-xx}}$;

ergo $\frac{fz}{c} = S\frac{fdx}{\sqrt{ff-xx}}$, quæ eſt arcus circularis, cujus radius $=f$, ſinus $=x$;

igitur $x = \frac{f}{2\sqrt{-1}}.(\frac{c}{f})^{\frac{z\sqrt{-1}}{f}} - (\frac{c}{f})^{\frac{-z\sqrt{-1}}{f}} = Sc.\frac{fz}{c}$. Q. E. D.

21. Quoad ultimam poſita $(\frac{c}{f})^{\frac{z\sqrt{-1}}{f}} + (\frac{c}{f})^{\frac{-z\sqrt{-1}}{f}} = \frac{2x}{f}$, pervenimus ad formulam $(\frac{c}{f})^{\frac{2z\sqrt{-1}}{f}} - \frac{2x}{f}(\frac{c}{f})^{\frac{z\sqrt{-1}}{f}} = -1$, & addito utrobique quadrato $\frac{xx}{ff}$, fiet $\left((\frac{c}{f})^{\frac{z\sqrt{-1}}{f}} - \frac{x}{f}\right)^2 = \frac{xx-ff}{ff} = \frac{ff-xx}{ff}.-1$, & extracta radice $(\frac{c}{f})^{\frac{z\sqrt{-1}}{f}} = \frac{x+\sqrt{ff-xx}.\sqrt{-1}}{f}$. Tranſi ad logarithmos, ut habeas $z\sqrt{-1} = l\,\overline{x+\sqrt{ff-xx}.\sqrt{-1}}$. Accipe differentias

$$dz\sqrt{-1} = c.\frac{dx - \frac{xdx\sqrt{-1}}{\sqrt{ff-xx}}}{x+\sqrt{ff-xx}.\sqrt{-1}} = c.\frac{dx\sqrt{ff-xx} - xdx\sqrt{-1}}{\sqrt{ff-xx}.\overline{x+\sqrt{ff-xx}.\sqrt{-1}}}.$$

Divide per $\sqrt{-1}$, ut ſit $dz = c.\frac{\frac{1}{\sqrt{-1}}dx\sqrt{ff-xx} - xdx}{\sqrt{ff-xx}.\overline{x+\sqrt{ff-xx}.\sqrt{-1}}}$; atqui $\frac{1}{\sqrt{-1}} = -\sqrt{-1}$; ergo

$$dz = c.\frac{-dx\sqrt{ff-xx}\sqrt{-1} - xdx}{\sqrt{ff-xx}.\overline{x+\sqrt{ff-xx}\sqrt{-1}}} = \frac{c}{f}.\frac{-fdx}{\sqrt{ff-xx}}; \text{ ergo } \frac{fz}{c} = S\frac{-fdx}{\sqrt{ff-xx}},$$

quæ exhibet arcum circularem radii $= f$, & coſinus $= x$; igitur

$$x = \frac{f}{2}.(\frac{c}{f})^{\frac{z\sqrt{-1}}{f}} + (\frac{c}{f})^{\frac{-z\sqrt{-1}}{f}} = Cc.\frac{fz}{c}. \text{ Q. E. D.}$$

22. Per calculum logarithmicum ſolvuntur aliquot problemata arithmetica, quæ alia ratione vix ſolvi poſſunt. Sit problema primum. Ex vaſe $= a$ vino repleto haurio menſuram $= b$; & aqua repleo; tum ſecundam menſuram adimo, & aqua ſuppleo, quod ademptum eſt; ita fit in tertia, quarta &c. menſura: quæritur quot menſuræ hauriendæ ſint, ut in vaſe reliqua ſit vini quantitas $= m$.

Poſt

| Poſt hauſtas menſuras | vinum remanet |
|---|---|
| 1 | $a-b$ |
| 2 | $\frac{\overline{a-b}^2}{a}$ |
| 3 | $\frac{\overline{a-b}^3}{a^2}$ |
| 4 | $\frac{\overline{a-b}^4}{a^3}$ |
| ⋮ | ⋮ |
| $n$ | $\frac{\overline{a-b}^n}{a^{n-1}}$ |

Igitur ſi numerus menſurarum extrahendarum ſit $=n$, ut vini quantitas $=m$ reliqua ſit, prodibit æquatio

$$\frac{\overline{a-b}^n}{a^{n-1}}=m,$$

in qua, quum incognita quantitas ſit in exponente, neceſſe eſt tranſire a numeris ad logarithmos, ut fiat coefficiens. Facto autem hoc tranſitu erit $n\,l\,\overline{a-b}-(\overline{n-1}.\,la=lm$; ergo $la-lm=n.\overline{la-l\,\overline{a-b}}$; ergo

$$\frac{la-lm}{la-l\,\overline{a-b}}=n,$$

quæ quæſitæ quantitatis exhibet valorem. Quoniam autem idem provenit quocumque poſito protonumero, & ſubtangente; hinc uti poſſumus quolibet logarithmorum ſyſtemate.

23. Problema alterum. Numerus hominum quotannis in provincia augetur parte ſui $p^{eſima}$, quæritur quot anni neceſſarii ſunt, ut numerus augeatur in ratione $1:m$. Numerus hominum ſit $=a$

| poſt annos | numerus erit |
|---|---|
| 1 | $\frac{a.\overline{p+1}}{p}$ |
| 2 | $\frac{a.\overline{p+1}^2}{p^2}$ |
| 3 | $\frac{a.\overline{p+1}^3}{p^3}$ |
| ⋮ | ⋮ |
| $n$ | $\frac{a.\overline{p+1}^n}{p^n}$ |

Sit numerus annorum quæſitus $=n$; ergo ex conditione problematis

$$\frac{a.\overline{p+1}^n}{p^n}=ma,$$

ſive

$$\frac{\overline{p+1}^n}{p^n}=m;$$

& facto tranſitu ad logarithmos $n.\,l\overline{p+1}-nlp=lm$, ſive

$$n=\frac{lm}{l\overline{p+1}-lp}.$$

24. Problema tertium. Cæſar convenit cum creditore, ſe ſingulis annis ſoluturum penſionem $=b$, donec omne debitum $=a$ extinctum fuerit. Supponentes pecuniæ fructum annuum eſſe ſortis partem $p^{eſimam}$, quærimus quot annis penſionem ſolvere debeat. Facto calculo invenies

| poſt annos | æs alienum reliquum |
|---|---|
| 1 | $a.\frac{p+1}{p} - b$ |
| 2 | $a.\frac{\overline{p+1}^2}{p^2} - 2b - \frac{b}{p}$ |
| 3 | $a.\frac{\overline{p+1}^3}{p^3} - 3b - \frac{3b}{p} - \frac{b}{pp}$ |
| 4 | $a.\frac{\overline{p+1}^4}{p^4} - 4b - \frac{6b}{p} - \frac{4b}{pp} - \frac{b}{p^3}$ |
| 5 | $a.\frac{\overline{p+1}^5}{p^5} - 5b - \frac{10b}{p} - \frac{10b}{p^2} - \frac{5b}{p^3} - \frac{b}{p^4}$ |
| | &c. |

Hæc tabula, ut commodior fiat, & ad generales formulas nos perducat; ita erit exponenda

| poſt annos | æs alienum reliquum eſt |
|---|---|
| 1 | $a.\frac{p+1}{p} - bp.\frac{p+1}{p} + bp$ |
| 2 | $a.\frac{\overline{p+1}^2}{p^2} - bp.\frac{\overline{p+1}^2}{p^2} + bp$ |
| 3 | $a.\frac{\overline{p+1}^3}{p^3} - bp.\frac{\overline{p+1}^3}{p^3} + bp$ |
| 4 | $a.\frac{\overline{p+1}^4}{p^4} - bp.\frac{\overline{p+1}^4}{p^4} + bp$ |
| 5 | $a.\frac{\overline{p+1}^5}{p^5} - bp.\frac{\overline{p+1}^5}{p^5} + bp$ atque ita deinceps. |

Quare generatim poſt numerum annorum $= n$ æs alienum redactum erit ad $a.\frac{\overline{p+1}^n}{p^n} - bp.\frac{\overline{p+1}^n}{p^n} + bp$. Quare ſi $n$ ſit ille annorum numerus, poſt quem nihil prorſus debet, inventa quantitas $= o$; ergo naſcitur æquatio $a.\frac{\overline{p+1}^n}{p^n} = bp.\frac{\overline{p+1}^n}{p^n} - bp$, ſeu facta multiplicatione per $p^n$, & terminorum

translatione $bp^{n+1} = \overline{bp-a}\cdot\overline{p+1}^{n}$.

25. Alia methodo ad hanc æquationem poſſumus pervenire. Nam poſt annum primum æs alienum $= a\cdot\frac{p+1}{p} - b$. In fine ſecundi anni invenietur $= a\cdot\frac{\overline{p+1}^2}{p^2} - 2b - \frac{b}{p} = a\cdot\frac{\overline{p+1}^2}{p^2} - b\cdot\frac{p+1}{p} - b$. Simili modo in fine anni tertii, quod debetur, erit $= a\cdot\frac{\overline{p+1}^3}{p^3} - b\cdot\frac{\overline{p+1}^2}{p^2} - b\cdot\frac{p+1}{p} - b$; in fine quarti $= a\cdot\frac{\overline{p+1}^4}{p^4} - b\cdot\frac{\overline{p+1}^3}{p^3} - b\cdot\frac{\overline{p+1}^2}{p^2} - b\cdot\frac{p+1}{p} - b$; atque ita deinceps. Quare generatim poſt quemlibet annorum numerum $= n$, æs alienum erit reliquum

$$a\cdot\frac{\overline{p+1}^{n}}{p^{n}} - b\cdot\overline{\frac{\overline{p+1}^{n-1}}{p^{n-1}} + \frac{\overline{p+1}^{n-2}}{p^{n-2}} + \frac{\overline{p+1}^{n-3}}{p^{n-3}} \ldots \frac{p+1}{p} + 1}.$$

Series geometrica, quæ multiplicat quantitatem $b$, per regulas in ſuperiore tomo a nobis traditas nullo negotio colligitur in ſummam. Ea autem eſt $= \frac{\overline{p+1}^{n} - p^{n}}{p^{n-1}}$. Igitur poſt annorum numerum $= n$ remanet æs alienum

$$a\cdot\frac{\overline{p+1}^{n}}{p^{n}} - b\cdot\frac{\overline{p+1}^{n} - p^{n}}{p^{n-1}}.$$

Quapropter ſi $n$ ſit numerus annorum, poſt quos omne ſolutum eſt, quantitas ultimo inventa debet $= 0$; ergo prodibit æquatio $a\cdot\frac{\overline{p+1}^{n}}{p^{n}} = b\cdot\frac{\overline{p+1}^{n} - p^{n}}{p^{n-1}}$, ſive $bp^{n+1} = \overline{bp-a}\cdot\overline{p+1}^{n}$, prorſus ut antea.

26. Ut ab æquatione inventa eruatur valor numeri $n$ per logarithmos accepto quocumque protonumero $= f$, in hunc modum ſcribatur æquatio

$$\frac{b\cdot\overline{pf}^{n+1}}{f^{n+1}} = \overline{bp-a}\cdot\frac{\overline{\overline{p+1}\cdot f}^{n}}{f^{n}}.$$

Fiat a numeris ad logarithmos tranſitus, ut fit $lb + \overline{n+1}\cdot lpf = l\overline{bp-a} + nl\overline{p+1}\cdot f$; ergo $\frac{lb + lpf - l\overline{bp-a}}{l\overline{p+1}\cdot f - lpf} = n$. Hæc ſatis ſint de calculo logarithmico.

27. Loquuti ſumus in ſuperiore tomo de perutili theoremate newtoniano docente, binomium quodlibet $x + a$ elatum ad poteſtatem $m$ eſſe æqualem

ſeriei

ſeriei $x^m + max^{m-1} + \frac{m.\overline{m-1}}{2}a^2x^{m-2} + \frac{m.\overline{m-1}.\overline{m-2}}{2.3}a^3x^{m-3}$ &c.

Veritatem hujuſce theorematis demonſtravimus non ſolum quum exponens $m$ eſt numerus integer, & poſitivus, quo in caſu ſeries abrumpitur, & finito conſtat numero terminorum, ſed etiam quum $m$ eſt negativus, & fractus, abeunte ſerie in infinitum. Verum ſi $m$ ſit numerus ſurdus, diximus, demonſtrationem perfici non poſſe ſine calculo infiniteſimali. Nunc poſt traditam logarithmorum theoriam promiſſa ſolvamus, & compleamus demonſtrationem. Quamobrem dividamus æquationem, de cujus veritate inquirimus per $f^{m-1}$, ut ſit

$$\frac{\overline{x+a}^m}{f^{m-1}} = \frac{x^m + max^{m-1} + \frac{m.\overline{m-1}}{2}a^2x^{m-2} + \frac{m.\overline{m-1}.\overline{m-2}}{2.3}a^3x^{m-3}}{f^{m-1}} \text{ \&c.}$$

Fiat tranſitus a numeris ad logarithmos, ut oriatur $ml\frac{\overline{x+a}}{f^{m-1}} =$

$$l\,\frac{x^m + max^{m-1} + \frac{m.\overline{m-1}}{2}a^2x^{m-2} + \frac{m.\overline{m-1}.\overline{m-2}}{2.3}a^3x^{m-3}}{f^{m-1}} \text{ \&c.}$$

Supponens $a$ conſtantem, $x$ variabilem, accipe logarithmorum differentias, & erit $\frac{mdx}{x+a} =$

$$\frac{mx^{m-1}dx + m.\overline{m-1}.\,x^{m-2}dx + \frac{m.\overline{m-1}.\overline{m-2}}{2}a^2x^{m-3}dx + \frac{m.\overline{m-1}.\overline{m-2}.\overline{m-3}}{2.3}a^3x^{m-4}dx \text{ \&c.}}{x^m + max^{m-1} + \frac{m.\overline{m-1}}{2}a^2x^{m-2} + \frac{m.\overline{m-1}.\overline{m-2}}{2.3}a^3x^{m-3} \text{ \&c.}}$$

Si ab hac æquatione ejicias diviſores facta, ut par eſt, congrua multiplicatione, invenies

$$mx^m dx + m^2ax^{m-1}dx + \frac{m^2.\overline{m-1}}{2}a^2x^{m-2}dx + m^2.\overline{m-1}.\overline{m-2}\,a^3x^{m-3}dx \text{ \&c.}$$

$$=$$

$$mx^m dx + m.\overline{m-1}.ax^{m-1}dx + \frac{m.\overline{m-1}.\overline{m-2}}{2}a^2x^{m-2}dx + \frac{m.\overline{m-1}.\overline{m-2}.\overline{m-3}}{2.3}a^3x^{m-3}dx \text{ \&c.}$$

$$+ max^{m-1}dx + m.\overline{m-1}\,a^2x^{m-2}dx + \frac{m.\overline{m-1}.\overline{m-2}}{2}a^3x^{m-3}dx \text{ \&c.}$$

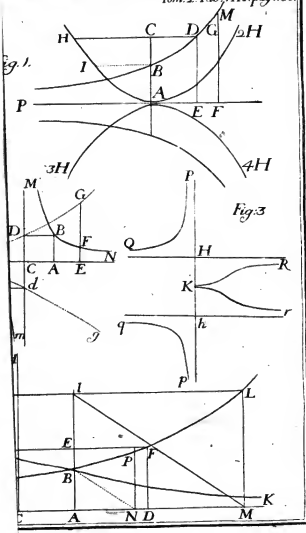
Collige in hac ſecunda parte æquationis in ſummam terminos ſimiles, & habebis

$$mx^m dx + mmax^{m-1}dx + \frac{m^2.\overline{m-1}}{2}a^2x^{m-2}dx + \frac{m^2.\overline{m-1}.\overline{m-2}}{2.3}a^3x^{m-3}dx \text{ \&c.},$$

quæ identica eſt cum prima æquationis parte. Identicitas autem hæc ſatis oſtendit, æquationem illam primam, ex qua colligitur, veriſſimam eſſe; nam ex hac identica æquatione regredi poſſum, iiſdemque veſtigiis inſiſtendo theorema newtonianum demonſtrare.

CA-

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Fig. 1.
H
C
D
G
M
2H
I
B
A
P
E
F
3H
4H
M
G
D
B
F
N
C
A
E
d
g
m
P
Fig: 3
Q
H
R
K
r
q
h
p
I
L
E
F
P
B
K
A
N
D
M

## CAPUT DECIMUM.

### De calculo ſinuum, & coſinuum, & de integratione formularum, in quibus differentia logarithmi analogi, aut arcus circularis multiplicatur aut dividitur per ſinus, & coſinus.

1. PErutiles ſane ſunt formulæ ſpectantes ad ſinus, & coſinus arcuum, & logarithmorum analogorum ductorum in numerum $n$, quas exhibuimus Tom. I. Cap. 12. L. 2. Verum demonſtrationes protulimus pro hypotheſi numeri $n$ rationalis vel integri vel fracti, vel poſitivi vel negativi. Diximus autem pro hypotheſi $n$ numeri ſurdi demonſtrationem perfici non poſſe niſi per calculum infiniteſimalem. Ut autem hanc in præſentia exponam ope theoriæ logarithmicæ, præmitto hoc lemma. Vocato ſinu toto $= r$, arcu aut logarithmo $= \mu$, valet æquatio $\frac{C.\mu.dS.\mu - S.\mu.dC.\mu}{r} = d\mu$, quæ eodem modo vera eſt cum in arcubus, tum in logarithmis analogis, cum in ſinibus, & coſinibus circularibus, tum in hyperbolicis. Demonſtratur autem inſpectis figuris in ſequentem modum.

2. In hyperbola ſector CAF æquat (*Fig.* 1.) triangulum CDF — ſemiſegmentum ADF, ſive $\frac{r\mu}{2} = \frac{Cb.\mu.Sb.\mu}{2} - \int Sb.\mu.dCb.\mu$, & ſumptis differentiis $d\mu = \frac{Cb.\mu.dSb.\mu - Sb.\mu.dCb.\mu}{r}$. Q. E. D.

3. In circulo autem ſector CAF $=$ (*Fig.* 2.) triang. CDF $+$ ſemiſeg. ADF, ſive $\frac{r\mu}{2} = \frac{Cc.\mu.Sc.\mu}{2} + \int - Sc.\mu.dCc.\mu$. Apponitur ſignum $-$, quia creſcente ſectore, decreſcit coſinus, ejuſque differentia eſt negativa. Igitur acceptis differentiis fit æquatio $d\mu = \frac{Cc.\mu.dSc.\mu - Sc.\mu.dCc.\mu}{r}$. Q. E. D.

4. Ex his aliud lemma deduco, quod his formulis continetur

$$dSb.\mu = \frac{d\mu.Cb.\mu}{r};\ dCb.\mu = \frac{d\mu.Sb.\mu}{r}$$

$$dSc.\mu = \frac{d\mu.Cc.\mu}{r},\ dCc.\mu = \frac{-d\mu.Sc.\mu}{r}.$$

5. In hyperbola conſtat $\overline{Cb.\mu}^2 = r^2 + \overline{Sb.\mu}^2$; ergo ſumptis differentiis $Cb.\mu.dCb.\mu = Sb.\mu.dSb.\mu$, ſive $dCb.\mu = \frac{Sb.\mu.dSb.\mu}{Cb.\mu}$, & $dSb.\mu = \frac{Cb.\mu.dCb.\mu}{Sb.\mu}$. Qui valores ſubſtituti in formula num. 2. dant

$$Cb.\mu.dSb.\mu - \frac{\overline{Sb.\mu}^2}{Cb.\mu}dSb.\mu = rd\mu,\ \frac{\overline{Cb.\mu}^2}{Sb.\mu}dCb.\mu - Sb.\mu.dCb.\mu = rd\mu;$$

atqui

atqui $\overline{Cb.\mu}^2 - \overline{Sb.\mu}^2 = r^2$; ergo facta ſubſtitutione, expurgatiſque formulis $dSb.\mu = \frac{d\mu\, Cb.\mu}{r}$, $dCb.\mu = \frac{d\mu\, Sb.\mu}{r}$.

6. Simili ratiocinio utens in circulo, uſurpata æquatione ipſi propria $\overline{Cc.\mu}^2 = r^2 - \overline{Sc.\mu}^2$ perveniam ad formulas $dSc.\mu = \frac{d\mu . Cc.\mu}{r}$, $dCc.\mu = \frac{- d\mu . Sc.\mu}{r}$. Quæ formulæ ex ipſa inſpectione figuræ deducuntur. Nam triangulum Fef infiniteſimum eſt ſimile FDC (*Fig.* 2); igitur GF : CD :: Ff : fe, & CF : DF :: Ff : Fe, $r : Cc.\mu :: d\mu : dSc.\mu$, & $r : Sc.\mu :: d\mu : dCc.\mu$, ex quibus analogiis ſuperiores formulæ ſtatim prodibunt.

7. His demonſtratis ad rem accedo propius. Quiſque videt, identicas eſſe ac proinde æquales hujuſmodi formulas $\frac{Sb.\mu + Cb.\mu}{Cb.\mu + Sb.\mu} = \frac{Sb.n\mu + Cb.n\mu}{Cb.n\mu + Sb.n\mu}$; denominatores enim iidem ſunt cum numeratoribus. Multiplicetur utraque per $n d\mu$, ut oriatur $n . \frac{d\mu . Sb.\mu + d\mu . Cb.\mu}{Cb.\mu + Sb.\mu} = \frac{n d\mu . Sb.n\mu + n d\mu . Cb.n\mu}{Cb.n\mu + Sb.n\mu}$; atqui ex probatis $d\mu . Sb.\mu = r . dCb.\mu$, $n d\mu . Sb.n\mu = r . dCb.n\mu$, $d\mu . Cb.\mu = r . dSb.\mu$, $n d\mu . Cb.n\mu = r . dSb.n\mu$; ergo factis ſubſtitutionibus, proveniet

$$n . \frac{r d . Cb.\mu + r . dSb.\mu}{Cb.\mu + Sb.\mu} = \frac{r . dCb.n\mu + r . dSb.n\mu}{Cb.n\mu + Sb.n\mu}.$$

Utraque expreſſio eſt logarithmica. Poſita ſyſtematis ſubtangente $= r$, fiat integratio, hac ſervata conditione, ut quum $\mu$, & $Sb.\mu = 0$, fit $Cb.\mu = r$. Habebitur $n l . \overline{Cb.\mu + Sb.\mu} = l \overline{Cb.n\mu + Sb.n\mu} + \overline{n - 1} . l r$. Si eſſet protonumerus $= r$, ut logarithmi acciperentur in ſyſtemate hyperbolico, ſuperflua eſſet additio conſtantis $\overline{n - 1} . l r$. Sed malo uti quocumque ſyſtemate, ut appareat, demonſtrationem a peculiari ſyſtemate non dependere. Fiat tranſitus a logarithmis ad numeros, & proveniet $\frac{\overline{Cb.\mu + Sb.\mu}^n}{r^{n-1}} = Cb.n\mu + Sb.n\mu$.

8. Aſſumpta prima identica formula $\frac{Sb.\mu - Cb.\mu}{Cb.\mu - Sb.\mu} = \frac{Sb.n\mu - Cb.n\mu}{Cb.n\mu - Sb.n\mu}$ eodem ratiocinio probabis $\frac{\overline{Cb.\mu - Sb.\mu}^n}{r^{n-1}} = Cb.n\mu - Sb.n\mu$.

9. Progredior ad ſinus, & coſinus circulares. Valet æquatio, in qua diviſores iidem ſunt, ac numeratores

$$\frac{\sqrt{-1} . Sc.\mu + Cc.\mu}{Cc.\mu + \sqrt{-1} . Sc.\mu} = \frac{\sqrt{-1} . Sc.n\mu + Cc.n\mu}{Cc.n\mu + \sqrt{-1} . Sc.n\mu}.$$

Utraque æquationis pars multiplicetur per $\sqrt{-1}$, & fiet

$$\frac{- Sc.\mu + \sqrt{-1} . Cc.\mu}{Cc.\mu + \sqrt{-1} . Sc.\mu} = \frac{- Sc.n\mu + \sqrt{-1} . Cc.n\mu}{Cc.n\mu + \sqrt{-1} . Sc.n\mu}.$$

Utraque multiplicetur

tur per $nd\mu$, ut naſcatur

$$n.\frac{-d\mu.Sc.\mu+d\mu.\sqrt{-1}.Cc.\mu}{Cc.\mu+\sqrt{-1}.Sc.\mu}=\frac{-nd\mu.Sc.n\mu+nd\mu\sqrt{-1}.Cc.n\mu}{Cc.n\mu+\sqrt{-1}.Sc.n\mu}.$$

Sed $-d\mu.Sc.\mu=r.dCc.\mu$, $-nd\mu.Sc.n\mu=r.dCc.n\mu$

$d\mu.Cc.\mu=r.dSc.\mu$, $nd\mu.Cc.n\mu=r.dSc.n\mu$.

Factis itaque ſubſtitutionibus, oritur

$$n.\frac{r.dCc.\mu+r\sqrt{-1}.dSc.\mu}{Cc.\mu+\sqrt{-1}.Sc.\mu}=\frac{r.dCc.n\mu+r\sqrt{-1}.dSc.n\mu}{Cc.n\mu+\sqrt{-1}.Sc.n\mu}.$$

Accepta ſyſtematis ſubtangente $=r$, fiat integratio per logarithmos, ſervata conditione, ut evadat $Cc.\mu=r$, quum $\mu$, & $Sc\mu$. nulleſcunt, erit

$n.l\overline{Cc.\mu+\sqrt{-1}.Sc.\mu}=l\overline{Cc.n\mu+\sqrt{-1}.Sc.n\mu}+\overline{n-1}.lr$ & facto tranſitu ad numeros $\frac{\overline{Cc.\mu+\sqrt{-1}.Sc.\mu}^{\,n}}{r^{n-1}}=Cc.n\mu+\sqrt{-1}.Sc.n\mu$.

10. Si primam formulam identicam ponas eſſe

$\frac{\sqrt{-1}.Sc.\mu-Cc.\mu}{Cc.\mu-\sqrt{-1}.Sc.\mu}=\frac{\sqrt{-1}.Sc.n\mu-Cc.n\mu}{Cc.n\mu-\sqrt{-1}.Sc.n\mu}$, idem ratiocinium inſtituens,

invenies $\frac{\overline{Cc.\mu-\sqrt{-1}.Sc.\mu}^{\,n}}{r^{n-1}}=Cc.n\mu-\sqrt{-1}.Sc.n\mu$.

11. Ex his omnibus formulis propter ambiguitatem ſignorum orientur formulæ quatuor

$$Cb.n\mu=\frac{\overline{Cb.\mu+Sb.\mu}^{\,n}+\overline{Cb.\mu-Sb.\mu}^{\,n}}{2r^{n-1}}$$

$$Sb.n\mu=\frac{\overline{Cb.\mu+Sb.\mu}^{\,n}-(\overline{Cb.\mu-Sb.\mu}^{\,n}}{2r^{n-1}}$$

$$Cc.n\mu=\frac{\overline{Cc.\mu+\sqrt{-1}.Sc.\mu}^{\,n}+\overline{Cc.\mu-\sqrt{-1}.Sc.\mu}^{\,n}}{2r^{n-1}}$$

$$Sc.n\mu=\frac{\overline{Cc\mu+\sqrt{-1}.Sc.\mu}^{\,n}-(\overline{Cc.\mu-\sqrt{-1}.Sc.\mu}^{\,n}}{2r^{n-1}.\sqrt{-1}}.$$

Hæ autem ſunt formulæ illæ ipſæ, quas in primo tomo probavimus, quoties $n$ eſſet numerus rationalis. Demonſtratio autem hæc eas veras pariter oſtendit, tametſi $n$ ſit numerus irrationalis & ſurdus.

12. Ex his inveniamus formulas pertinentes ad tangentes. Conſtat tam in logarithmis, quam in arcubus eſſe coſinum ad ſinum ut ſinus totus ad tangentem; ergo $S=\frac{C.T}{r}$. Quapropter ſi in duabus primis formulis pro $Sb.\mu$ ſubſti-

 tuas

tuas $\frac{Cb.\mu.Tb.\mu}{r}$, invenies

$$Cb.n\mu = \frac{\overline{Cb.\mu}^n}{r^n} \cdot \frac{\overline{r+Tb.\mu}^n + \overline{r-Tb.\mu}^n}{2r^n}$$

$$Sb.n\mu = \frac{\overline{Cb.\mu}^n}{r^n} \cdot \frac{\overline{r+Tb.\mu}^n - \overline{r-Tb.\mu}^n}{2r^n}$$; ergo

$$\frac{r.Sb.n\mu}{Cb.n\mu} = Tb.n\mu = r.\frac{\overline{r+Tb.\mu}^n - \overline{r-Tb.\mu}^n}{\overline{r+Tb.\mu}^n + \overline{r-Tb.\mu}^n}.$$ Simili facta ſubſtitutione, hoc eſt pro $Sc.\mu$ poſito $\frac{Cc.\mu Tc.\mu}{r}$, provenies ad formulam

$$\frac{r.Sc.\mu}{Cc.\mu} = Tc.\mu = \frac{r}{\sqrt{-1}} \cdot \frac{\overline{r+\sqrt{-1}.Tc.\mu}^n - \overline{r-\sqrt{-1}.Tc.\mu}^n}{\overline{r+\sqrt{-1}.Tc.\mu}^n + \overline{r-\sqrt{-1}.Tc.\mu}^n}$$

Cotangentium formulas quiſque ſibi comparare poteſt ope theorematis docentis, tangentem eſſe ad ſinum totum, ut ſinus totus ad cotangentem.

13. Per eas formulas, quas poſui num. 4. demonſtrantur quatuor theoremata, quæ ſequuntur

$$m S\overline{Cb.\mu}^m.d\mu = \overline{m-1}.r^2.S\overline{Cb.\mu}^{m-2}d\mu + r\overline{Cb.\mu}^{m-1}Sb.\mu$$

$$m S\overline{Sb.\mu}^m.d\mu = -(\overline{m-1}.r^2 S\overline{Sb.\mu}^{m-2}d\mu + r\overline{Sb.\mu}^{m-1}Cb.\mu$$

$$m S\overline{Cc.\mu}^m.d\mu = \overline{m-1}.r^2 S\overline{Cc.\mu}^{m-2}d\mu + r\overline{Cc.\mu}^{m-1}Sc.\mu$$

$$m S\overline{Sc.\mu}^m.d\mu = \overline{m-1}.r^2 S\overline{Sc.\mu}^{m-2}d\mu - r\overline{Sc.\mu}^{m-1}Cc.\mu.$$

Omnium demonſtrationes eadem methodo perficiuntur.

14. Quoad primum manifeſtum eſt

$$D\overline{Cb.\mu}^{m-1}Sb.\mu = \overline{m-1}.\overline{Cb.\mu}^{m-2}.Sb.\mu.dCb.\mu + \overline{Cb.\mu}^{m-1}dSb.\mu.$$

Subſtitue in hac pro $dCb.\mu$, $dSb.\mu$ eorum valores, & invenies

$$D.\overline{Cb.\mu}^{m-1}Sb.\mu = \frac{\overline{m-1}.\overline{Cb.\mu}^{m-2}.\overline{Sb.\mu}^2.d\mu + \overline{Cb.\mu}^m.d\mu}{r}.$$ Pro

$\overline{Sb.\mu}^2$ ſubſtitue $\overline{Cb.\mu}^2 - r^2$, ut habeas

$$r D.\overline{Cb.\mu}^{m-1}Sb.\mu = \overline{m-1}.\overline{Cb.\mu}^m d\mu - (\overline{m-1}.r^2.\overline{Cb.\mu}^{m-2}d\mu$$
$$+ \overline{Cb.\mu}^m d\mu$$

igitur

$$m\overline{Cb.\mu}^m d\mu = \overline{m-1}.r^2.\overline{Cb.\mu}^{m-2}d\mu + rD.\overline{Cb.\mu}^{m-1}Sb.\mu,$$ ſive

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ſive integrando

$m\,\mathrm{S}\,\overline{Cb.\mu}^{m}\,d\mu = \overline{m-1}.r^{2}\,\mathrm{S}\,\overline{Cb.\mu}^{m-2}\,d\mu + r.\overline{Cb.\mu}^{m-1}.Sb.\mu$. Q. E. D.

15. Alterum theorema oſtenditur per formulam

$D.\overline{Sb.\mu}^{m-1}\,Cb.\mu = \overline{m-1}.\overline{Sb.\mu}^{m-2}.Cb.\mu.dSb.\mu + \overline{Sb.\mu}^{m-1}\,dCb.\mu$,

factiſque ut antea ſubſtitutionibus perveniemus ad

$rD.\overline{Sb.\mu}^{m-1}\,Cb.\mu = \overline{m-1}.r^{2}.\overline{Sb.\mu}^{m-2}\,d\mu + \overline{m-1}.\overline{Sb.\mu}^{m}\,d\mu$

$+\overline{Sb.\mu}^{m}\,d\mu$;

ergo translatis terminis, factaque integratione

$m\mathrm{S}.\overline{Sb.\mu}^{m}\,d\mu = -(\overline{m-1}.r^{2}\,\mathrm{S}\,\overline{Sb.\mu}^{m-2}\,d\mu + r\overline{Sb.\mu}^{m-1}\,Cb.\mu$.
Q. E. D.

16. Similem methodum applica reliquis duabus formulis, quæ quantitates circulares complectuntur. Nam

$D.\overline{Cc.\mu}^{m-1}\,Sc.\mu = \overline{m-1}.\overline{Cc.\mu}^{m-2}.Sc.\mu\,dCc.\mu + \overline{Cc.\mu}^{m-1}\,dSc.\mu$.

Pro $dCc.\mu$, $dSc.\mu$ valores ſubſtitue

$D.\overline{Cc.\mu}^{m-1}\,Sc.\mu = -(\frac{\overline{m-1}.\overline{Cc.\mu}^{m-2}.\overline{Sc.\mu}^{2}.d\mu + \overline{Cc.\mu}^{m}.d\mu}{r}$.

Pro $\overline{Sc.\mu}^{2}$ ſcribe $r^{2} - \overline{Cc.\mu}^{2}$, ut facta multiplicatione per $r$ proveniat

$rD.\overline{Cc.\mu}^{m-1}\,Sc.\mu = -\overline{m-1}.r^{2}\,\overline{Cc.\mu}^{m-2}\,d\mu + \overline{m-1}.\overline{Cc.\mu}^{m}\,d\mu$

$+\overline{Cc.\mu}^{m}\,d\mu$;

ergo $m\,\mathrm{S}\,\overline{Cc.\mu}^{m}\,d\mu = \overline{m-1}.r^{2}\,\mathrm{S}\,\overline{Cc.\mu}^{m-2}\,d\mu + r\,\overline{Cc.\mu}^{m-1}.Sc.\mu$.
Q. E. D.

17. Ultimum theorema eadem ratione oſtenditur. Etenim ex formula

$D.\overline{Sc.\mu}^{m-1}\,Cc.\mu = \overline{m-1}.\overline{Sc.\mu}^{m-2}.Cc.\mu.d\mu + \overline{Sc.\mu}^{m-1}\,dCc.\mu$

invenies

$m\mathrm{S}.\overline{Sc.\mu}^{m}\,d\mu = \overline{m-1}.r^{2}\,\mathrm{S}\,\overline{Sc.\mu}^{m-2}\,d\mu - r\,\overline{Sc.\mu}^{m-1}.Cc.\mu$. Q. E. D.

18. Ex quatuor, quæ demonſtravi, theorematibus aliquot primum corollaria maxime ſimplicia deducamus. Si ſupponas $m=o$, invenies

$r\,\mathrm{S}\frac{d\mu}{\overline{Cb.\mu}^{2}} = \frac{Sb.\mu}{Cb.\mu}$, $-r\,\mathrm{S}\frac{d\mu}{\overline{Sb.\mu}^{2}} = \frac{Cb.\mu}{Sb.\mu}$

$r\,\mathrm{S}\frac{d\mu}{\overline{Cc.\mu}^{2}} = \frac{Sc.\mu}{Cc.\mu}$, $-r\,\mathrm{S}\frac{d\mu}{\overline{Sc.\mu}^{2}} = \frac{Cc.\mu}{Sc.\mu}$; ex quibus conſtat $d\mu$ diviſam per quadratum ſinus aut coſinus integrabilem eſſe, vel quantitates hyperbolicæ ſint vel circulares. Si $m=1$, habebimus

$S.Cb.\mu.d\mu = r.Sb.\mu$ , $S.Sb.\mu.d\mu = r.Cb.\mu$

$S.Cc.\mu.d\mu = r.Sc.\mu$ , $S.Sc.\mu.d\mu = -r.Cc.\mu$, quæ formulæ illæ ipſæ ſunt, quas ſuppoſuimus. Demum ſi $m = 2$, naaciſcemur

$2S.\overline{Cb.\mu}^2.d\mu = r^2.\mu + rCb.\mu.Sb.\mu$, $2S.\overline{Sb.\mu}^2.d\mu = -r^2\mu + r.Sb.\mu.Cb.\mu$

$2S.\overline{Cc.\mu}^2.d\mu = r^2.\mu + rCc.\mu.Sc.\mu$, $2S.\overline{Sc.\mu}^2.d\mu = r^2\mu - rSc.\mu.Cc.\mu$

Quare differentia logarithmi $\mu$ in quadratum ſinus, vel coſinus dependet a quadratura hyperbolæ, & differentia arcus circularis multiplicata per quadratum ſinus, vel coſinus dependet a quadratura circuli.

19. Inter has formulas non invenies quatuor maxime ſimplices, nempe $\frac{d\mu}{Cb.\mu}$, $\frac{d\mu}{Sb.\mu}$, $\frac{d\mu}{Cc.\mu}$, $\frac{d\mu}{Sc.\mu}$. Nam ſi ſupponeres $m = -1$, iſtæ quidem prodirent in primo termino, ſed conjungerentur cum quatuor altioribus $\frac{d\mu}{\overline{Cb.\mu}^3}$, $\frac{d\mu}{\overline{Sb.\mu}^3}$, $\frac{d\mu}{\overline{Cc.\mu}^3}$, $\frac{d\mu}{\overline{Sc.\mu}^3}$; quod oſtendit has ab illis dependere. Ut autem noſtrorum theorematum uſus amplior efficiatur, neceſſe eſt prorſus, ut per aliam methodum, harum formularum integratio inveniatur. Ordiamur a prima, in qua pro $d\mu$ ſubſtituamus ejus valorem, ut fiat

$$\frac{r\,d\mu}{Cb.\mu} = \frac{r^2 d\,Sb.\mu}{\overline{Cb.\mu}^2} = \frac{r^2 d\,Sb.\mu}{r^2 + \overline{Sb.\mu}^2}$$

. Hujus formulæ integratio exhibetur a ſectore circulari diviſo per $\frac{r}{2}$, ſeu ab arcu circulari, cujus tangens $= Sb.\mu$. Quare radio HK $= r$ deſcripto circulo HML (*Fig.* 1, 3); ductaque tangente, in eaque ſecta HI $=$ DF $= Sb.\mu$, agatur KMI, erit

$$S\frac{r\,d\mu}{Cb.\mu} = \frac{2.HKM}{r} = HM.$$

Formula itaque $\frac{d\mu}{Cb.\mu}$ dependet a quadratura circuli.

20. Applicantes eamdem methodum formulæ ſecundæ inveniemus

$$\frac{r\,d\mu}{Sb.\mu} = \frac{r^2 d\,Cb.\mu}{\overline{Sb.\mu}^2} = \frac{r^2 d\,Cb.\mu}{\overline{Cb.\mu}^2 - r^2}$$

, quæ dependet ab hyperbolæ quadratura. Semiaxibus KH, KL $= r$ deſcribatur hyperbola æquilatera HM (*Fig.* 1, 4). Ex puncto L parallela rectæ KH agatur LO, quæ quamquam non tangit hyperbolam; tamen, ut ſervetur circuli analogia, cotangens vocatur. In hac abſcindatur LO $=$ CD $= Cb.\mu$, & agatur KMO, erit $S\frac{r\,d\mu}{Sb.\mu} = \frac{2KHM}{r}$. Superfluum eſt advertere, hyperbolam HM, per quam formula integratur, eamdem eſſe cum hyperbola AF, in qua ſumuntur ſinus & coſinus.

21. Reliquæ duæ formulæ continentes quantitates circulares eadem methodo tractentur. Fiet itaque $\frac{r\,d\mu}{Cc.\mu} = \frac{r^2 d\,Sc.\mu}{\overline{Cc.\mu}^2} = \frac{r^2 d\,Sc.\mu}{r^2 - \overline{Sc.\mu}^2}$, quæ pariter ab hyperbolæ quadratura dependet; deſcripta enim eadem hyperbola HM (*Fig.* 2, 4)

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z, 4) duæ tangentes $HI = DF = Sc.\mu$. Age KIM; erit

$S\,\frac{r\,d\,\mu}{Cc.\mu} = \frac{2\,KHM}{r}$. Simili pacto $\frac{r\,d.\mu}{Sc.\mu} = \frac{-r^2 d\,Cc.\mu}{\overline{Sc.\mu}^2} = \frac{-r^2 d\,Cc.\mu}{r^2 - \overline{Cc.\mu}^2}$, quæ

pariter hyperbolæ poſcit quadraturam. In eadem hyperbolæ tangente ſecta $HI = CD = Cc.\mu$; habebimus $S\,\frac{-r\,d.\mu}{Sc.\mu} = \frac{2.KHM}{r}$. Q. E. I.

22. Doctiſſimus Eulerus duas ultimas formulas integrat per logarithmos; atque vocato angulo recto $= \omega$, ita integrationem exponit

$S\,\frac{r\,d\,\mu}{Cc.\mu} = l\,Tt.\frac{\omega + \mu}{2}$, $S\,\frac{r\,d\,\mu}{Sc.\mu} = l\,Tc.\frac{\mu}{2}$. Eleganter iſtæ integrationes cum noſtris apprime cohærent, dummodo memineris Eulerum uti logarithmis hyperbolicis, in quibus ſcilicet tam ſubtangens, quam protonumerus $= r$. Quoniam vero utile eſt, ſæpenumero exprimere integrationes per hos logarithmos, non ſolum has, ſed etiam eam, quæ habetur num. 10, ad logarithmos perducere juvabit. Quam ob rem neceſſe eſt præmittere aliquot facillima lemmata, per quæ demonſtratio omnis perficitur. Ac primum data tangente HI (*Fig. 4.*) determinandus ſit numerus logarithmi analogi $\frac{2\,KHM}{r}$. Sumpta $HG = KH = r$, ducatur & producatur KG, quæ erit aſymptotum hyperbolæ. Ex puncto M ducatur MQ, normalis aſymptoto: erit KQ numerus logarithmi quæſitus. Per M agatur RMS parallela HG, & vocetur $HI = z$. Facile invenies $KR = RS = \frac{rr}{\sqrt{rr - zz}}$; præterea $RM = \frac{rz}{\sqrt{rr - zz}}$; ergo $MS = \frac{rr - rz}{\sqrt{rr - zz}}$; ergo $SQ = \frac{rr - rz}{\sqrt{2}.\sqrt{rr - zz}}$; ſed $KS = \frac{rr\sqrt{2}}{\sqrt{rr - zz}}$; igitur

$$KQ = \frac{rr\sqrt{2}}{\sqrt{rr - zz}} + \frac{rz - rr}{\sqrt{2}.\sqrt{rr - zz}} = \frac{rr + rz}{\sqrt{2}.\sqrt{rr - zz}} = \frac{r.\sqrt{r + z}}{\sqrt{2}.\sqrt{r - z}},$$

qui erit numerus logarithmi analogi $\frac{2.HKM}{r}$; ergo $\frac{2.HKM}{r} = l\,\frac{r.\sqrt{r + z}}{\sqrt{2}.\sqrt{r - z}}$. Si vero non tangens HI, ſed cotangens LO data fuerit, hæc vocetur $= u$. Facta ſubſtitutione invenies $\frac{2.HKM}{r} = l\,\frac{r}{\sqrt{2}}.\frac{\sqrt{u + r}}{\sqrt{u - r}}$. Hi autem logarithmi ſunt analogi, in quibus ſcilicet ſubtangens $= r$, protonumerus $= \frac{r}{\sqrt{2}}$.

23. Invenire numerum, cujus logarithmus hyperbolicus ſit æqualis logarithmo analogo dati numeri. Duc HP normalem aſymptoto, abſcinde $KN = HK = r$, & duc NT normalem aſymptoto. Datus numerus ſit KQ, cujus logarithmus analogus eſt $\frac{2\,HPQM}{r}$; Quæſitus numerus ſit KZ, cujus logarithmus hyperbolicus erit $\frac{2.\,TNZV}{r}$. Ergo ex conditione problematis $HPQM = TNZV$, &

 demo-

dempto communi TNQM, erit HPNT $=$ MQZV; ergo ex hyperbolæ proprietate KP : KN :: KQ : KZ; atqui KP : KN :: $\frac{r}{\sqrt{2}} : r :: 1 : \sqrt{2}$; ergo KQ : KZ :: $1 : \sqrt{2}$, & KQ$\sqrt{2} =$ KZ. Itaque numerus logarithmi analogi multiplicatus per $\sqrt{2}$ dat numerum æqualis logarithmi hyperbolici.

24. His præmiſſis quum demonſtratum ſit num. 19 $S\frac{r\,d\rho}{Sb.\rho} = \frac{2.KHM}{r}$, exiſtente cotangente LO $= Cb.\rho$, erit ex num. 21 $S\frac{r\,d\rho}{Sb.\rho} = l\frac{r}{\sqrt{2}}.\frac{\sqrt{Cb.\rho+r}}{\sqrt{Cb.\rho-r}}$, qui logarithmus eſt analogus; ergo ex ſuperiore $S\frac{r\,d\rho}{Sb.\rho} = lr.\frac{\sqrt{Cb.\rho+r}}{\sqrt{Cb.\rho-r}}$ ſumpto logarithmo hyperbolico.

25. $S\frac{r\,d\rho}{Cc.\rho} = \frac{2.KHM}{r}$ exiſtente HI $= Sc.\rho$; ergo $S\frac{r\,d\rho}{Cc.\rho} = l\frac{r}{\sqrt{2}}.\frac{\sqrt{r+Sc.\rho}}{\sqrt{r-Sc.\rho}}$; atqui ex T. 1, lib. 1. cap. 10. num. 29. $r+Sc.\rho = \frac{Cc.\rho}{r}.Tc.\frac{\upsilon+\rho}{2}$, & $r-Sc.\rho = \frac{r.Cc.\rho}{Tc.\frac{\upsilon+\rho}{2}}$; ergo $\frac{r+Sc.\rho}{r-Sc.\rho} = \frac{\overline{Tc.\frac{\upsilon+\rho}{2}}^2}{r^2}$; ergo $S\frac{r\,d\rho}{Cc.\rho} = l\frac{1}{\sqrt{2}}.Tc.\frac{\upsilon+\rho}{2}$. Hi autem logarithmi omnes in analogorum ſyſtemate accipiendi ſunt. Ut ad hyperbolicos tranſeamus, ſatis eſt numeros multiplicare per $\sqrt{2}$, & habebimus $S\frac{r\,d\rho}{Cc.\rho} = lr.\frac{\sqrt{r+Sc.\rho}}{\sqrt{r-Sc.\rho}} = lTc.\frac{\upsilon+\rho}{2}$, acceptis logarithmis hyperbolicis.

Similiter quum ſit $S\frac{-r\,d\rho}{Sc.\rho} = \frac{2.KHM}{r}$ exiſtente HI $= Cc.\rho$; erit $S\frac{-r\,d\rho}{Sc.\rho} = l\frac{r}{\sqrt{2}}.\frac{\sqrt{r+Cc.\rho}}{\sqrt{r-Cc.\rho}}$; atqui ex citato capite num. 28. $r+Cc.\rho = \frac{r\,Sc.\rho}{Tc.\frac{\rho}{2}}$, & $r-Cc.\rho = \frac{Sc.\rho\,Tc.\frac{\rho}{2}}{r}$; ergo $\frac{r+Cc.\rho}{r-Cc.\rho} = \frac{r^2}{\overline{Tc.\frac{\rho}{2}}^2}$; ergo $S\frac{-r\,d\rho}{Sc.\rho} = l\frac{r^2}{\sqrt{2}.Tc.\frac{\rho}{2}}$; ergo

[illegible] Tr. [illegible]

$S\frac{r\,d\mu}{Sc.\mu} = l\frac{\ldots}{\sqrt{2}}$. Hi ſunt logarithmi analogi, quorum numeros ſi multiplices per $\sqrt{2}$, habebis logarithmos hyperbolicos, per quos formula integratur.

26. His patefactis ajo primum, ſi $m$ ſit numerus affirmativus, & impar, formulas omnes $S.\overline{Cb.\mu}^{m}.d\mu$, $S.\overline{Sb.\mu}^{m}.d\mu$, $S.\overline{Cc.\mu}^{m}.d\mu$, $S.\overline{Sc.\mu}^{m}.d\mu$ algebraicam integrationem recipere. Etenim hæ dependent a ſimilibus formulis, in quibus exponens eſt $m-2$, iſtæ ab illis, in quibus exponens eſt $m-4$; atque ita deinceps, donec deveniamus ad formulas habentes unitatem in exponente; atqui iſtæ ex dictis algebraicam integrationem recipiunt; ergo etiam primæ propoſitæ.

27. Ut indoles ſeriei, quæ ex hac methodo provenit, melius cognoſcatur, ſatis erit unam ex prædictis formulis evolvere, ſeries enim in omnibus eodem paſſu procedunt. Hanc ſeligo $S.\overline{Cc.\mu}^{m}.d\mu$. Habemus

$$S.\overline{Cc.\mu}^{m}.d\mu = \frac{r}{m}\overline{Cc.\mu}^{m-1} Sc.\mu + \frac{m-1}{m} r^2 S.\overline{Cc.\mu}^{m-2}.d\mu\text{; atqui}$$

$$S.\overline{Cc.\mu}^{m-2}.d\mu = \frac{r}{m-2}\overline{Cc.\mu}^{m-3} Sc.\mu + \frac{m-3}{m-2} r^2 S.\overline{Cc.\mu}^{m-4}.d\mu\text{; ergo}$$

$$S.\overline{Cc.\mu}^{m}.d\mu = \frac{r}{m}\overline{Cc.\mu}^{m-1}.Sc.\mu + \frac{m-1}{m.m-2} r^3.\overline{Cc.\mu}^{m-3}.Sc.\mu +$$
$$\frac{\overline{m-1}.\overline{m-3}}{m.m-2} r^4.S.\overline{Cc.\mu}^{m-4}.d\mu\text{; atqui}$$

$$S.\overline{Cc.\mu}^{m-4}.d\mu = \frac{r}{m-4}\overline{Cc.\mu}^{m-5}.Sc.\mu + \frac{m-5}{m-4} r^2.S.\overline{Cc.\mu}^{m-6}.d\mu\text{; ergo}$$

$$S.\overline{Cc.\mu}^{m}.d\mu = \frac{r}{m}\overline{Cc.\mu}^{m-1} Sc.\mu + \frac{m-1}{m.m-2}.r^3.\overline{Cc.\mu}^{m-3}.Sc.\mu +$$
$$\frac{\overline{m-1}.\overline{m-3}}{m.m-2.m-4} r^5.\overline{Cc.\mu}^{m-5}.Sc.\mu + \frac{\overline{m-1}.\overline{m-3}.\overline{m-5}}{m.m-2.m-4.m-6} r^5.S.\overline{Cc.\mu}^{m-6}.d\mu$$

Atque ita progrediens invenies ſeriem, in qua omnes termini multiplicantur per $Sc.\mu$; exponentes autem $Cc.\mu$ procedunt per ſeriem $m-1$, $m-3$, $m-5$ uſque ad 0, in quo tamquam in ultimo termino ſiſtes, coefficientes vero terminorum erunt $\frac{1}{m}$, $\frac{m-1}{m.m-2}$, $\frac{\overline{m-1}.\overline{m-3}}{m.m-2.m-4}$, $\frac{\overline{m-1}.\overline{m-3}.\overline{m-5}}{m.m-2.m-4.m-6}$ &c. Supple autem dimenſiones per poteſtates ſinus totius.

28. Si $m$ ſit numerus poſitivus, & par, duæ primæ formulæ pendent ab hyperbolæ quadratura, duæ ultimæ a quadratura circuli. Nam facto eodem progreſſu tandem perveniemus ad formulam $S\,d\mu$, quæ in primis dat duplum ſectorem hyperbolicum, in aliis duplum ſectorem circularem diviſum per ſinum totum.

29. Si evolvas eamdem formulam $S.\overline{Cc.\mu}^{m}.d\mu$, invenies eamdem ſeriem,

in

in cujus ultimo termino $C c.\mu$ habet exponentem $m = 1$. Huic autem ſeriei addendus eſt terminus $r^{m} . \mu$, cujus coefficiens idem eſt ac coefficiens ultimi termini ſeriei, ſcilicet $\frac{\overline{m-1}.\overline{m-3}\ldots\ldots 3}{m.\overline{m-2}.\overline{m-4}\ldots 2}$. Si ſummatoria debeat nulleſcere, nulleſcente arcu $\mu$, ejuſque ſinu, completa eſt, neque ei ulla addenda eſt conſtant. Quare ſi fiat $C c.\mu = o$, & $S c.\mu = \pm r$ proveniet $S. \overline{C c.\mu}^{\,m} . d\mu = \frac{\overline{m-1}.\overline{m-3}\ldots\ldots 3}{m.\overline{m-2}.\overline{m-4}\ldots 2} r^{m} \mu$. Eſt autem $\mu$, vel quadrans, vel tres quadrantes, vel quinque &c., ut omnibus notum eſt.

30. Ut facilius tractem caſus, in quibus $m$ eſt numerus negativus, paullulum tranſmutandæ ſunt formulæ. Quomodo hoc faciendum ſit aperiam dumtaxat in prima; reliquarum enim ratio eadem eſt. Muta ſignum ſpeciei $m$, ut ex negativa fiat poſitiva, & oritur

$$-mS\frac{d\mu}{\overline{Cb.\mu}^{\,m}} = \overline{-m-1}.r^{2}.S\frac{d\mu}{\overline{Cb.\mu}^{\,m+2}} + \frac{rSb.\mu}{\overline{Cb.\mu}^{\,m+1}}.$$

Tranſfer opportune terminos $\overline{m+1}.r^{2}S\frac{d\mu}{\overline{Cb.\mu}^{\,m+2}} = mS\frac{d\mu}{\overline{Cb.\mu}^{\,m}} + \frac{rSb.\mu}{\overline{Cb.\mu}^{\,m+1}}$. Pone $m+2 = n$, ut fiat $\overline{n-1}.r^{2}S\frac{d\mu}{\overline{Cb.\mu}^{\,n}} = \overline{n-2}\,S\frac{d\mu}{\overline{Cb.\mu}^{\,n-2}} + \frac{rSb.\mu}{\overline{Cb.\mu}^{\,n-1}}$.

Simili modo aliæ provenient

$$\overline{n-1}.r^{2}S\frac{d\mu}{\overline{Sb.\mu}^{\,n}} = -(\overline{n-2}.S\frac{d\mu}{\overline{Sb.\mu}^{\,n-2}} - \frac{rCb.\mu}{\overline{Sb.\mu}^{\,n-1}}$$

$$\overline{n-1}.r^{2}S\frac{d\mu}{\overline{Cc.\mu}^{\,n}} = \overline{n-2}.S\frac{d\mu}{\overline{Cc.\mu}^{\,n-2}} + \frac{rSc.\mu}{\overline{Cc.\mu}^{\,n-1}}$$

$$\overline{n-1}.r^{2}S\frac{d\mu}{\overline{Sc.\mu}^{\,n}} = \overline{n-2}.S\frac{d\mu}{\overline{Sc.\mu}^{\,n-2}} - \frac{rCc.\mu}{\overline{Sc.\mu}^{\,n-2}}.$$

31. Si $n$ eſt par, manifeſtum eſt formulas omnes $S\frac{d\mu}{\overline{Cb.\mu}^{\,n}}$, $S\frac{d\mu}{\overline{Sb.\mu}^{\,n}}$, $S\frac{d\mu}{\overline{Cc.\mu}^{\,n}}$, $S\frac{d\mu}{\overline{Sc.\mu}^{\,n}}$ algebraica integratione gaudere. Namque theoremata inventa demonſtrant, formulas iſtas dependere a ſimilibus formulis, in quibus exponens eſt $n-2$; iſtæ ab aliis exponentis $n-4$, atque ita deinceps, donec in diviſoribus reliquus ſit exponens $= 2$; ſed ſupra vidimus, formulas affectas exponente $= 2$ integrabiles eſſe; ergo & propoſitæ integrabiles ſunt. Si $n$ ſit impar, ſimili ratiocinio oſtendam, formulas dependere a ſequentibus $\frac{d\mu}{Cb.\mu}$, $\frac{d\mu}{Sb.\mu}$, $\frac{d\mu}{Cc.\mu}$, $\frac{d\mu}{Sc.\mu}$; ſed prima ex his ad ſuimet integratio-

ſed

nem poſtulat quadraturam circuli, reliquæ quadraturam hyperbolæ, ut ſupra probavi: ergo prima ex propoſitis dependet a quadratura circuli, reliquæ tres ab hyperbolæ quadratura.

31. Nihil videtur deeſſe integrationi formularum, in quibus differentia logarithmi analogi, ſive arcus multiplicatur vel dividitur aut per ſolum ſinum, aut per ſolum coſinum elatum ad quamlibet integram poteſtatem. Reſtat ut ad integrationem perducamus formulas, in quas non minus ſinus, quam coſinus ingrediuntur. Quamobrem in eadem methodo inſiſtens præmitto quatuor hæc generaliora theoremata

$$\overline{m+n}.S\,\overline{Sb.\mu}^{\,n-1}\,\overline{Cb.\mu}^{\,m+1}.d\mu = r\,\overline{Sb.\mu}^{\,n}.\overline{Cb.\mu}^{\,m} +$$
$$mr^2.S\,\overline{Sb.\mu}^{\,n-1}\,\overline{Cb.\mu}^{\,m-1}\,d\mu$$

$$\overline{m+n}.S\,\overline{Cb.\mu}^{\,m-1}.\overline{Sb.\mu}^{\,n+1}\,d\mu = r\,\overline{Cb.\mu}^{\,m}.\overline{Sb.\mu}^{\,n} -$$
$$nr^2.S\,\overline{Cb.\mu}^{\,m-1}\,\overline{Sb.\mu}^{\,n-1}\,d\mu$$

$$\overline{m+n}.S\,\overline{Sc.\mu}^{\,n-1}\,\overline{Cc.\mu}^{\,m+1}\,d\mu = r\,\overline{Sc.\mu}^{\,n}.\overline{Cc.\mu}^{\,m} +$$
$$mr^2.S\,\overline{Sc.\mu}^{\,n-1}\,\overline{Cc.\mu}^{\,m-1}\,d\mu$$

$$\overline{m+n}.S\,\overline{Cc.\mu}^{\,m-1}.\overline{Sc.\mu}^{\,n+1}\,d\mu = -r\,\overline{Cc.\mu}^{\,m}.\overline{Sc.\mu}^{\,n} +$$
$$nr^2.S\,\overline{Cc.\mu}^{\,m-1}.\overline{Sc.\mu}^{\,n-1}\,d\mu.$$

32. Prima duo theoremata demonſtrabis, ſi quantitatis $\overline{Cb.\mu}^{\,m}.\overline{Sb.\mu}^{\,n}$ differentiam accipias hoc modo

$$D.\overline{Cb.\mu}^{\,m}.\overline{Sb.\mu}^{\,n} = m\,\overline{Sb.\mu}^{\,n}.\overline{Cb.\mu}^{\,m-1}\,dCb.\mu + n\,\overline{Cb.\mu}^{\,m}.\overline{Sb.\mu}^{\,n-1}\,dSb.\mu;$$

atqui $dCb.\mu = \frac{d\mu.Sb.\mu}{r}$, $dSb.\mu = \frac{d\mu.Cb.\mu}{r}$; ergo factis ſubſtitutionibus, obvia veniet formula

$$rD\overline{Cb.\mu}^{\,m}.\overline{Sb.\mu}^{\,n} = m\,\overline{Sb.\mu}^{\,n+1}.\overline{Cb.\mu}^{\,m-1}\,d\mu + n\,\overline{Cb.\mu}^{\,m+1}.\overline{Sb.\mu}^{\,n-1}\,d\mu.$$

Si in hac pro $\overline{Sb.\mu}^{\,n+1}$ ponas $\overline{Sb.\mu}^{\,n-1}.\overline{Cb.\mu}^{\,2} - r^2$, & opportune transferas terminos, orietur theorema primum. Si vero pro $\overline{Cb.\mu}^{\,m+1}$ ſcribas $\overline{Cb.\mu}^{\,m-1}.r^2 + \overline{Sb.\mu}^{\,2}$, ſecundum theorema proveniet. Simili modo reliqua duo theoremata demonſtrabis. Nam

$$D.\overline{Cc.\mu}^{\,m}.\overline{Sc.\mu}^{\,n} = m\,\overline{Sc.\mu}^{\,n}.\overline{Cc.\mu}^{\,m-1}\,dCc.\mu + n\,\overline{Cc.\mu}^{\,m}\,\overline{Sc.\mu}^{\,n-1}\,dSc.\mu;$$

$dCc.\mu = \frac{-d\mu Sc.\mu}{r}$, $dSc.\mu = \frac{d\mu Cc.\mu}{r}$, ergo, peracta ſubſtitutione, orietur

$$rD.\overline{Cc.\mu}^{\,m}.\overline{Sc.\mu}^{\,n} = -m\,\overline{Sc.\mu}^{\,n+1}.\overline{Cc.\mu}^{\,m-1}\,d\mu + n\,\overline{Cc.\mu}^{\,m+1}.\overline{Sc.\mu}^{\,n-1}\,d\mu.$$

Si pro $\overline{Sc.\mu}^{\,n+1}$ ſubſtituas $\overline{Sc.\mu}^{\,n-1}.r^2 - \overline{Cc.\mu}^{\,2}$, factis congruis transpoſitio-

ſtionibus, invenies theorema tertium. Demum ſi pro $\overline{Cc.\mu}^{\,m+1}$ ſcribas $\overline{Cc.\mu}^{\,m-1}.r^2 - \overline{Sc.\mu}^{\,2}$ tranſpoſitis terminis theorema quartum, & ultimum ſeſe offeret.

33. Si exiſtente $n$ numero integro vel poſitivo vel negativo, ſit $m$ numerus integer, & affirmativus, in quantitatibus hyperbolicis utere theoremate primo, in circularibus tertio vel $m$ ſit impar vel par. Etenim ſi $m$ ſit impar & $m+1$ par, integratio formulæ $\overline{S.\mu}^{\,n-1}.\overline{C.\mu}^{\,m+1}d\mu$ dependet ab integratione $\overline{S.\mu}^{\,n-1}.\overline{C.\mu}^{\,m-1}d\mu$; hæc ab alia $\overline{S.\mu}^{\,n-1}.\overline{C.\mu}^{\,m-3}d\mu$, atque ita deinceps, donec $C.\mu$ exponens fiat $=o$; ergo propoſitæ integratio dependebit ab integratione $\overline{S.\mu}^{\,n-1}d\mu$, quæ ſupra tradita eſt. Si vero $m$ par ſit, & $m+1$ impar, methodus producenda eſt, donec deveniamus ad terminum, in quo $C.\mu$ habeat exponentem $=1$; ergo propoſita ab hac dependebit $\overline{S.\mu}^{\,n-1}.C.\mu.d\mu$; ſed $C.\mu.d\mu = rdS.\mu$; ergo ultima formula, a qua propoſita dependet, fiet $r\overline{S.\mu}^{\,n-1}dS.\mu$, quæ integrabilis eſt excepto caſu, in quo $n=o$, quia in hoc integratur per logarithmos.

34. Si exiſtente $m$ numero integro vel poſitivo vel negativo, ſit $n$ poſitivus, adhibe pro quantitatibus hyperbolicis theorema ſecundum, quartum pro circularibus. Per ea enim ſi $n$ ſit impar, & $n+1$ par, demonſtrabis propoſitam dependere a ſummatoria formulæ ſimplicioris $\overline{C.\mu}^{\,m-1}d\mu$, quæ paullo ante tradita eſt. Si vero $n$ ſit par, & $n+1$ impar, propoſitam reduces ad $\overline{C.\mu}^{\,m-1}.S.\mu.d\mu$; atqui $S.\mu.d\mu = \pm rdC.\mu$, ſignum $+$ valet in hyperbolicis quantitatibus, ſignum $-$ in circularibus; ergo reductio fiet ad formulam $\pm r\overline{C.\mu}^{\,m-1}dC.\mu$, quæ integrabilis eſt algebraice, vel per logarithmos ſi $m=o$. Si tam $m$, quam $n$ eſſet numerus integer & affirmativus, duplex theorema duplicem præberet integrationem.

35. Hypotheſis, in qua ambo numeri $m, n$ negativi ſunt, nullam adhuc accepit ſolutionem. Ut hanc quoque inveniamus, opus eſt aliquantulum invertere theoremata. Quomodo hoc præſtetur, oſtendam in primo theoremate; nam reliqua eadem methodo tractantur. Mutentur ſigna ſpeciebus $m, n$, ut ex negativis fiant poſitivæ, & orietur

$$\overline{-m-n}\,S\frac{d\mu}{\overline{Sb.\mu}^{\,n+1}.\overline{Cb.\mu}^{\,m-1}} = \frac{r}{\overline{Sb.\mu}^{\,n}.\overline{Cb.\mu}^{\,m}} - mr^2$$

$$S\frac{d\mu}{\overline{Sb.\mu}^{\,n+1}.\overline{Cb.\mu}^{\,m+1}};$$ ergo opportune tranſlatis terminis

$$mr^2 S\frac{d\mu}{\overline{Sb.\mu}^{\,n+1}.\overline{Cb.\mu}^{\,m+1}} = \frac{r}{\overline{Sb.\mu}^{\,n}.\overline{Cb.\mu}^{\,m}} + \overline{m+n}$$

$$S\frac{d\mu}{\overline{Sb.\mu}^{\,n+1}.\overline{Cb.\mu}^{\,n-1}}.$$ Simili ratione in ſequentia convertes alia theore-

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$$nr^2 S \frac{d\mu}{\overline{Cb.\mu}^{m+1}.\overline{Sb.\mu}^{n+1}} = \frac{-r}{\overline{Cb.\mu}^{m}.\overline{Sb.\mu}^{n}} - (\overline{m+n}) S \frac{d\mu}{\overline{Sc.\mu}^{n+1}.\overline{Cc.\mu}^{m-1}}$$

$$mr^2 S \frac{d\mu}{\overline{Sc.\mu}^{n+1}.\overline{Cc.\mu}^{m+1}} = \frac{r}{\overline{Sc.\mu}^{n}.\overline{Cc.\mu}^{m}} + \overline{m+n}\, S \frac{d\mu}{\overline{Sc.\mu}^{n+1}.\overline{Cc.\mu}^{m-1}}$$

$$nr^2 S \frac{d\mu}{\overline{Cc.\mu}^{m+1}.\overline{Sc.\mu}^{n+1}} = \frac{-r}{\overline{Cc.\mu}^{m}.\overline{Sc.\mu}^{n}} + \overline{m+n}\, S \frac{d\mu}{\overline{Cc.\mu}^{m+1}.\overline{Sc.\mu}^{n-1}}.$$

36. Quatuor hæc theoremata adhibenda ſunt eo prorſus modo, quo ſuperiora. Nam $m$ exiſtente integro & affirmativo primum, & tertium theorema in uſum eſt traducendum. Si $m$ ſit impar, & $m+1$ par, formula, in qua coſinus exponens $= m+1$, dependet ab ea, in qua exponens $= m-1$; hæc ab ea, cujus exponens $= m-3$, atque deinceps. Hunc progreſſum produc, donec exponens $= 0$; quo facto ſummatoria propoſitæ formulæ dependebit ab ea, in qua $d\mu$ dividitur per ſolum ſinum elatum ad poteſtatem $n+1$, de qua antea actum eſt ſatis. Si $m$ ſit par, & $m+1$ impar, res eſt difficilior. Etenim eo tantum produci poteſt calculus, donec coſinus exponens $= 1$. Si enim ulterius produceretur, ut hic exponens $= -1$, prodirent coefficientes $= 0$, qui dum tranſeunt in diviſores, reddunt quantitates infinitas, & integrationem inutilem.

37. Eodem modo uſurpanda ſunt theoremata ſecundum, & quartum, ſi $n$ ſit integer affirmativus. Nam ſi $n$ ſit impar, & $n+1$ par, per eumdem progreſſum devenies ad formulam, in qua ſinus exponens $= 0$, & habetur $d\mu$ diviſa per coſinus poteſtatem $m+1$, cujus integrationem ſupra exhibui. Si $n$ ſit par, & $n+1$ impar, devenies ad formulam, in qua coſinus exponens $= 1$. Inutile enim eſt ulterius progredi propter diviſores $= 0$.

38. Si alteruter ex numeris $m$, $n$ ſit impar, conſtat, quomodo integranda ſit formula. Sed ſi uterque par ſit, nondum res confecta eſt. In hoc caſu per primum aut tertium theorema deveni ad formulas $\frac{d\mu}{\overline{Sb.\mu}^{n+1}.Cb.\mu}$, $\frac{d\mu}{\overline{Sc.\mu}^{n+1}.Cc.\mu}$. Deinde has ipſas formulas per theorema ſecundum aut quartum reduc ad ſequentes $\frac{d\mu}{Sb.\mu.Cb.\mu}$, $\frac{d\mu}{Sc.\mu.Cc.\mu}$. Quapropter quicumque has duas noverit integrare, & propoſitas integrabit.

39. Quæro primum integrationem formulæ $\frac{r^2 d\mu}{Sb.\mu.Cb.\mu}$. Pro $r^2$ ſubſtituo ejus valorem $\overline{Cb.\mu}^2 - \overline{Sb.\mu}^2$, ut fiat

$$\frac{r^2 d\mu}{Sb.\mu.Cb.\mu} = \frac{d\mu.\overline{Cb.\mu}^2 - d\mu\,\overline{Sb.\mu}^2}{Sb.\mu.Cb.\mu} = \frac{d\mu\, Cb.\mu}{Sb.\mu} - \frac{d\mu\, Sb.\mu}{Cb.\mu};$$ ſed

$d\mu\, Cb.\mu = r\,dSb.\mu$, & $d\mu\, Sb.\mu = r\,dCb.\mu$; ergo

$$\frac{r^2 d\mu}{Sb.\mu.Cb.\mu} = \frac{r\,dSb.\mu}{Sb.\mu} - \frac{r\,dCb.\mu}{Cb.\mu};$$ & integrando facta ſyſtematis ſub-

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tangente $=r$, fiet $r^2 S \frac{d\mu}{Sb.\mu.Cb.\mu} = lSb.\mu - lCb.\mu$, quam ſi dividas per $r^2$ habebis $S \frac{d\mu}{Sb.\mu.Cb.\mu} = \frac{lSb.\mu - lCb.\mu}{r^2}$ Q. E. Inv.

40. Venio ad aliam $\frac{r^2 d\mu}{Sc.\mu.Cc.\mu}$, in qua pro $r^2$ ſubſtitue $\overline{Cc.\mu}^2 + \overline{Sc.\mu}^2$, ut obtineas $\frac{r^2 d\mu}{Sc.\mu Cc.\mu} = \frac{d\mu.\overline{Cc.\mu}^2 + d\mu.\overline{Sc.\mu}^2}{Sc.\mu.Cc.\mu} = \frac{d\mu Cc.\mu}{Sc.\mu} + \frac{d\mu Sc.\mu}{Cc.\mu}$; ſed $d\mu Cc.\mu = rdSc.\mu$, $d\mu Sc.\mu = -rdCc.\mu$; ergo $\frac{r^2 d\mu}{Sc.\mu.Cc.\mu} = \frac{rdSc.\mu}{Sc.\mu} - \frac{rdCc.\mu}{Cc.\mu}$; factaque ut antea integratione, & diviſione per $r^2$ $S \frac{d\mu}{Sc.\mu.Cc.\mu} = \frac{lSc.\mu - lCc.\mu}{r^2}$. Q. E. Inven.

41. Quoniam formulæ omnes differentiales, quæ ex ſinibus, coſinibuſque coaleſcunt reduci poſſunt ad binomia differentialia, neque binomia omnia differentialia per ſinus, & coſinus exprimi poſſunt; iccirco de ipſis binomiis agendum eſt neceſſario, & docendum, quando nam ſint algebraice integrabilia quando autem ſuppoſitis ſolis quadraturis circuli, & hyperbolæ. Demonſtravimus antea binomium $x^n dx.\overline{f+gx^p}^m$ ſemper eſſe algebraice integrabile ſi $m$ ſit numerus integer & affirmativus, quicumque ſint numeri $n$, $p$ vel poſitivi, vel negativi vel integri, vel fracti. Quam formulam deinceps dicam algebraice integrabilem, ſemper exceptum volo caſum illum, in quo ſeſe offert terminus logarithmicus a quadratura hyperbolæ dependens. Binomium vero $\frac{x^n dx}{\overline{f+gx^p}^m}$ iiſdem poſitis conditionibus integratur ſaltem per quadraturas circuli, & hyperbolæ. Si $p$ ſit integer, & affirmativus, vel $n$ ſit poſitivus vel negativus dummodo integer, res demonſtrata eſt Cap. ſuperiore. Nunc caſus alios evolvamus.

42. Sit primo $p$ integer & negativus; ergo formula excluſo exponente negativo hæc erit $\frac{x^n dx}{\overline{f+\frac{g}{x^p}}^m} = \frac{x^{pm+n} dx}{\overline{fx^p+g}^m}$, quam conſtat eſſe integrabilem, ſi $m$ eſt integer & poſitivus, $n$ integer vel poſitivus vel negativus. Sint deinde $n$, $p$ aut ambo, aut alteruter fractus ponatur $n=\frac{t}{q}$, $p=\frac{i}{r}$; exponente, expreſſi per ſuas fractiones; formula erit $\frac{x^{\frac{t}{q}} dx}{\overline{f+gx^{\frac{i}{r}}}^m}$.

Pone

Pone $x = y^{qr}$, & ſumptis differentialibus $dx = qry^{qr-1}dy$; ergo peractis ſubſtitutionibus formula vertetur in hanc $\frac{qry^{nr+qr-1}dy}{\overline{f+gy^{pq}}^{m}}$, in qua nullus exponens fractus eſt. Quapropter integratio formulæ $\frac{x^n dx}{\overline{f+gx^p}^{m}}$ exiſtente $m$ integro, & affirmativo, quicumque ſint numeri $n$, $p$ non ſuperat circuli, & hyperbolæ quadraturas.

43. Duas formulas brevitati conſulentes in unam contrahamus $x^n dx \cdot \overline{f+gx^p}^{m}$, quam conſtat eſſe algebraice integrabilem, ſi $m$ ſit integer & affirmativus; eſſe integrabilem ſaltem per quadraturas circuli, & hyperbolæ, ſi $m$ ſit integer negativus. Si $m$ ſit fractus, duo exiſtunt caſus, in quibus idem accidit. Ut oſtendamus primum utamur ſubſtitutione $f+gx^p = zx^p$, ex qua $f \cdot \overline{z-g}^{-1} = x^p$, & $f^{\frac{1}{p}} \cdot \overline{z-g}^{\frac{-1}{p}} = x$; ergo $f^{\frac{n+1}{p}} \cdot \overline{z-g}^{\frac{-n-1}{p}} = x^{n+1}$, & differentiando $\frac{-1}{p} \cdot f^{\frac{n+1}{p}} dz \cdot \overline{z-g}^{\frac{-n-p-1}{p}} = x^n dx$: atqui

$$f+gx^p = f+\frac{gf}{z-g} = \frac{fz}{z-g}; \text{ ergo } \overline{f+gx^p}^{m} = \frac{f^m z^m}{\overline{z-g}^{m}}: \text{ igitur}$$

$$x^n dx \cdot \overline{f+gz^p}^{m} = \frac{-f^{mp+n+1}}{p} \cdot z^m dz \cdot \overline{z-g}^{\frac{-mp-n-p-1}{p}};$$ atqui hæc algebraice integrabilis eſt, ſi $\frac{-mp-n-p-1}{p}$ ſit integer affirmativus, aut $\frac{mp+n+p+1}{p}$ negativus; eſt integrabilis ſaltem quadratis circulo & hyperbola, ſi $\frac{-mp-n-p-1}{p}$ ſit integer negativus, aut $\frac{mp+n+p+1}{p}$ poſitivus: ergo etiam prima.

44. Caſus alter patefiet, ſi adhibeatur ſubſtitutio $f+gx^p = z$, ex qua

$$x = \frac{\overline{z-f}^{\frac{1}{p}}}{g^{\frac{1}{p}}}, \ \& \ x^{n+1} = \frac{\overline{z-f}^{\frac{n+1}{p}}}{g^{\frac{n+1}{p}}}; \text{ ergo differentiando}$$

$$x^n dx = \frac{1}{pg^{\frac{n+1}{p}}} \cdot dz \cdot \overline{z-f}^{\frac{n-p+1}{p}}; \text{ igitur } x^n dx \cdot \overline{f+gx^p}^{m} = \frac{1}{pg^{\frac{n+1}{p}}} \cdot z^m dz \cdot \overline{z-f}^{\frac{n-p+1}{p}};$$

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atqui hæc altera integratur aut algebraice, ſi $\frac{n-p+1}{p}$ ſit integer & poſitivus, aut ſaltem quadratis circulo, & hyperbola, ſi $\frac{n-p+1}{p}$ ſit integer negativus; igitur etiam prima. Has reductiones accepimus a diſſertatione edita in Ludovici a Ripa miſcellaneis, quæ inſcribitur *Demonſtratio theorematum ad quadraturas ſpectantium a Joanne Bernoullio expoſitorum*.

45. Ut horum theorematum planior fiat uſus, pauca aliquot exempla proponenda cenſeo. Sit primum $\frac{a^2 dx}{\overline{a^{\frac{1}{3}}+x^{\frac{1}{3}}}^{6}}$, ſeu $a^2 dx \cdot \overline{a^{\frac{1}{3}}+x^{\frac{1}{3}}}^{-6}$. Quare ſi hæc conferatur cum formula canonica, invenies $m=-6$, $n=0$, $p=\frac{1}{3}$. Videamus utrum alter ex duobus numeris ſancitis ſit integer. Primus erit $\frac{-mp-n-p-1}{p} = \frac{2-\frac{1}{3}-1}{\frac{1}{3}} = 2$, qui eſt integer & poſitivus. Quare formula erit algebraice integrabilis. Si ſecundum numerum ſpectes habebis $\frac{n-p+1}{p} = \frac{2}{3\cdot\frac{1}{3}} = 2$, qui pariter eſt integer & poſitivus, & formulam algebraice integrabilem oſtendit. Quapropter integrationem formulæ invenies adhibita utralibet ſubſtitutione. Prima erit $a^{\frac{1}{3}}+x^{\frac{1}{3}} = \frac{z x^{\frac{1}{3}}}{a}$, ſecunda erit $a^{\frac{1}{3}}+x^{\frac{1}{3}} = \frac{z}{a^{\frac{2}{3}}}$. Hac autem adhibita pervenies ad formulam $\frac{3a^4 dz \cdot \overline{z-a}^2}{z^6}$, quam patet eſſe algebraice integrabilem.

46. Secundum exemplum præbeat formula $\frac{dx}{x^{\frac{1}{4}} \cdot \overline{x^{\frac{1}{2}}-a^{\frac{1}{2}}}^{\frac{3}{2}}}$, in qua $n=\frac{-1}{4}$, $m=\frac{-3}{2}$, $p=\frac{1}{2}$. Secundus numerus $\frac{n-p+1}{p}$ fiet $=\frac{1}{2}$, qui non eſt integer. At primus numerus $\frac{-mp-n-p-1}{p}$ invenitur $=-1$,

qui

qui eſt integer, & negativus, oſtenditque quadratis circulo, & hyperbola formulam integrari. Ob rem hanc uſurpanda eſt ſubſtitutio $x^{\frac{1}{2}} - a^{\frac{1}{2}} = \frac{z x^{\frac{1}{2}}}{a^{\frac{1}{2}}}$ quæ propoſitam mutat in ſequentem formulam $\frac{-2a^{\frac{3}{2}}}{z^{\frac{3}{2}}\sqrt{a-z}}dz$. Ut omnes exponentes fracti arceantur, ponatur $z = \frac{tt}{a}$, & formula ſeſe offeret

$$\frac{-4dt}{tt\cdot\overline{aa-tt}} = \frac{-4dt}{tt} \cdot \frac{-4dt}{aa-tt},$$

quarum prima integrabilis eſt, altera pendet ab hyperbolæ quadratura.

47. Tertium, & ultimum exemplum præbeat formula $\frac{x\,dx}{\overline{a^{\frac{2}{3}} - x^{\frac{2}{3}}}^{\frac{11}{3}}}$, in qua $m = -\frac{11}{3}$, $p = \frac{2}{3}$, $n = 1$. Primus numerus prodit fractus nempe $-\frac{1}{3}$; ſecundus autem fit $= 2$ integer & poſitivus, indicans formulam algebraice integrabilem. Adhibe ſubſtitutionem $a^{\frac{2}{3}} - x^{\frac{2}{3}} = \frac{z}{a^{\frac{1}{3}}}$, quæ convertet formulam in ſequentem $-\frac{3}{2} \cdot \frac{a^{\frac{2}{9}}dz\cdot\overline{a-z}^{2}}{z^{\frac{11}{3}}}$, quam novimus integrare. Hæc hactenus dicta ſunt de binomiis. De illis vero, quæ ſuperiores canones non comprehendunt, nihil poſſumus pronunciare. Quare multa poſteris relinquuntur, in quibus induſtriam exerceant.

48. De polynomiis ferme nihil detectum eſt, quia illa tantum in poteſtate ſunt, quæ aut ad binomia prædita neceſſariis conditionibus, aut ad functiones rationales reduci poſſunt. Primum inſpiciamus quando nam reduci poſſit ad binomia formula $\frac{x^n dx}{\overline{bx^{2p} + gx^p + f}^{m}}$ exiſtente $m$ numero integro & poſitivo; ſi enim eſſet negativus, ea nullo negotio algebraice integraretur. Pono $x^p = z + A$; ergo $x^{n+1} = \overline{z+A}^{\frac{n+1}{p}}$, & differentiando $x^n dx = \frac{1}{p}dz.\overline{z+A}^{\frac{n+1-p}{p}}$; igitur peractis ſubſtitutionibus, erit

$\frac{1}{p} \cdot \frac{dz \cdot \overline{z+A}^{\frac{n+1-p}{p}}}{\overline{bz^2 + 2bAz + bA^2 + gz + gA + f}^{\,m}}$. Ut diviſor ex trinomio fiat binomium, ponatur $2bA + g = 0$, ſeu $A = \frac{-g}{2b}$, & habebimus $\frac{1}{p} \cdot \frac{dz \cdot \overline{z - \frac{g}{2b}}^{\frac{n+1-p}{p}}}{\overline{bz^2 - \frac{gg}{4b} + f}^{\,m}}$. Si $\frac{n+1-p}{p}$ ſit numerus integer, & poſitivus, elevato binomio $z - \frac{g}{2b}$ ad integram poteſtatem, formula in plura binomia differentialia convertetur, quorum integratio non ſuperabit quadraturas circuli, & hyperbolæ.

49. Unicum exemplum propono in formula $\frac{x^{\frac{1}{2}}dx}{\overline{x + a^{\frac{1}{2}}x^{\frac{1}{2}} + a}^{\,m}}$, in qua quum $n = \frac{1}{2}$, $p = \frac{1}{2}$, erit $\frac{n+1-p}{p} = 2$, qui eſt numerus integer & affirmativus. Fac $x^{\frac{1}{2}} = \frac{z - \frac{1}{2}a}{a^{\frac{1}{2}}}$, & formula in hanc mutabitur $\frac{a^{m-\frac{3}{2}} dz \cdot \overline{z - \frac{1}{2}a}^{\,2}}{\overline{zz + \frac{3}{4}aa}^{\,m}}$, quæ elevato binomio $z - \frac{1}{2}a$ ad ſecundam poteſtatem convertitur in tria binomia, quarum integratio non ſuperat quadraturas circuli, & hyperbolæ.

50. Deinde oſtendo, quomodo $x^n dx$ diviſa per polynomium quodlibet elatum ad poteſtatem $m$ reduci poſſit ad fractionem rationalem, ſi $n$, $m$ ſint numeri integri & affirmativi. Invenienda eſt ſeries arithmetica, quæ complectatur omnes exponentes variabilis $x$ in polynomio. Hujus ſeriei terminus, qui propior eſt $0$ ſit $\frac{m}{q}$, ut hic ſit minimus exponens $x$. Tum fac $x^{\frac{m}{q}} = z^m$, tum facta ſubſtitutione nullus remanebit exponens fractus in polynomio. Præterea fiet $x = z^q$ & $dx = qz^{q-1}dz$ & $x^n = z^{qn}$. Quare completis omnibus ſubſtitutionibus fractio ſeſe offert rationalis. Hæc exemplis reddenda ſunt clariora.

51. Exemplum primum præbeat formula $\frac{x^3 dx}{x^{\frac{1}{2}} + ax^{\frac{1}{3}} + b}$. Ut ſeriem arithmeticam inveniam, multiplico diviſorem exponentium, ut habeam 6, qui numerus

rus mihi offert ſeriem arithmeticam $\frac{3}{6}$, $\frac{2}{6}$, $\frac{1}{6}$, $0$, cujus duo primi termini ſunt æquales $\frac{1}{2}$, $\frac{1}{3}$. Quoniam terminus propior $0$ eſt $\frac{1}{6}$, pono $x^{\frac{1}{6}} = z$; ergo $x = z^6$, & $dx = 6z^5 dz$, & $x^2 = z^{12}$. Quare facta ſubſtitutione formula in hanc rationalem convertitur $\frac{6z^{17}dz}{z^3 + az^2 + b}$. Methodus eodem modo procederet, ſi diviſor affectus eſſet exponente integro $= m$.

52. Ad ſecundum exemplum propono formulam $\frac{x^2 dx}{x^2 + ax^{\frac{4}{3}} + b}$. Series arithmetica continens omnes exponentes denominatoris eſt $\frac{6}{3}$, $\frac{4}{3}$, $\frac{2}{3}$, $0$. Pono itaque $x^{\frac{2}{3}} = z^2$, ex quo $x = z^3$, & $x^2 = z^6$; ergo $x^2 dx = 3z^8 dz$. Quare formula in hanc mutatur $\frac{3z^8 dz}{z^6 + az^4 + b}$, quæ eſt fractio rationalis. Idem eveniet, tametſi diviſor præditus fuerit exponente integro quocumque.

53. Appoſui conditionem, ut numeratoris exponens ſit integer, ut facilius methodum indicarem. Caiterum etiamſi ſit fractus, voti compos fiet; dummodo ſeriem arithmeticam invenias, quæ contineat non minus exponentes denominatoris, quam exponentem numeratoris. Sit exemplum $\frac{x^{\frac{-1}{5}} dx}{(x^2 + ax + b)^m}$. Series omnes exponentes comprehendens eſt hujuſmodi
$\frac{10}{5}$, $\frac{9}{5}$, $\frac{8}{5}$, $\frac{7}{5}$, $\frac{6}{5}$, $\frac{5}{5}$, $\frac{4}{5}$, $\frac{3}{5}$, $\frac{2}{5}$, $\frac{1}{5}$, $0$. Quare ponenda eſt $x^{\frac{1}{5}} = z$, & $x = z^5$, & $dx = 5z^4 dz$; ergo $x^{\frac{-1}{5}} dx = 5z^3 dz$. Peractis itaque ſubſtitutionibus exurget formula rationalis $\frac{5z^3 dz}{(z^{10} + az^5 + b)^m}$.

54. Ultimum exemplum ſit $\frac{x^{\frac{-1}{2}} dx}{(x^{\frac{1}{3}} + ax^{\frac{1}{5}} + b)^m}$. Series omnes exponentes

continens habet duos primos terminos $\frac{15}{30}$, $\frac{14}{30}$ & duos ultimos $\frac{1}{30}$, $o$. Pono $x^{\frac{1}{30}} = z$, & facta ſubſtitutione oritur formula rationalis $\frac{30 z^{14} dz}{\overline{z^{10} + a z^6 + b}^{\,m}}$. In hiſce, quæ expoſuimus, quidquid hactenus de æquationibus, quorum integratio non ſuperat quadraturam circuli, & hyperbolæ, ab auctoribus detectum eſt clara, & diſtincta methodo continetur.

## *CAPUT UNDECIMUM.*

### De rectificatione curvarum, ejuſque uſu in conſtruendis formulis differentialibus unicam variabilem continentibus.

1. Rectificare curvam nihil aliud eſt, quam invenire lineam rectam ejus arcui æqualem. Ut exhibeatur methodus, qua hoc præſtari queat, curvæ DC ( *Fig.* 1. ) abſciſſæ ſint AB, ordinatæ autem normales BC; primæ ſint $= x$, aliæ $= y$. Infinite proxima BC agatur parallela IE, & ex puncto C ducatur CV parallela AB, & intelligatur ſubtenſa chorda EC. Erit $CV = BI = dx$, $VE = dy$, & propter angulum rectum V erit chorda $CE = \sqrt{dx^2 + dy^2}$. Quoniam vero arcus CE elementum curvæ DC non differt a chorda CE niſi quantitate ad ipſas relate minore quacumque data, erit quoque arcus $CE = \sqrt{dx^2 + dy^2}$. Itaque vocato $DC = s$, habebimus $ds = \sqrt{dx^2 + dy^2}$. Facta itaque integratione proveniet valor arcus $DC = s$ datus aut per abſciſſam $x$, aut per ordinatam $y$, prout commodum fuerit. Memento, in integratione ſæpe addendam, vel detrahendam conſtantem, ut monuimus alias, & in exemplis præſtabimus.

2. Habes in formula ſuperiore elementum arcus expreſſum per elementa coordinatarum, ſi iſtæ ad angulos rectos conveniant; verum ſi earum angulus non eſſet rectus, formula non levem ſubiret mutationem. Nam ſint $AB = x$, $BC = y$ non ad angulos rectos. Agatur IE ( *Fig.* 2. ) parallela, & infinite proxima BC, & CV parallela BI. Erit $BI = CV = dx$, $VE = dy$. Ex puncto C duc CF normalem IE; voca angulum coſtantem $CVF = \mu$. Erit $r : Cc.\mu :: CV = dx : FV = \frac{Cc.\mu}{r}.dx$. Atqui $CE^2 = CV^2 + VE^2 \pm 2 EV.VF$. Signum $+$ valet, ſi angulus CVE ſit obtuſus; ſignum $-$, ſi ſit acutus; igitur $CE = ds = \sqrt{dx^2 + dy^2 \pm \frac{2 Cc.\mu}{r}.dx dy}$ elementum curvæ, cujus integrale dat ejus rectificationem.

3. Hæc valent in curvis, ſi referantur ad earum coordinatas vel orthogonales, vel ſecus. Quod ſi ipſæ referuntur ad focum, ferme eadem provenit formu

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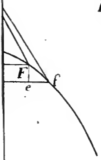
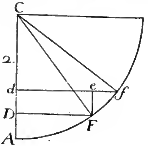
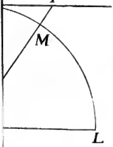
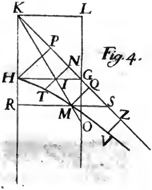



Fig. 2.




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mula. Sit curva DC, (*Fig.* 3.) ad quam feruntur ordinatæ AC ex eodem puncto A diſcendentes; agatur AE faciens cum AC angulum infiniteſimum; tum centro A intervallo AC minimus arcus circuli CV. Vocata $AC = y$, erit $VE = dy$; vocetur arcus $CV = dx$; age CF perpendicularem AE. Conſtat $CE^2 = CF^2 + EF^2$; ſed $EV = EF$, & arcus $CV = CF$, quia inter ſe non diferunt niſi differentia relate ad ipſas minore quacumque: ergo facta ſubſtitutione $CE^2 = CV^2 + EV^2$, ſeu $CE = ds = \sqrt{dx^2 + dy^2}$. Adverte ad integrationem faciendam oportere, ut ejiciatur $dx$, inveniendo formulam datam per $y$, $dy$, quia quum $dx$ ſit arcus radio variabili $= y$ deſcriptus, per ipſum obtineri integratio non poteſt, ut alias monuimus.

4. Veniamus ad exempla. Sit AC (*Fig.* 4) parabola ſecunda cubica, in qua, poſitis ordinatis orthogonalibus $AB = x$, $BC = y$, valeat æquatio $ax^2 = y^3$, quæ opportune differentiata præbet $dx = \frac{3}{2} dy \sqrt{\frac{y}{a}}$, &

$ds^2 = \sqrt{\frac{9ydy^2}{4a} + dy^2} = \frac{dy\sqrt{9a \cdot y + \frac{4a}{9}}}{2a}$, igitur CE : EV ::

$\frac{dy\sqrt{9a \cdot y + \frac{4}{9}a}}{2a} : dy :: \sqrt{9a \cdot y + \frac{4}{9}a} : 2a$. Per punctum A ducatur parallela ordinatis AM, quæ producatur, in D, donec $AD = \frac{4}{9}a$. Vertice D, diametro DM, parametro $= 9a$ deſcribatur parabola appolloniana DRP, erit quæcumque $DM = \frac{4}{9}a + y$, & quæcumque $MP = \sqrt{9a \cdot y + \frac{4}{9}a}$; ergo $CE : EV :: MP : 2a$, adeoque $2a \cdot CE = MP \cdot EV = MP \cdot MN$; igitur acceptis ſummis $2a \cdot AC =$ ſpatio parabolico RAMP; atqui hoc ſpatium eſt æquale $\frac{2}{3} DM.MP - \frac{2}{3} AD.AR$; ergo

$2a.AC = \frac{2}{3} DM.MP - \frac{2}{3} AD.AR$; adeoque

$AC = \frac{DM.MP - AD.AR}{3a}$, ac proinde arcus AC remanet rectificatus. Si deſideres exprimere arcum AC per $y$, facile invenies

$AC = \frac{\overline{9y + 4a} \cdot \sqrt{a \cdot \overline{9y + 4a}} - 8aa}{27a}$, quæ pariter inventa fuiſſet, ſi per regulas traditas integrata fuiſſet formula $\frac{dy\sqrt{9a \cdot y + \frac{4a}{9}}}{2a}$.

5. Proponatur rectificanda parabola apolloniana, cujus æquatio eſt $2ax = yy$.

Sumptis differentiis erit $a\,dx = y\,dy$, ſeu $dx^2 = \frac{y^2 dy^2}{a^2}$; ergo

$ds = \frac{dy}{a}\sqrt{yy+aa}$, ſeu integrando $as = S\,dy\sqrt{yy+aa}$, quæ indicat rectificationem parabolæ dependere ab hyperbolæ quadratura. Sit parabola propoſita 2FCP, (*Fig.* 5) quæ tangatur in præcipuo vertice C a recta 2BM, in qua acceptis CH $= y$, valet $as = S\,dy\sqrt{y^2+a^2}$. Rectæ CH normalis agatur CA $= a$, & centro C vertice A deſcribatur hyperbola æquilatera 2EAK. Ex puncto H ordinetur parabolæ recta HI, quæ producatur uſque ad hyperbolam in K: erit CAKH $= S\,dy\sqrt{y^2+a^2}$; ergo $as =$ CAKH: ergo parabolæ arcus CI æquat ſpatium hyperbolicum CAKH diviſum per demiparametrum $= a$.

6. Quamquam parabola rectificari algebraice non poteſt, tamen habet arcus infinitos, quorum differentia algebraice rectificatur. Hoc primum demonſtravit non ſolum in parabola, ſed etiam in ellypſi, ac hyperbola Comes Julius Carolus de Fagnanis. Inventum hoc propter ſubobſcuram demonſtrationem in obſcuro diu jacuit, donec Vincentius Riccatus Tom. 2. opuſculorum opuſc. 2, illud clariori, ac ſimpliciori demonſtratione munivit. Hunc ſequemur, loquentes hic de parabola, in capite ſequenti de ellypſi, & hyperbola. Suppoſita ſuperiori conſtructione agatur alia ordinata PMN; erit ex dictis $a$.CP $=$ CANM, ſicuti eſt $a$. CI $=$ CAKH; igitur deducta hac æquatione ab alia, fiet $a$. IP $=$ HKNM: quapropter quilibet arcus parabolicus ductus in ſemiparametrum æquat ſpatium hyperbolicum clauſum inter eaſdem ordinatas.

7. Per punctum C ducatur hyperbolæ aſymptotus CT faciens cum axe CA angulum ſemirectum. Acceptis in eo CG, CL, CQ continue proportionalibus, ductiſque ordinatis aſymptoto GE, LK, QN, ſcimus ſpatia hyperbolica GEKL, LKNQ æqualia eſſe. Ex punctis E, K, N agantur ordinatæ EBF, KHI, NMP. Spatium hyperbolicum

HKMN $=$ HSTM $+$ SLK $+$ LKMQ $-$ TNQ

BEKH $=$ BOSH $+$ OEG $+$ GEKL $-$ SKL. Igitur

$a$. IP $=$ HSTM $+$ SLK $+$ LKNQ $-$ TNQ

$a$. FI $=$ BOSH $+$ OEG $+$ GEKL $-$ SKL. Si altera ex his æquationibus detrahatur ab altera, deletis ſpatiis illis hyperbolicis, quæ propter æqualitatem eliduntur, remanebit $a$. IP $-$ FI $=$ HSTM $-$ BOSH $+$ 2SKL $-$ OEG $-$ TNQ, in quam quum ſolæ figuræ rectilineæ ingrediantur, conſtat, differentiam arcuum IP, FI eſſe algebraice rectificabilem.

8. Ut trapetia ad triangula redigantur, advertendum eſt

HSTM $=$ CMT $-$ CHS

BOSH $=$ CHS $-$ CBO. ergo

$a$. IP $-$ FI $=$ CMT $-$ 2.CHS $+$ CBO

$-$ TNQ $+$ 2.SKL $-$ OEG. Si ſpeciebus analyticis uti velis, CB $=$ BO $= b$; CH $=$ HS $= x$, CM $=$ MT $= z$, invenies

CG $= \frac{\sqrt{aa+bb}}{\sqrt{2}}$, CL $= \frac{x+\sqrt{aa+xx}}{\sqrt{2}}$, CQ $= \frac{z+\sqrt{aa+zz}}{\sqrt{2}}$.

Præ-

Præterea $GE = GO = \frac{-b+\sqrt{aa+bb}}{\sqrt{2}}$, $LK = LS = \frac{-x+\sqrt{aa+xx}}{\sqrt{2}}$, $QN = QT = \frac{-z+\sqrt{aa+zz}}{\sqrt{2}}$. Quare in ultimam æquationem introductis hiſce valoribus, obtinebis

$$a.\overline{IP-FI} = \frac{1}{2}zz - xx + \frac{1}{2}bb - \frac{1}{4}.\overline{\sqrt{aa+zz}-z}^2 + \frac{1}{2}.\overline{\sqrt{aa+xx}-x}^2 - \frac{1}{4}.\overline{\sqrt{aa+bb}-b}^2,$$

quæ redacta ad ſimpliciorem formam in hanc mutatur

$$a.\overline{IP-FI} = \frac{z\sqrt{aa+zz}}{2} - x\sqrt{aa+xx} + \frac{b\sqrt{aa+bb}}{2}.$$

Oportet jam determinare $z$ per $x$. Quoniam eſt $CG : CL :: CL : CQ$, habebimus

$$b+\sqrt{aa+xx} : x+\sqrt{aa+xx} :: x+\sqrt{aa+xx} : z+\sqrt{aa+zz};$$ ergo

$$\frac{\overline{x+\sqrt{aa+xx}}^2}{b+\sqrt{aa+bb}} = z+\sqrt{aa+zz},$$ ſive tranſpoſita $z$, ſumptiſque quadratis

$$\frac{\overline{x+\sqrt{aa+xx}}^4}{\overline{b+\sqrt{aa+bb}}^2} - 2z.\frac{\overline{x+\sqrt{aa+xx}}^2}{b+\sqrt{aa+bb}} + zz = aa+zz,$$ peractiſque opportunis analyſeos operationibus

$$\frac{\overline{x+\sqrt{aa+xx}}^4 - a^2.\overline{b+\sqrt{aa+bb}}^2}{\overline{x+\sqrt{aa+xx}}^2} = 2z.\overline{b+\sqrt{aa+bb}};$$ quæ æquatio exprimi etiam poteſt in hunc modum

$$\frac{\overline{x+\sqrt{aa+xx}}^2 + a.\overline{b+\sqrt{aa+bb}}}{x+\sqrt{aa+xx}} . \frac{\overline{x+\sqrt{aa+xx}}^2 - a.\overline{b+\sqrt{aa+bb}}}{x+\sqrt{aa+xx}} = 2z.\overline{b+\sqrt{aa+bb}}.$$

In hac æquatione remanet per $x$ determinata $z$.

9. Hinc colligitur huiuſmodi theorema. Ducta ad principem verticem parabolæ $CIP$ tangente $CM$, ſi accipiantur pro libito $CB = b$, $CH = x$, tum tertia abſciſſa $CM = z$, quæ data ſit per $x$, ut ſupra definitum eſt, duorum arcuum $IP$, $FI$ differentia erit rectificabilis algebraice, & æqualis

$$\frac{z\sqrt{aa+zz}}{2a} - \frac{x\sqrt{aa+xx}}{2a} + \frac{b\sqrt{aa+bb}}{2a}.$$

Ex puncto $A$ ducatur $AD$ normalis aſymptoto hyperbolæ. Hactenus accepimus $CG > CD$; ſed ſi accipiatur $C_2G < CD$, tunc recta $C_2B$ fieret negativa, adeoque $= -b$. Quod ſi ſumatur $CG = CD$, fiet $b = 0$. Caſus hic, cujus ſolummodo mentionem facit Comes de Fagnanis, dignus eſt, qui penitius conſideretur.

10. Si

10. Si ſupponamus tres lineas CD, CL, CQ eſſe continue proportionales, conſtat eſſe $a.\overline{\mathrm{IP}-\mathrm{CI}}=\frac{z\sqrt{aa+zz}}{2}-x\sqrt{aa+xx}$. Invenietur

$$z=\frac{1}{2a}\cdot\frac{\overline{x+\sqrt{aa+xx}}^{2}+aa}{x+\sqrt{aa+xx}}\cdot\frac{\overline{x+\sqrt{aa+xx}}^{2}-aa}{x+\sqrt{aa+xx}}$$, & quantitatibus ad ſecundam poteſtatem elevatis,

$$z=\frac{1}{2a}\cdot\frac{2aa+2xx+2x\sqrt{aa+xx}}{x+\sqrt{aa+xx}}\cdot\frac{2xx+2x\sqrt{aa+xx}}{x+\sqrt{aa+xx}}$$, factaque diviſione (uterque enim factor diviſionem recipit) fiet

$$z=\frac{1}{2a}\cdot 2\sqrt{aa+xx}\,.\,2x=\frac{2x\sqrt{aa+xx}}{a}$$, quæ formula maxima eſt ſimplicitate donata. Progrediens invenio $aa+zz=\frac{a^{4}+4a^{2}x^{2}+4x^{4}}{a^{2}}$,

& $\sqrt{aa+zz}=\frac{aa+2xx}{a}$. Quare

$$z\sqrt{aa+zz}=\frac{aa+2xx}{aa}\cdot 2x\sqrt{aa+xx}\ ;\ \text{ergo}$$

$$a.\overline{\mathrm{IP}-\mathrm{CI}}=\frac{aa+2xx}{aa}\cdot x\sqrt{aa+xx}-x\sqrt{aa+xx}=\frac{2x^{3}\sqrt{aa+xx}}{aa}.$$

Itaque ſumpta ex arbitratu abſciſſa $\mathrm{CH}=x$, accipiatur $\mathrm{CM}=\frac{2x\sqrt{aa+xx}}{a}$, invenietur differentia arcuum $\mathrm{IP}-\mathrm{CI}=\frac{2x^{3}\sqrt{aa+xx}}{a^{3}}$. Si $x=a$, adverte fore $\mathrm{CM}=2a\sqrt{2}$, & diferentiam arcuum $\mathrm{IP}-\mathrm{IC}=2a\sqrt{2}$, atque adeo æqualem CM.

11. Ex hoc theoremate alia infinita deduci poſſunt. Aliquot breviter indicemus, quæ methodum patefaciant, reliqua aliorum induſtriæ relinquemus. Accipe CQ quartam proportionalem poſt CG, CL, CQ, & age QN normalem aſymptoto, & ordina rectam $M^{\prime}N^{\prime}P^{\prime}$. Voca $\mathrm{C}M^{\prime}=z$. Theorema duas tibi æquationes ſufficiet

$$\mathrm{P}P^{\prime}-\mathrm{IP}=\frac{z^{\prime}\sqrt{aa+z^{\prime}z^{\prime}}}{2a}-\frac{z\sqrt{aa+zz}}{a}+\frac{x\sqrt{aa+xx}}{2a}$$

$$\mathrm{IP}-\mathrm{FI}=\frac{z\sqrt{aa+zz}}{2a}-\frac{x\sqrt{aa+xx}}{a}+\frac{b\sqrt{aa+bb}}{2a}$$, quibus duobus additis, fiet

$$\mathrm{P}P^{\prime}-\mathrm{FI}=\frac{z^{\prime}\sqrt{aa+zz^{\prime}}}{2a}-\frac{z\sqrt{aa+zz}}{2a}-\frac{x\sqrt{aa+xx}}{2a}+\frac{b\sqrt{aa+bb}}{2a}.$$

Detracta vero ſecunda æquatione ex prima orietur $\mathrm{P}P^{\prime}-2\mathrm{IP}+\mathrm{FI}$ æqualis

quan-

quantitati algebraicæ. Hoc modo novas proportionales in hyperbolæ aſympto-to capienti combinatio ſufficiet quamplurimos arcus parabolicos, quorum diffe-rentia algebraicam recipit rectificationem. Tametſi CG, CL, CQ, CQ' non ſint continue proportionales, dummodo ſit CG :: CL :: CQ : CQ', ſpatia GEKL, QNN'Q' erunt æqualia, atque adeo differentia arcuum PP', IF rectificabitur algebraice.

12. Antequam parabolæ rectificationem linquamus, non inutile erit verba facere de curva, quæ ejus rectificatrix vocari poteſt. Parabola CI (*Fig. 6.*) tangatur in præcipuo vertice a recta CH, & HI ſit una ex ordinatis tangen-ti. Moveatur planum CHI motu parallelo, ita ut CH planum deſcribat nor-male plano CHI. Hoc motu generatur cylinder parabolicus HCIich. Sum-pta Hh = ch cylinder ſecetur per planum hCi, quod cum plano HCI effi-ciet angulum ſemirectum. Planum hoc ſignabit in concava ſuperficie cylindrica curvam Ci. Arcus parabolicus CI (*Fig. 6, 7*) extendentur in rectam CI', & ſuperficies concava CiI in planam C'i'I' conformetur; mutabitur curva Ci in C'i', cujus abſciſſæ C'I' æquant arcum parabolicum CI, ordinatæ vero I'i' æ-quant CH, quæ æqualis eſt ex conſtructione Ii. Hac ratione delineetur cur-va C'i', per quam hyperbola rectificatur. Nam accepta in hac ordinata I'i', æ-qualis CH rectæ in tangente parabolæ, abſciſſa CI æquabit arcum paraboli-cum CI. Quum autem rectificata parabola quadretur hyperbola, conſtat cur-vam deſcriptam, quam diximus parabolæ rectificatricem, eſſe unam ex hyper-bolæ quadratricis. Si planum hCi efficeret cum HCI angulum diverſum a ſemirecto, curva, facta eadem conſtructione, orietur, ut exiſtente C'I' = CI, ordinata I'i' non prodiret æqualis CH, ſed haberet ad CH rationem datam. Poſſet etiam parabolicus cylinder ita ſecari, ut facta complanatio-ne curva naſceretur cujus, abſciſſæ æquarent arcus parabolicos, ordinatæ vero aut eſſent æquales, aut datam haberent rationem cum abſciſſis parabolæ acce-ptis in diametro. Sed hæc ſatis eſt indicaviſſe.

13. Poſt parabolam proponimus rectificandam ſpiralem archimideam. Hæc deſcribitur a puncto C, (*Fig. 8*) quod motu æquabili fertur verſus punctum A, inte-rim dum linea CA motu æquabili circulariter movetur circa centrum C. Sit AB circulus ille, cujus circumferentia a puncto A deſcribitur eo tempore, quo punctum C motu rectilineo percurrit radium CA. Spiralis ſit CDA. A-gatur quælibet CDB, cum qua angulum infiniteſimum faciat Cdb. Centro C intervallo CD deſcribatur minimus arcus Dm. Radius AC vocetur $= r$, integra circumferentia $ABA = 4u$, arcus $AB = z$, ejus elementum $Bb = dz$, $CD = y$, ejus elementum $md = dy$, arcus $Dm = dx$. Quoniam mD, Bb eo-dem tempore percurruntur, erit $dy : dz :: r : 4u$,

ſed ex ſectorum ſimilitudine $dz : dx :: r : y$

ergo rationibus compoſitis $dy : dx :: r^2 : 4uy$;

igitur

igitur $dx = \frac{4uy\,dy}{r^2}$, & elementum $Dd = \frac{4u\,dy\sqrt{y^2 + \frac{r^4}{16u^2}}}{r^2}$. Quæ expreſſio eadem eſt cum expreſſione arcus parabolæ, cujus ſemiparameter $= \frac{r^2}{4u}$, ſumpta abſciſſa $y$ in tangente: Ex his ſtatim cognoſces, rectificationem ſpiralis archimedeæ dependere ab hyperbolæ quadratura. Sed præterea advertendum eſt, neceſſariam eſſe quadraturam circuli, ut determinetur conſtans $\frac{rr}{4u}$, ſcilicet tertia proportionalis poſt circumferentiam, & radium.

14. Inquiro in rectificationem ſpiralis logarithmicæ, cujus naturam ac geneſim ita expono. Protonumerus ſyſtematis logarithmici ſit C A (*Fig.9*), quo radio deſcribo circulum A B K. Tum ſumpto arcu A B æquali logarithmo, & C D ejus numero, punctum D erit in logarithmica ſpirali. Curva hæc tranſit per punctum A, & per ramum A D circulo externum per infinitas ſpiras magis magiſque recedit a circulo in infinitum, per ramum vero internum A C ad centrum accedit, & infinitis ſpiris illud ambit, quin unquam ad illud perveniat. Vocato radio, ſeu protonumero $CA = r$, arcu $AB = u$, $CD = y$, erit æquatio curvæ $u = ly$; ergo differentiando, ſuppoſita ſyſtematis ſubtangente $= c$, erit $du = \frac{c\,dy}{y}$; atqui vocato $Dm = dx$ eſt $du = \frac{c\,dy}{y} : dx :: r : y$; ergo $r\,dx = c\,dy$, ſeu $dx : dy :: c : r$, hoc eſt in ratione conſtanti; quod oſtendit ordinatam C D cum curva eumdem ubique efficere angulum, quæ eſt pulcherrima ſpiralis logarithmicæ proprietas. His poſitis arcus minimus $Dd = \frac{dy\sqrt{rr+cc}}{r}$: ergo integrando ea dempta conſtante, quæ efficiat, ut facta $y = r$ arcus A D (*Fig. 9, 10*) evaneſcat, erit $AD = \frac{\overline{y-r}\cdot\sqrt{rr+cc}}{r}$. Itaque ſtatue ad angulos rectos $LM = r$, $IM = c$, & junge L I Q. Abſcinde $LN = CD$, & ordina N Q. Per punctum I duc I P parallelam L N: manifeſtum eſt fore $PI = y - r$; igitur $IQ = \frac{\overline{y-r}\cdot\sqrt{rr+cc}}{r}$; igitur curva A D æquabit rectam I Q. Si ordinata ſit minor A C, ut C 2 D, huic pone æqualem 2 N 2 L, & erit arcus $A\,2D = I\,2Q$; ergo facta $C\,2D = o$, fiet integer ſpiralis arcus æqualis I L. Quamobrem licet curva per infinitos gyros ambiat centrum; tamen ejus longitudo finita eſt, & æqualis I L. Si $c = r$, hoc eſt protonumerus ſubtangenti ſyſtematis, angulus, quem ordinata facit cum curva erit ſemirectus.

15. Quandoquidem ſpiralis logarithmicæ mentionem fecimus, non injucundum erit, obſervare quomodo per hanc curvam hyperbolæ quadratura cum quadratura circuli conjungatur. Ducta C G normali C A, inter aſymptota C G, C H deſcribatur hyperbola, in qua coordinatarum rectangulum $= aa$. Parallelas C G duc A E, & quamlibet H F. Deinde centro C radio C H deſcribatur circuli arcus ſecans ſpiralem in puncto D; vocato $CH = CD = y$, conſtat ſpa-

tium

tium $AEFH = S\frac{aady}{y}$, & arcum $AB = S\frac{cdy}{q}$; ergo $a.AB = S\frac{acdy}{y}$; igitur $AEFH : a.AB :: a : c$. ergo $AEFH = a.\frac{a}{c}.AB$. Sumatur AK qui habeat ad AB rationem $a : c$; ſpatium hyperbolicum $AEFH = a.AK$. Si $a = c$, punctum K cadit in B. Si punctum D eſt in prima ſpira curvæ, arcus AB ſumendus eſt minor circumferentia; ſi in ſecunda, arcus coaleſcit ex tota circumferentia & arcu minori; ſi in tertia, arcus componitur ex duabus circumferentiis, & arcu circumferentia minore; atque ita deinceps. Idem dicendum eſt de ramo, qui intra circulum continetur.

16. Hic exempla plura non addimus, quia alia non pauca proponemus deinceps. Nunc quem uſum habeat rectificatio curvarum in conſtruendis formulis differentialibus, quæ carent algebraica integratione oſtendendum eſt. Docuimus quo pacto ea conſtruantur per quadraturas; ſed conſtructio per rectificationes huic anteferenda videtur, quia curvæ, cujus conſtructio nota eſt, filo circumplicato, ejus longitudinem in praxi nullo negotio determinamus. Verum reducere generatim formulam quamlibet differentialem ad curvæ algebraicæ rectificationem, problema eſt difficillimæ ſolutionis. Utilitatem non exiguam habet theorema propoſitum a Joanne Bernoullio viro de mathematicis optime merito; ſed, fatendum eſt enim, non ita late patet quemadmodum viro acutiſſimo viſum eſt, nam ſæpe ſeſe intrudunt quantitates imaginariæ, quæ inutilem methodum reddunt, & de univerſalis ſolutionis poſſibilitate ſuſpicionem inducunt. Theorema proponemus, & huic maximo incommodo tutum remedium afferemus. Verum quum Bernoullius utatur in ſua demonſtratione differentiarum differentiis, quas ſecundas vocant, de quibus nondum egimus, ad has vitandas, neceſſe nobis fuit aliam excogitare demonſtrationem, in qua ſolæ differentiæ primæ ſeſe offerunt. Sed ad rem propius accedimus.

17. Data $p$ per $x$ & conſtantes, formulam $pdx$, quæ ſupponitur redacta ad linearem dimenſionem, oportet reducere ad aliam formulam, quæ exprimat elementum arcus curvæ, cujus æquatio ſit algebraica. Hæc quaſi non vocata ſeſe methodus offert. Supponantur curvæ quæſitæ coordinatæ orthogonales $y$, $u$: erit elementum curvæ $= \sqrt{dy^2 + du^2}$; ergo ſit opportet $pdx = \sqrt{dy^2 + du^2}$; & $p^2dx^2 = dy^2 + du^2$. Hinc colligimus, quantitatem $p^2dx^2$ ſecandam eſſe in duas partes, quarum radices quadratæ ſint reales, & algebraicæ integrabiles & quarum una ponenda eſt $= dy$, altera $= du$; & earum ſummatoriæ erunt $y$, $u$ coordinatæ curvæ quæſitæ.

18. Ponatur $p = \frac{\sqrt{4xx + aa}}{a}$, erit $p^2dx^2 = \frac{4x^2dx^2}{a^2} + dx^2 = dy^2 + du^2$. Fiat $\frac{4x^2dx^2}{a^2} = dy^2$, & $dx^2 = du^2$; ergo $\frac{2xdx}{a} = dy$, $dx = du$, & facta integratione $\frac{x^2}{a} = y$, $x = u$; ex quibus $uu = ay$, quæ eſt æquatio parabolæ parametri $= a$. Quare $S\frac{dx\sqrt{4xx + aa}}{a}$ obtinetur per rectificationem arcus parabolici.

19. Formulam $\frac{dx\sqrt{q^2x^{2q-2}+a^{2q-2}}}{a^{q-1}}$ reducere oporteat ad elementum curvæ algebraicæ. Quantitas multiplicans $dx$ elevetur ad quadratum, ut ſit $\frac{q^2x^{2q-2}}{a^{2q-2}}+1$. Hoc dividatur in partes duas $\frac{q^2x^{2q-2}}{a^{2q-2}}$, 1, ex quibus extrahantur radices quadratæ, nempe $\frac{qx^{q-1}}{a^{q-1}}$, 1. Fiat multiplicatio per $dx$, & exurgent formulæ $\frac{qx^{q-1}dx}{a^{q-1}}$, $dx$ ambæ integrabiles algebraice. Facta integratione reſultant $\frac{x^q}{a^{q-1}}$, $x$. Prima fiat $=u$, ſecunda $=y$, ut ſit $x^q=a^{q-1}u$, ac proinde $a^{q-1}u=y^q$, quæ eſt æquatio exprimens omnes parabolas, & hyperbolas inter aſymptotos. Harum itaque curvarum rectificatio exhibet propoſitæ formulæ integrationem.

20. Proponatur æquatio $dx\sqrt{\frac{x}{a}+\frac{b}{a}+\frac{x^3}{a^2c}}$ reducenda ad elementum curvæ algebraicæ. Elevetur ad quadratum quantitas multiplicans $dx$, & dividatur in duo quadrata, nempe $\frac{x+b}{a}$, $\frac{x^3}{a^2c}$. Horum radices ambæ reales ducantur in $dx$, ut fiant $\frac{dx\sqrt{x+b}}{\sqrt{a}}$, $\frac{x^{\frac{3}{2}}dx}{a\sqrt{c}}$, quæ integratæ & factæ æquales $u$, $y$ præbent $\frac{2}{3}\cdot\frac{\overline{x+b}^{\frac{3}{2}}}{a^{\frac{1}{2}}}=u$, $\frac{2}{5}\cdot\frac{x^{\frac{5}{2}}}{ac^{\frac{1}{2}}}=y$. Arcus itaque curvæ his coordinatis deſcriptæ æquabit ſummatoriam formulæ propoſitæ, nempe

$$Sdx\sqrt{\frac{x}{a}+\frac{b}{a}+\frac{x^3}{a^2c}}.$$

21. Juvat aliquando quadrato formulæ propoſitæ addere, & ſubtrahere eamdem quantitatem differentialem, ut poſſit commodius reduci ad rectificationem curvæ algebraicæ. Ad exemplum quadrato formulæ $\frac{adx}{2\sqrt{ax-xx}}$, quæ conſtru-

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cunda ſit, addatur, & detrahatur quantitas $\frac{\overline{ax-xx}.dx^2}{ax-xx}$, ut fiat

$$\frac{\overline{\tfrac{1}{4}aa-ax+xx+ax-xx}.dx^2}{ax-xx}=\frac{\overline{\tfrac{1}{2}a-x}^2.dx^2}{ax-xx}+dx^2.$$

Ita quadratum formulæ datæ apparet diviſum in duo quadrata $\frac{\overline{\tfrac{1}{2}a-x}^2.dx^2}{ax-xx}$, $dx^2$, cujus radices ſunt $\frac{\tfrac{1}{2}a-x.dx}{\sqrt{ax-xx}}$, $dx$ ambæ reales, dummodo $x$ minor ſit $a$, & integrabiles. Integratione autem facta erunt $\sqrt{ax-xx}=y$, $x=u$, ex quibus $au-uu=yy$, quæ eſt æquatio ad circulum, cujus rectificatio dat ſummatoriam formulæ $\frac{adx}{\sqrt{ax-xx}}$ quando $x$ major eſt $a$ tunc etiam formula propoſita eſt imaginaria.

22. Sit conſtruenda formula $\frac{dx\sqrt{f-pxx}}{\sqrt{g-qx^2}}$. Quantitas multiplicans $dx$ efferatur ad quadratum, tum multiplicetur, & dividatur per $g$, ut ſit $\frac{fg-gpx^2}{g.\overline{g-qx^2}}$. Huic addatur & dematur $\frac{fqx^2}{g.\overline{g-qx^2}}$, ut proveniat

$$\frac{fg-gpx^2+fqx^2-fqx^2}{g.\overline{g-qx^2}}=\frac{f.\overline{g-qx^2}}{g.\overline{g-qx^2}}+\frac{\overline{fq-gp}.x^2}{g.\overline{g-qx^2}}=\frac{f}{g}+\frac{\overline{fq-gp}.x^2}{g.\overline{g-qx^2}}.$$

Ita diviſum eſt quadratum in quadrata duo nempe $\frac{f}{g}$, $\frac{\overline{fq-gp}.x^2}{g.\overline{g-qx^2}}$, cujus radices reales ſunt, ſi ſupponatur $fq>gp$, nempe $\frac{\sqrt{f}}{\sqrt{g}}$, $\frac{-x\sqrt{fq-gp}}{\sqrt{g}.\sqrt{g-qx^2}}$, accipio ſecundam radicem negativam, quia commodum calculo accidit. Radices inventas multiplico per $dx$, & proveniunt formulæ duæ algebraicæ integrabiles; factaque integratione ponenda eſt prima $=y$, ſecunda $=u$, & habebuntur per $x$ coordinatæ curvæ quæſitæ. Æquationes autem ſunt $\frac{x\sqrt{f}}{\sqrt{g}}=y$,

$\frac{\sqrt{fq-gp}.\sqrt{g-qx^2}}{q\sqrt{g}}=u$; ergo $x=\frac{y\sqrt{g}}{\sqrt{f}}$, & $g-qx^2=\frac{gq^2u^2}{fq-gp}$; igitur $g-\frac{gqy^2}{f}=\frac{gq^2u^2}{fq-gp}$, ſive $\frac{f}{q}-y^2=\frac{fqu^2}{fq-gp}$; ergo $\frac{f}{q}-y^2:u^2::\frac{f}{q}:\frac{fq-gp}{q^2}$;

quæ æquatio eſt ad ellypſim, cujus ſemiaxis major $=\frac{\sqrt{f}}{\sqrt{q}}$, minor $=\frac{1}{q}\sqrt{fq-gp}$.

Hujus itaque ellypſis rectificatione conſtruitur formula $dx\frac{\sqrt{f-px^2}}{\sqrt{g-qx^2}}$. Sed de hac formula plura ſequenti capite erunt dicenda.

23. Quoniam raro contingit, ut formula propoſita $pdx$ ſit præciſe æqualis arcui curvæ algebraicæ, ſed plerumque æquet arcum curvæ algebraicæ addita vel dempta algebraica quantitate; propterea fas eſt addere formulæ $pdx$ ejuſmodi quantitatem $qdx$ algebraice integrabilem, ut per methodum ſuperiorem formula $p+q.dx$ ad elementum curvæ algebraicæ reducatur. Ita enim, vocato $=L$ arcu, erit $Spdx+Sqdx=L$; ergo $Spdx=L-Sqdx$; atqui $Sqdx$ algebraice habetur; ergo $Spdx$ ſuppoſita rectificatione curvæ obtinetur.

24. Ad exemplum propono $\frac{adx}{4x}$. Addatur huic formula differentialis algebraice integrabilis $\frac{x^4dx+a^4dx}{8ax^3}$, ut formula fiat

$$\frac{x^4+2a^2x^2+a^4}{8ax^3}.dx=\frac{x^2+a^2}{2ax}.\frac{x^2+a^2}{4x^2}.dx$$ Elevetur ad quadratum

$$\frac{x^4+2a^2x^2+x^4}{4a^2x^2}.\frac{\overline{x^2+a^2}^2}{16x^4}.dx^2=\overline{\frac{\overline{x^2-a^2}^2}{4a^2x^2}+1}.\frac{\overline{x^2+a^2}^2}{16x^4}.dx^2.$$ Quadratum hoc ſcindatur in duas partes nempe

$$\frac{\overline{x^2-a^2}^2}{4a^2x^2}.\frac{\overline{x^2+a^2}^2}{16x^4}.dx^2,\ \frac{\overline{x^2+a^2}^2}{16x^4}.dx^2.$$ Extrahantur radices

$$\frac{x^2-a^2}{2ax}.\frac{x^2+a^2}{4x^2}.dx,\ \frac{x^2+a^2}{4x^2}.dx.$$ Hæ ſunt algebraice integrabiles, nempe $\frac{x^4+a^4}{16ax^2}$; $\frac{x^2-a^2}{4x}$, quæ ſunt coordinatæ $u$, $y$ curvæ cujus arcus æquat ſummatoriam propoſitæ addita quantitate algebraica $\frac{x^4-a^4}{16ax^2}$. Sed determinemus naturam curvæ. Duæ ſunt æquationes $\frac{x^4+a^4}{16ax^2}=u$, $\frac{x^2-a^2}{4x}=y$. Hæc ſecunda elevetur ad quadratum & dividatur per $a$, ut fiat

$$\frac{x^4-2a^2x^2+a^4}{16ax^2}=\frac{y^2}{a}.$$ Hæc dematur ex prima, ut oriatur $\frac{a}{8}=u-\frac{y^2}{a}$, ſeu

ſeu $y^2 = a.\overline{u - \frac{a}{8}}$, quæ eſt æquatio ad parabolam. Parametro $= a$ deſcribatur parabola AFC, (*Fig.* 11) cujus abſciſſæ ſumptæ in diametro nempe $AB = u - \frac{a}{8}$, $BC = y$; ergo ſecta $AD = \frac{a}{8}$, erunt $DB = u$. Ponamus formulam $\frac{a\,dx}{4x}$ ita integrandam eſſe, ut facta $x = f$ evaneſcat ſummatoria. Abſcinde $DE = \frac{f^4 + a^4}{16\,af^2}$, & duc ordinatam EF. Arcus in integratione adhibendus incipiat a puncto F; ita enim evaneſcet facta $x = f$. Secta $DB = \frac{x^4 + a^4}{16\,ax^2}$, & ordinata BC, erit arcus $FC = S\frac{a\,dx}{4x} + \frac{x^4 - a^4}{16\,ax^2} + \frac{a^4 - f^4}{16\,af^2}$. Hac in integratione addita conſtante, facta $x = f$, omnia evaneſcunt. Ex his, conſtat, quomodo formula propoſita, ſuppoſita rectificatione arcus parabolici, integretur.

25. Verum nulla nos certa methodus docet, quænam quantitas differentialis addenda ſit formulæ propoſitæ, ut ejus quadratum in duo quadrata ſecari poſſit, quorum radices & reales ſint, & algebraice integrabiles. Cui incommodo ut mederetur Joannes Bernoullius, hujuſmodi theorema propoſuit. Si poſita $dx = s\,dp$, deſcribatur curva coordinatarum $s.\overline{1 - pp}^{\frac{3}{2}}$, $s.\overline{p - p^3} - x$, erit arcus hujus curvæ, quem dicam L, $= s.\overline{1 - pp} - Sp\,dx$. Demonſtratio. Inveniantur coordinatarum elementa, nempe

$$ds.\overline{1-pp}^{\frac{3}{2}} - 3p\,dx.\overline{1-pp}^{\frac{1}{2}} = \overline{1-pp}^{\frac{1}{2}}.\overline{ds.\overline{1-pp} - 3p\,dx}$$

$$ds.\overline{p - p^3} - 3pp\,dx \quad = \quad p.\overline{ds.\overline{1-pp} - 3p\,dx}.$$

Si hæc elementa eleves ad quadrata, & quadratorum ſummam accipias, invenies $\overline{ds.\overline{1-pp} - 3p\,dx}^2$, cujus radix quadrata $ds.\overline{1-pp} - 3p\,dx$ præbebit elementum arcus $= dL$. Addatur & detrahatur quantitas differentialis $s.D\,\overline{1-pp}$, ut fiat $ds.\overline{1-pp} + s.D\,\overline{1-pp} + 2sp\,dp - 3p\,dx = dL$ pro $2s\,dp$ ſcribatur $2\,dx$, ut fit $ds.\overline{1-pp} + s.D\,\overline{1-pp} - p\,dx = dL$, & facta integratione $s.\overline{1-pp} - Sp\,dx = L$. Q. E. D. Itaque $Sp\,dx = s.\overline{1-pp} - L$ dabitur rectificato arcu L. Advertendum eſt in hoc calculo, liberum eſſe coordinatas aſſumptas accipere vel negative, vel poſitive, quia idem quadratum reſultat, quum earum elementa ad ſecundam poteſtatem elata ſimul conjuguntur, ut habeatur $dL^2$. Similiter dum hæc radix extrahitur, incertum eſt, utrum elementum $dL$ debeat affici ſigno +, an ſigno —, quod dicendum eſt de arcu L facta integratione. Quapropter in ſingulis caſibus opportunum erit præfigere arcui L ſignum ambiguum, tum determinare, utrum + an — ſit accipiendum. Memento non omittendam conſtantem ſi opus fuerit.

26. Ope hujus theorematis bernoulliani reducamus denuo ad rectificationem curvæ

curvæ algebraicæ formulam logarithmicam $\frac{adx}{x}$. Statim facta collatione habemus $p=\frac{a}{x}$, $dp=\frac{-adx}{xx}$, & $s=\frac{dx}{dp}=\frac{-x^2}{a}$. Præterea $1-pp=\frac{xx-aa}{xx}$, $s.\overline{1-pp}=\frac{-1}{a}.\overline{xx-aa}$. Coordinatæ curvæ, cujus arcus ſumendus eſt, inveniuntur $-(\frac{\overline{xx-aa}^{\frac{3}{2}}}{ax}$, $\frac{-2xx+aa}{x}$, ſive $\frac{\overline{xx-aa}^{\frac{3}{2}}}{ax}$, $\frac{2xx-aa}{x}$; ergo ex theoremate erit $\frac{-1}{a}.\overline{xx-aa}\mp L=S\frac{adx}{x}$. Ut definias, quodnam ſignum ſumere oporteat, accipe duarum coordinatarum differentias, eleva ad quadratum, ſimul duo quadrata conjunge, & invenies $\frac{4x^4+4a^2x^2+a^4}{a^2x^2}.dx^2=dL^2$. Extracta radice $\frac{2x^2+a^2}{ax}.dx=dL$. Differentia $\frac{-1}{a}.\overline{x^2-a^2}$, ut habeas $\frac{-2xdx}{a}$; quam ſi ſuperiori addas, provenit formula propoſita; ergo vera formula eſt $\frac{-1}{a}.\overline{xx-aa}+L=S\frac{adx}{x}$, in qua ita accipitur L, ut creſcat creſcente $x$.

27. Exemplum ſecundum præbeat formula $\frac{dx\sqrt{xx-aa}}{x}$. Habemus $p=\frac{\sqrt{xx-aa}}{x}$, & $dp=\frac{a^2dx}{x^2\sqrt{xx-aa}}$; ergo $s=\frac{dx}{dp}=\frac{x^2\sqrt{x^2-a^2}}{a^2}$, & $1-pp=\frac{a^2}{x^2}$, & $s.\overline{1-pp}=\sqrt{xx-aa}$. Coordinatæ curvæ quæſitæ erunt $s.\overline{1-pp}^{\frac{2}{3}}=\frac{a}{x}\sqrt{xx-aa}$, $s.\overline{p-p^3}-x=\frac{-aa}{x}$. Prima vocetur $=u$, ſecunda mutato ſigno $=y$, habebimus $uu=aa-\frac{a^4}{xx}$, $y^2=\frac{a^4}{x^2}$; ergo $uu=aa-yy$, qui eſt circulus deſcriptus radio $=a$, cujus coordinatæ ſunt $CF=y=\frac{aa}{x}$, (*Fig.* 12) $FD=u=\frac{a}{x}\sqrt{xx-aa}$. Ut arcum quo indigemus, recte determinemus, inveniamus in figura lineam æqualem $x$. Ex puncto B agatur tangens circuli BE, & ducatur ſecans CDE, erit $CE=x$; quia ut $CF:CD::CB:CE$, ſeu $\frac{a^2}{x}:a::a:CE=x$, & $BE=\sqrt{xx-aa}$; ergo BD erit arcus ille qui creſcit creſcente $x$. Hujus autem arcus differentia erit $=\frac{a^2dx}{x\sqrt{xx-aa}}$: Cape differens

ferentiam $\sqrt{xx-aa}$, quæ eſt quantitas conjungenda cum arcu ad inveniendam integrationem propoſitæ; ea autem erit $\frac{x\,dx}{\sqrt{x^2-a^2}}$, ex qua differentia arcus detrahenda eſt, ut proveniat formula propoſita: quare habebimus

$$S\,\frac{dx\sqrt{xx-aa}}{x}=\sqrt{xx-aa}-BD=BE-BD.\ Q.\ E.\ I.$$

28. Exemplum tertium conſtruat formulam $\frac{-a^m dx}{\overline{a^{2m}+x^{2m}}^{\frac{1}{2}}}$. Facta collatione cum theorematis formula canonica invenio $p=\frac{-a^m}{\overline{a^{2m}+x^{2m}}^{\frac{1}{2}}}$,

$1-pp=\frac{x^{2m}}{a^{2m}+x^{2m}}$, $dp\ \frac{m a^m x^{2m-1}dx}{\overline{a^{2m}+x^{2m}}^{\frac{3}{2}}}$. Præterea

$s=\frac{dx}{dp}=\frac{\overline{a^{2m}+x^{2m}}^{\frac{3}{2}}}{m a^m x^{2m-1}}$, $s.\overline{1-pp}=\frac{x.\overline{a^{2m}+x^{2m}}^{\frac{1}{2}}}{m a^m}$. Coordinatæ ejus curvæ, cujus arcus formulam integrat inveniuntur eſſe $\frac{x^{m+1}}{m a^m}$, $\frac{-x\ \overline{n+1}}{m}$, ſive $\frac{m+1}{m}x$. Hujus curvæ arcus qui creſcit creſcente $x$ vocetur L. Calculus vero docebit, formulam hoc modo ſcribendam eſſe

$$S\,\frac{-a^m dx}{\overline{a^{2m}+x^{2m}}^{\frac{1}{2}}}=\frac{x.\overline{a^{2m}+x^{2m}}^{\frac{1}{2}}}{m x^m}-L.$$

Si curvam quæras, quam inventæ coordinatæ ſufficiunt, pone $\frac{x^{m+1}}{m a^m}=y$, $\frac{m+1}{m}.x=u$. Hanc eleva ad poteſtatem $m+1$, ut fiat $\pm x^{m+1}=\frac{m^{m+1}u^{m+1}}{\overline{m+1}^{m+1}}$; ſignum $+$ valet, ſi $m$ ſit impar, ſignum $-$, ſit par; ergo $\frac{\pm a^m.\overline{m+1}^{m+1}y}{m^m}=u^{m+1}$, quæ eſt ad infinitas parabolas, & hyperbolas.

29. Quamquam theorema propoſitum maxima gaudeat utilitate, nihilominus

nus aliquando inutile redditur, quum eſt $pp > 1$; tunc enim ordinata $s.\overline{1-pp}^{\frac{3}{2}}$ evadit imaginaria. Ut huic incommodo remedium tutum afferamus, relictis alijs, quæ aut univerſalitate carebant, aut difficilem calculum poſtulabant, ſequentem methodum elegimus. Poſita $dx = sdp$ deſcribatur curva coordinatarum $s.\overline{pp-1}^{\frac{3}{2}} = y$, $s.\overline{p^3-p} + x = u$; ajo fore $S\sqrt{dx^2 - dy^2} = s.\overline{pp-1} + Spdx$. Hoc theorema eodem modo demonſtratur ac bernoullianum. Sumantur enim coordinatarum elementa

$$ds.\overline{pp-1}^{\frac{3}{2}} + 3spdp.\overline{pp-1}^{\frac{1}{2}} = dy$$

$$ds.\overline{p^3-p} + 3sp^2dp - sdp + dx = du.$$ In his ſubſtituatur $dx$ pro $sdp$, ut evadant

$$ds.\overline{pp-1}^{\frac{3}{2}} + 3pdx.\overline{pp-1}^{\frac{1}{2}} = dy,\ \text{ſive}\ \overline{pp-1}^{\frac{1}{2}}.\overline{ds.\overline{p^2-1} + 3pdx} = dy$$

$$ds.\overline{p^3-p} + 3p^2dx = du \qquad p.\ \overline{ds.\overline{p^2-1} + 3pdx} = du$$

Eleva ad ſecundam poteſtatem

$$\overline{pp-1}.\overline{ds.\overline{pp-1} + 3pdx}^2 = dy^2$$

$$p^2.\ \overline{ds.\overline{pp-1} + 3pdx}^2 = du^2$$

Quoniam ſemper eſt $pp > pp-1$, liquet fore ſemper $ds^2 > dy^2$.

Quadratum primum detrahe a ſecundo, ut habeas $\overline{ds.\overline{pp-1} + 3pdx}^2 = du^2 - dy^2$, & $ds.\overline{pp-1} + 3pdx = \sqrt{du^2 - dy^2}$. In prima æquationis parte addatur, & dematur $s\,D.\overline{pp-1}$, ut ſit $ds.\overline{pp-1} + s\,D.\overline{pp-1} - 2spdp + 3pdx = \sqrt{du^2 - dy^2}$. Pro $sdp$ ſubſtitue $dx$, & invenies $ds.\overline{pp-1} + s\,D.\overline{pp-1} + pdx = \sqrt{du^2 - dy^2}$, & facta integratione $s.\overline{pp-1} + Spdx = S\sqrt{du^2 - dy^2}$. Q. E. D.

30. Hoc primum invento advoca theorema bernoullianum ad inveniendam integrationem formulæ $\sqrt{du^2 - dy^2}$ per rectificationem curvæ algebraicæ. Ut hoc fiat, pone $dy = qdu$, ut formula evadat $du\sqrt{1-qq}$, in qua eſt $q^2 < 1$, quia $du^2 > dy^2$. Ergo quæ in theoremate bernoulliano eſt $p$, nunc fit $= \sqrt{1-qq}$; ergo $qq = 1 - pp$, quæ poſitiva eſt. Quæ vero in theoremate bernoulliano eſt $x$, hic $= u$. Quæ vero ibi eſt $s$, hic vocetur $= z$. Quare in eas formulas introductis his valoribus habebis $du = \frac{-zqdq}{\sqrt{1-qq}}$. Curvæ autem coordinatæ erunt $zq^3$, $zq^2.\sqrt{1-qq} - u$. Demum habebis $L = z.q^2 - S\,du\sqrt{1-qq}$, ſive $zq^2 - L = S\,du\sqrt{1-q^2} = S\sqrt{du^2 - dy^2}$. Hunc valorem colloca in for-

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formula numeri ſuperioris, & habebis $s.\overline{pp-1}+\mathrm{S}p\,dx=zq^2-\mathrm{L}$, ſive $\mathrm{S}p\,dx=zq^2-s.\overline{pp-1}-\mathrm{L}$, quam in hypotheſi $pp>1$ imaginaria nulla turbare poſſunt. Hoc modo patefactum eſt ſine dubitatione, quamlibet formulam differentialem per rectificationem curvæ algebraicæ conſtrui poſſe.

31. Exemplo unico & facili methodum declarabo. Sit conſtruenda formula $\frac{-dx\sqrt{x}}{\sqrt{x-a}}$. Theoremate bernoulliano uti non licet, quia poſita $p=\frac{-\sqrt{x}}{\sqrt{x-a}}$, fieret $1-pp=\frac{-a}{x-a}$ quantitas negativa. Quare confugiendum eſt ad methodum num. 29. Erit itaque $pp-1=\frac{a}{x-a}$, & $dp=\frac{a\,dx}{2.x^{\frac{1}{2}}.\overline{x-a}^{\frac{3}{2}}}$, &

$s=\frac{dx}{dp}=\frac{2x^{\frac{1}{2}}.\overline{x-a}^{\frac{3}{2}}}{a}$; ergo $s.\overline{pp-1}=2\sqrt{x}.\sqrt{x-a}$. Præterea coordinatæ curvæ adhibendæ inveniuntur eſſe $2\sqrt{ax}=y$, $-x=u$, ſive $x=u$; Ergo ex theoremate habebimus $2\sqrt{x}\ \sqrt{x-a}+\mathrm{S}\frac{-dx\sqrt{x}}{\sqrt{x-a}}=\mathrm{S}\sqrt{du^2-dy^2}$. Huic radici calculus indicabit, ſignum $+$ apponendum eſſe.

32. Jam vero per ea, quæ dicta ſunt num. 29, conſtruamus formulam $\sqrt{du^2-dy^2}$ rectificato arcu curvæ algebraicæ. Quoniam $u=x$, & $y=2\sqrt{ax}$, & ſumptis differentijs $du=dx$, $dy=\frac{dx\sqrt{a}}{\sqrt{x}}$, formula in ſequentē mutabitur

$dx\sqrt{1-\frac{a}{x}}=\frac{dx\sqrt{x-a}}{\sqrt{x}}$; ergo $q=\frac{\sqrt{a}}{\sqrt{x}}$, & $1-qq=\frac{x-a}{x}$; ergo

$\mathrm{D}\sqrt{1-qq}=-\frac{a\,dx}{2x^{\frac{3}{2}}.\overline{x-a}^{\frac{1}{2}}}$, & $z=\frac{du}{\mathrm{D}\sqrt{1-qq}}=\frac{2x^{\frac{3}{2}}.\overline{x-a}^{\frac{1}{2}}}{a}$. Coordinatæ curvæ invenientur $2\sqrt{a}.\sqrt{x-a}$, $x-2a$. Deſcripta hac curva, ejus arcus, qui ita accipiendus eſt, ut creſcat creſcente $x$, vocetur $=\mathrm{L}$; Subſtitutis congruis ſpeciebus in formula $zq^2-\mathrm{L}=\mathrm{S}\sqrt{du^2-dy^2}$, erit $2\sqrt{x}.\sqrt{x-a}-\mathrm{L}=\mathrm{S}\sqrt{du^2-dy^2}$. Quare hoc valore ſubſtituto in formula numeri ſuperioris $2\sqrt{x}\ \sqrt{x-a}+\mathrm{S}\frac{-dx\sqrt{x}}{\sqrt{x-a}}=2\sqrt{x}.\sqrt{x-a}-\mathrm{L}$, ſive

$\mathrm{S}\frac{-dx\sqrt{x}}{\sqrt{x-a}}=-\mathrm{L}$, ſive $\mathrm{S}\frac{dx\sqrt{x}}{\sqrt{x-a}}=\mathrm{L}$, quæ oſtendit formulam integrari per ſolum arcum curvæ algebraicæ ſine ulla additione quantitatis algebraicæ: Determinemus curvam. Prima ordinata vocetur $=m$, ſecunda $=n$. Erit ergo



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go $4a . \overline{x-a} = mm$, $x - a = n + a$; ergo $4a . \overline{n+a} = mm$, quæ eſt ad parabolam apollonianam. Parametro $= 4a$ deſcribatur parabola AFC, (*Fig.* 11) cujus abſciſſæ $AE = n + a$, ergo ſecta $AI = a$, erunt $IE = n$, ordinatæ $EF = m$. Produc EA in D, donec $AD = a$, erunt $DE = n + 2a = x$. Quare exiſtentibus $DE = x$ erit $AF = S \frac{dx\sqrt{x}}{\sqrt{x-a}}$.

33. Quamquam theorema bernoullianum aut ipſum per ſeſe, aut conjunctum cum illo artificio, quod nos addidimus, patefaciat nullam eſſe formulam differentialem, quæ conſtrui non poſſit rectificato arcu curvæ algebraicæ; tamen quia calculus ſæpe evadit maxime implexus, ſæpe deducit ad curvas altiores quam par eſt, utile eſſe judicavimus, aliquot theoremata bernoulliano ſimilia proponere, & ſimul methodum indicare, qua alia nullo negotio conſtrui poſſint. Sit curva algebraica, cujus coordinatæ ſint $u = sP$, $y = sQ + mx$; P, Q ſunt quantitates variabiles determinandæ per $p$, & conſtantes; $p$ data ſupponitur utcumque per $x$, & conſtantes, atque $s = \frac{dx}{dp}$; $m$ eſt coefficiens conſtans determinandus ex arbitratu. Sumantur coordinatarum differentiæ nempe $sdP + Pds$, $Qds + sdQ + mdx$. Ponatur $dP = Mdp$, $dQ = Ndp$, & facta ſubſtitutione elementa coordinatarum erunt $Pds + Msdp$, $Qds + Nsdp + mdx$, ſed $sdp = dx$; ergo $Pds + Mdx$, $Qds + \overline{N+m} . dx$. Harum quadrata erunt $P^2 ds^2 + 2PMdsdx + M^2 dx^2$, $Q^2 ds^2 + 2Q . \overline{N+m} . dsdx + \overline{N+m}^2 . dx^2$, quorum quadratorum ſumma hæc proveniet

$$\overline{P^2 + Q^2} . ds^2 + \overline{PM + Q . \overline{N+m}} . 2dsdx + \overline{M^2 + \overline{N+m}^2} . dx^2.$$ Radix quadrata hujus ſummæ $= \sqrt{du^2 + dy^2}$ eſt elementum arcus curvæ algebraicæ. Ut autem ex hac radix quadrata actu extrahai poſſit, oportet, ut

$$\sqrt{PP + QQ} . \sqrt{M^2 + \overline{N+m}^2} = PM + Q . \overline{N+m},$$ & quadrando

$$\overline{PP + QQ} . M^2 + \overline{PP + QQ} . \overline{N+m}^2 = P^2M^2 + 2PMQ . \overline{N+m} + QQ . \overline{N+m}^2;$$

ergo $QQM^2 - 2PMQ . \overline{N+m} + PP . \overline{N+m}^2 = 0$. Quoniam hoc eſt quadratum completum, extrahatur ejus radix, ut fit $QM - P . \overline{N+m} = 0$, ſeu $\frac{N}{Q} + \frac{m}{Q} = \frac{M}{P}$, & ſubſtitutis valoribus M, N, erit $\frac{dQ}{Q} + \frac{mdp}{Q} = \frac{dP}{P}$. Si ita determines Q per $p$, ut $\frac{mdp}{Q}$ fit differentiale logarithmicum, palam eſt, integrata ſuperiore æquatione, factoque tranſitu a logarithmis ad numeros, inveniri P datum algebraice per $p$. Datis vero algebraice P, Q per $p$ invenientur etiam M, N.

34. Hoc modo peracta determinatione ſpeciei Q, & vocato curvæ arcu $= L$, extrahatur quantitatis inventæ radix quadrata

$$ds\sqrt{PP + QQ} + dx\sqrt{M^2 + \overline{N+m}^2} = dL.$$ Addatur $sD\sqrt{P^2 + Q^2}$, &

de-

dematur æquale $\frac{dx\,D\sqrt{P^2+Q^2}}{dp}$, ut fiat

$$ds\sqrt{P^2+Q^2}+s.D\sqrt{P^2+Q^2}-\frac{dx\,D\sqrt{P^2+Q^2}}{dp}+dx\sqrt{M^2+\overline{N+m}^2}=dL,$$

quæ integrata exhibet

$$s\sqrt{P^2+Q^2}-S\,dx.\overline{\frac{D\sqrt{P^2+Q^2}}{dp}-\sqrt{M^2+\overline{N+m}^2}}=L.$$ Ex hac, & ſuperioribus formulis quamplurima theoremata bernoulliano ſimilia proficiſcuntur.

35. Sit primo $Q=p$, erit $\overline{m+1}.\frac{dp}{p}=\frac{dP}{P}$, & integrando $\overline{m+1}\,lp=lP$, factoque tranſitu a logarithmis ad numeros, ponentes cauſſa facilioris calculi protonumerum $=1$, erit $P=p^{m+1}$; ergo

$M=\frac{dP}{dp}=\overline{m+1}.p^m$, & $N=\frac{dQ}{dp}=1$, &

$\sqrt{PP+QQ}=\sqrt{p^{2m+2}+pp}=p\sqrt{p^{2m}+1}$, &

$\frac{D\sqrt{PP+QQ}}{dp}=\sqrt{p^{2m}+1}+\frac{mp^{2m}}{\sqrt{p^{2m}+1}}=\frac{\overline{m+1}.p^{2m}+1}{\sqrt{p^{2m}+1}}$, &

$\sqrt{M^2+\overline{N+m}^2}=\overline{m+1}.\sqrt{p^{2m}+1}$. Itaque hiſce ſubſtitutis orietur theorema

$sp\sqrt{p^{2m}+1}+S\frac{m\,dx}{\sqrt{p^{2m}+1}}=L$ exiſtente L arcus ejus curvæ, cujus coordinatæ ſunt $sp^{m+1}$, $sp+mx$.

36. Reducenda ſit ad rectificationem curvæ algebraicæ formula $\frac{ma^m dx}{\sqrt{x^{2m}+a^{2m}}}$. Facta comparatione cum $\frac{m\,dx}{\sqrt{p^{2m}+1}}$, invenies $p=\frac{x}{a}$; ergo $dp=\frac{dx}{a}$, & $s=\frac{dx}{dp}=a$. Itaque fiet $\frac{x\sqrt{x^{2m}+a^{2m}}}{a^m}+S\frac{ma^m dx}{\sqrt{x^{2m}+a^{2m}}}=L$.

Coordinatæ autem curvæ, cujus arcus L accipiendus eſt, invenientur $\frac{x^{m+1}}{a^m}=u$, $\overline{m+1}.x=y$, ex quibus æquatio curvæ $y^{m+1}=\overline{m+1}^{m+1}a^m u$, quæ eſt ad infinitas parabola, ſi $m+1$ ſit poſitiva, ad infinitas hyperbolas ſi $m+1$ ſit negativa. Propoſitæ autem formulæ reductio ad arcum curvæ algebraicæ, ferme nihil diſtat ab ea, quam deduximus ex theoremate bernoulliano num. 27.

37. Si poneres $Q=p-p^3$, & $m=-1$, fieret
$\frac{dQ}{Q}+\frac{m\,dp}{Q}=\frac{-3p\,dp}{1-pp}=\frac{dP}{P}$; ergo integrando $\frac{3}{2}\,l\,\overline{1-pp}=lP$, & tranſeundo ad numeros $\overline{1-pp}^{\frac{3}{2}}=P$. Præterea $dQ=dp-3p^2dp$,
$dP=-3\,\overline{p\,dp}.\overline{1-pp}^{\frac{1}{2}}$; ergo $N=\frac{dQ}{dp}=1-3p^2$,
$M=\frac{dP}{dp}=-3p.\overline{1-pp}^{\frac{1}{2}}$. Inſuper $\sqrt{P^2+Q^2}=1-p^2$, &
$\frac{D\sqrt{P^2+Q^2}}{dp}=-2p$. Similiter $\sqrt{M^2+\overline{N-1}^2}=-3p$. His poſitis naſcitur
$s.\overline{1-pp}-S\,p\,dx=L$ exiſtente L arcu ejus curvæ cujus coordinatæ ſunt
$s.\overline{1-pp}^{\frac{3}{2}}$, $s.\overline{p-p^3}-x$; quod idipſum eſt bernoullianum theorema, de quo ſupra.

38. Novum theorema conſtitues, ſi ponas $Q=pp-\frac{mm}{4}$; ex qua
$dQ=2p\,dp$; ergo $\frac{dQ}{Q}+\frac{m\,dp}{Q}=\frac{2p\,dp}{pp-\frac{mm}{4}}+\frac{m\,dp}{pp-\frac{mm}{4}}=\frac{2\,dp}{p-\frac{m}{2}}=\frac{dP}{P}$;
ergo integrando $lP=2\,l\overline{p-\frac{m}{2}}$, & $P=\overline{p-\frac{m}{2}}^2$. Ex his
$M=\frac{dP}{dp}=2p-m$, $N=\frac{dQ}{dp}=2p$. Itaque $\sqrt{PP+QQ}=$
$\sqrt{\overline{p-\frac{m}{2}}^4+\overline{pp-\frac{mm}{4}}^2}=\overline{p-\frac{m}{2}}.\sqrt{\overline{p-\frac{m}{2}}^2+\overline{p+\frac{m}{2}}^2}=$
$p-\frac{m}{2}.\sqrt{2pp+\frac{mm}{2}}$. Ex quibus invenies
$\frac{D\sqrt{PP+QQ}}{dp}$ — $\sqrt{M^2+\overline{N+m}^2}=\frac{-m.\overline{p+\frac{m}{2}}}{\sqrt{2pp+\frac{mm}{2}}}$; ergo proveniet theorema $s.\overline{p-\frac{m}{2}}.\sqrt{2pp+\frac{mm}{2}}-S\frac{-m\,dx.\overline{p+\frac{m}{2}}}{\sqrt{2pp+\frac{mm}{2}}}=L$, ſive

S

$s.\overline{p-\frac{m}{2}}.\sqrt{2pp+\frac{mm}{2}}+S\frac{mdx.\overline{p+\frac{m}{2}}}{\sqrt{2pp+\frac{mm}{2}}}=L$ exiſtente L arcu curvæ, cujus coordinatæ ſunt $s.\overline{p-\frac{m}{2}}^2$, $s.\overline{pp-\frac{mm}{4}}+mx$.

39. Si eſſet $p=\frac{x}{a}$, habemus $S\frac{mdx.\overline{x+\frac{ma}{2}}}{\sqrt{2xx+\frac{mma^2}{2}}}=$ $L-\frac{\overline{x-\frac{ma}{2}}\sqrt{2xx+\frac{mma^2}{2}}}{a}$. Coordinatæ autem curvæ, cujus arcus $=L$ ſunt $\frac{\overline{x-\frac{m}{2}a}^2}{a}=u$, $\frac{xx-\frac{m^2a^2}{4}}{a}+mx=y$; ergo $x-\frac{ma}{2}=\sqrt{au}$, & $x+\frac{ma}{2}=\sqrt{ay+\frac{m^2a^2}{2}}$; ergo detracta prima a ſecunda æquatione $ma=\sqrt{ay+\frac{m^2a^2}{2}}-\sqrt{au}$, ſeu $m\sqrt{a}+\sqrt{u}=\sqrt{y+\frac{m^2a}{2}}$, quæ liberata a radicalibus fiet $4m^2au=\overline{y-u-\frac{m^2a}{2}}^2$, quam conſtat eſſe ad parabolam. Igitur propoſita formula dependet a rectificatione parabolæ.

40. Poſtremum theorema propono faciendo $Q=\frac{pp+1}{p}$; ergo $\frac{dQ}{Q}+\frac{mpdp}{pp+1}=\frac{dP}{P}$, & integrando $lQ+\frac{m}{2}l\overline{pp+1}=lP$, & facto tranſitu ad numeros $\overline{pp+1}^{\frac{m}{2}}.Q=P$, & ſubſtituto valore Q erit $\frac{\overline{pp+1}^{\frac{m+2}{2}}}{p}=P$. Ex quibus facto opportuno calculo colliges

$$M=\frac{dP}{dp}=\frac{\overline{pp+1}^{\frac{m}{2}}}{p^2}.\overline{m+1.p^2-1},\ \&\ N=\frac{dQ}{dp}=\frac{p^2-1}{p^2}.$$

Præterea $\sqrt{PP+QQ}=\frac{pp+1}{p}\sqrt{\overline{pp+1}^m+1}$, &



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$$D\sqrt{PP+QQ} = \frac{\overline{\overline{m+1}.p^2-1}.\overline{pp+1}^m + pp-1}{pp\sqrt{\overline{pp+1}^m+1}}.dp.$$ Inſuper

$$\sqrt{M^2+\overline{N+m}^2} = \frac{\sqrt{\overline{pp+1}^m.\overline{\overline{m+1}.p^2-1}^2+\overline{\overline{m+1}.p^2-1}^2}}{p^2} =$$

$$\frac{\overline{\overline{m+1}.p^2-1}.\overline{p^2+1}^m+\overline{m+1}.p^2-1}{pp\sqrt{\overline{p^2+1}^m+1}}.$$ Quapropter theorema hoc naſcetur

$$s.\frac{pp+1}{p}\sqrt{\overline{p^2+1}^m+1} - S\frac{-m\,dx}{\sqrt{\overline{p^2+1}^m+1}} = L,$$ ſive

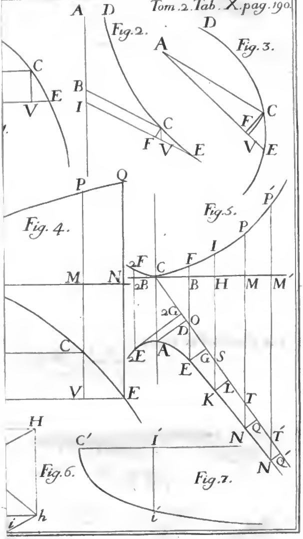
$$s.\frac{pp+1}{p}\sqrt{\overline{p^2+1}^m+1} + S\frac{m\,dx}{\sqrt{\overline{p^2+1}^m+1}} = L,$$ exiſtente L arcu curvæ cujus coordinatæ ſunt $s.\frac{\overline{pp+1}^{\frac{m+1}{2}}}{p}$, & $s.\frac{pp+1}{p}+mx$.

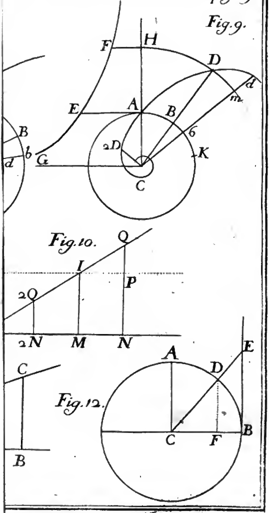
Eadem methodo ita determinans Q, ut $\frac{m\,dp}{Q}$ ſit quantitas differentialis logarithmica, alia theoremata poteris invenire; quæ ad conſtruendas formulas per curvarum rectificationem ingentem præbebunt utilitatem.

## CAPUT DUODECIMUM.

### De integratione formulæ $\frac{dz\sqrt{f+gzz}}{\sqrt{p+qzz}}$ per arcus ellypticos, & hyperbolicos.

1. QUemadmodum jure optimo utile viſum eſt analyſtis, eas formulas quæ algebraicam integrationem non admittunt ad rectificationem arcus circularis, vel ad logarithmi inventionem reducere, quæ quantitates poſt algebraicas ſunt omnium ſimpliciſſimæ: ita pari jure utile eſſe arbitramur (poſtquam egimus de methodo generali reducendi formulas differentiales unicam variabilem


Fig.2.
Fig.3.
Fig.4.
Fig.5.
Fig.6.
Fig.7.


Fig. 9.
F
H
D
d
m
E
A
B
B
6
2D
K
b
d
G
C
Fig. 10.
Q
I
P
2Q
2N
M
N
C
A
D
E
Fig. 12.
C
F
B
B

lem continentes ad rectificationem curvarum algebraicarum, perpendere rectificationes magis ſimplices, ellypſis ſcilicet & hyperbolæ atque ad iſtas reducere formulas, quæ per quadraturam circuli, & hyperbolæ minime integrantur. Omnium primus cæpit de hac re cogitare Comes Julius Carolus de Fagnanis acutiſſimus analyſta, qui arcum lemniſcatæ exhibuit per arcum ellypticum, & hyperbolicum; atque hoc pacto elegantiorem reddidit conſtructionem curvæ Iſocronæ paracentricæ, quæ a ſummis viris Jacobo, & Joanne Bernoulliis per rectificationem lemniſcatæ abſolvitur. Ut videre eſt in ejus opuſculo edito in tomo vigeſimo nono diarii Italici; & in tomo ſecundo ejus operum pag. 343.

2. Poſt Fagnanum Theoriam hanc vel maxime amplificavit Mac-Laurinus geometra maximus, qui præter rectificationem lemniſcatæ a Fagnano antea traditam, demonſtravit, tum binomia $\frac{dz\sqrt{z}}{\sqrt{1\mp zz}}$, $\frac{dz}{\sqrt{z}\cdot\sqrt{1\mp zz}}$, $\frac{dz}{\overline{1\pm zz}^{\frac{1}{4}}}$, $\frac{dz}{\overline{1\mp zz}^{\frac{5}{4}}}$; tum trinomia $\frac{dz\sqrt{z}}{\sqrt{zz+2ez-bb}}$, $\frac{dz\sqrt{z}}{\sqrt{bb+2ez-zz}}$, $\frac{dz\sqrt{z}}{\sqrt{2ez-zz-bb}}$, $\frac{dz\sqrt{z}}{\sqrt{z}.\sqrt{bb\pm 2ez-zz}}$ ſuppoſita rectificatione ellypſeos, & hyperbolæ obtineri: quæ inventa expoſuit in fluxionum tractatu libro ſecundo, capite tertio. Mac-Laurinum ſecutus eſt Allembertus vir cum paucis comparandus, qui oſtendit trinomia $\frac{dz\sqrt{z}}{\sqrt{a+bz+czz}}$, $\frac{dz}{\sqrt{z}\cdot\sqrt{a+bz+czz}}$, quæcumque ſint quantitates $a$, $b$, $c$ vel poſitivæ, vel negativæ, quoties imaginariæ non ſunt, ſemper integrari poſita rectificatione arcus ellyptici, & hyperbolici. Hoc demonſtratum invenies in Acad. Berolinenſi anni 1746. Deinde utilem inventionem adaugens incredibile dictum eſt, quantam formularum copiam in eadem Academia tum anni 1746, tum 1748. ad ea binomia perduxerit, adeo ut Theoriam hanc vir ingenioſus propemodum perfeciſſe dicendus ſit.

3. At nemo adhuc peculiarem ſermonem inſtituit de formula $\frac{dz\sqrt{f+gzz}}{\sqrt{p+qzz}}$ poſitis $f$, $g$, $p$, $q$, vel poſitivis, vel negativis, quæ tamen digna eſt, ut pro virili parte tractetur, tum quia pluribus varietatibus obnoxia eſt, tum præſertim quia ad eam, quoiquot per rectificationem hyperbolæ, & ellypſeos integrantur formulæ, revocentur, neceſſe eſt. De hac egit P. Vincentius Riccati in Opuſculo 2. Tom. 2 Opuſculorum; Propoſuit enim ſibi integrandum formulam $\frac{dz\sqrt{f+gzz}}{\sqrt{p+qzz}}$, in qua $f$, $g$, $p$, $q$ ſunt quantitates quæcumque vel poſitivæ, vel negativæ; dummodo nulla ex ipſis $=0$, & in pleriſque caſibus locum non habeat æqualitas $fq=pg$, in illis ſcilicet, in quibus poſita hac æqualitate radices per diviſionem eliduntur. Etenim in hiſce ſuppoſitionibus formula evaderet algebraice integrabilis, aut per quadraturam circuli, vel hyperbolæ conſtrueretur.

4. In quantitatibus $f$, $g$, $p$, $q$ ſi ſigna omnia mutentur valor formulæ non mutatur

mutatur, quia formula huic æquivalet $\frac{dz\sqrt{-f-gzz}.\sqrt{-1}}{\sqrt{-q-gzz}.\sqrt{-1}}$, factaque diviſione reſultat $\frac{dz\sqrt{-f-gzz}}{\sqrt{-p-qzz}}$. Verum ſi haberem formulam

$\frac{dz}{\sqrt{f+gzz}\,.\,\sqrt{p+qzz}}$ mutatis omnibus ſignis in quantitatibus, quæ ſubſunt duabus radicibus, formulæ ſignum $-$ eſſet præfigendum. Nam ipſa huic æquivalet $\frac{dz}{\sqrt{-f-gzz}.\sqrt{-1}.\sqrt{-p-qzz}.\sqrt{-1}}$, multiplicatiſque invicem duabus $\sqrt{-1}$ habebimus $\frac{-dz}{\sqrt{-f-gzz}.\sqrt{-p-qzz}}$. Hæc animadverſio probat, formulam non imaginariam eſſe, ſed realem, ſi utraque radix ſit imaginaria, quia mutatis ſignis omnibus in quantitatibus, quæ ſigno radicali afficiuntur, utraque radix realis evadit. Verumtamen ſi una radix realis, imaginaria altera, formula ſine dubio erit imaginaria. His prænotatis demonſtratur propoſitam formulam ſemper integrari rectificata ellypſi, & hyperbola.

5. Sit quadrans ellypticus ADB, (*Fig.* 1) in quo axis major $CA=a$ minor $CB=b$. In axe majore accipiatur abſciſſa $CF=x$, notum eſt arcum ellypticum $BD=S\frac{dx\sqrt{aa-xx.\frac{aa-bb}{aa}}}{\sqrt{aa-xx}}$. Adverte formulam hanc integrari quidem ellypſi rectificata, ſi abſciſſa $x$ poſita ſit intra limites $\pm x=o$, $\pm x=a$, ſi autem ex his limitibus egrediatur, nondum conſtat utrum formula per rectificationem ellypſis integretur. Neque putes eam imaginariam eſſe, ſi valor $x$ extra eos limites poſitus ſit. Nam quamquam imaginaria eſt, ſi $x$ conſtituta ſit intra limites $\pm x=a$, $\pm x=\frac{aa}{\sqrt{aa-bb}}$; tamen realis eſt, ſi ſtatuatur intra novos hos limites $\pm x=\frac{aa}{\sqrt{aa-bb}}$ $\pm x=\infty$. Namque in primo caſu radix ſuperior realis eſt, inferior imaginaria; in altero cum utraque radix imaginaria ſit, mutatis omnibus ſignis, ut antea dictum eſt, utraque realis evadit. Pone $x=cz$, & formula inventa in hac mutabitur

$$BD=S\frac{dz\sqrt{aa-cczz.\frac{aa-bb}{aa}}}{\sqrt{\frac{aa}{cc}-zz}}$$

. Hanc divide per $\sqrt{c}$, & habebis

$$\frac{BD}{\sqrt{c}}=S\frac{dz\sqrt{aa-cczz.\frac{aa-bb}{aa}}}{\sqrt{\frac{aac}{cc}-czz}}$$

. Quæ formula rectificata ellypſi integra-

tur

tur, ſi $z$ poſita ſit intra limites $\pm z = 0$, $\pm z = \frac{a}{c}$; imaginaria eſt ſi limites ſint $\pm z = \frac{a}{c}$, $\pm z = \frac{aa}{c\sqrt{aa-bb}}$; ſi vero limites fuerint $\pm z = \frac{aa}{c\sqrt{aa-bb}}$, $\pm z = \infty$ realis erit, ſed nondum conſtat utrum ad rectificationem ellypſis pertineat.

6. Proponatur jam integranda formula $\frac{dz\sqrt{f-gzz}}{\sqrt{p-qzz}}$. Conferatur cum ea, quæ paulo ante inventa eſt, & habebitur $a=\sqrt{f}$, $e=q$, $c=\frac{\sqrt{fq}}{\sqrt{p}}$, $b=\frac{\sqrt{fq-gp}}{\sqrt{q}}$; qui valor indicat, non poſſe formulam referri ad rectificationem ellypſis niſi fuerit $fq > gp$, quia ſecus eſſet imaginarius. Itaque hac conditione habita deſcribatur ellypſis ADB, (*Fig.* 1) cujus ſemiaxis major $AC = \sqrt{f}$, minor $CB = \frac{\sqrt{fq-gp}}{\sqrt{q}}$. In axe majore abſcinde $CF = \frac{z\sqrt{fq}}{\sqrt{g}}$, erit $\frac{BD}{\sqrt{q}} = S\,\frac{dz\sqrt{f-gzz}}{\sqrt{p-qzz}}$. Hæc conſtructio valet, ſi $z$ ſit intra hos fines $\pm z = 0$, $\pm z = \frac{\sqrt{p}}{\sqrt{q}}$. Si limites fuerint $\pm z = \frac{\sqrt{p}}{\sqrt{q}}$, $\pm z = \frac{\sqrt{f}}{\sqrt{g}}$ formula erit imaginaria; ſi vero ſint $\pm z = \frac{\sqrt{f}}{\sqrt{g}}$, $\pm z = \infty$, realis, ſed quomodo integretur nondum compertum eſt.

7. Si quæramus, quænam formula conſtruatur per rectificationem ejus ellypſis, in qua ſemiaxis major eſt ad minorem ut $\sqrt{2}:1$, fiet $f = \frac{2fq-2pg}{q}$, ſive $q = \frac{2pg}{f}$, quo valore ſubſtituto, factaque opportuna reductione, habebimus $\frac{BD}{\sqrt{2g}} = S\,\frac{dz\sqrt{f-gzz}}{\sqrt{f-2gzz}}$.

8. Iiſdem poſitis accipiatur in axe minore CB abſciſſa $CG = x$ erit arcus $AD = S\,\frac{dx\sqrt{bb+xx.\frac{aa-bb}{bb}}}{\sqrt{bb-xx}}$. Hæc autem formula integratur ellypſi rectificata, ſi $x$ poſita ſit intra limites $\pm x = 0$, $\pm x = b$; imaginaria eſt ſemper, ſi $x$ extra hos limites egrediantur. Fiat ut ſupra, $x = ez$, factaque ſubſtitutione dividatur æquatio per $\sqrt{e}$, & orietur $\frac{AD}{\sqrt{e}} = S\,\frac{dz\sqrt{bb+cczz.\frac{aa-bb}{bb}}}{\sqrt{\frac{bbe}{ec}-ezz}}$;



quæ

quæ formula exhibetur ab arcu ellyptico, ſi $z$ media ſit intra fines $\pm z = o$, $\pm z = \frac{b}{c}$; imaginaria eſt, ſi $\pm z > \frac{b}{c}$.

9. Oporteat integrare formulam $\frac{dz\sqrt{f+gzz}}{\sqrt{p-qzz}}$. Si comparetur cum ea, quæ ſuper inventa eſt, naſcentur hæ determinationes $b=\sqrt{f}$, $e=q$, $c=\frac{\sqrt{fq}}{\sqrt{p}}$, demum $a=\frac{\sqrt{fq+gp}}{\sqrt{q}}$. Quare deſcribe ellypſim, cujus ſemiaxis major CA $=\frac{\sqrt{fq+gp}}{\sqrt{q}}$, (*Fig.* 1) minor CB $=\sqrt{f}$. In hoc ſume abſciſſam CG $= z\frac{\sqrt{fq}}{\sqrt{p}}$, erit $\frac{AD}{\sqrt{q}} = S\frac{dz\sqrt{f+gzz}}{\sqrt{p-qzz}}$. Quapropter hæc formula reducitur ad rectificationem ellypſis ſi $z$ contineatur intra fines $\pm z=o$, $\pm z=\frac{\sqrt{p}}{\sqrt{q}}$; erit imaginaria ſi $\pm z > \frac{\sqrt{p}}{\sqrt{q}}$.

10. Si ſit $2f = \frac{fq+gp}{q}$, hoc eſt quadratum axis minoris bis ſumptum æquale quadrato axis majoris, ſive $q=\frac{gp}{f}$, fiet $\frac{AD}{\sqrt{g}} = S\frac{dz\sqrt{f+gzz}}{\sqrt{f-gzz}}$.

11. Tranſeo ad hyperbolam, cujus primus ſemiaxis KL $=a$, (*Fig.* 2.) ſecundus KM $=b$. In primo ſumatur KP $=x$. Quiſque cognoſcit arcum

$$LN = S\frac{dx\sqrt{-aa+xx.\frac{aa+bb}{aa}}}{\sqrt{-aa+xx}}.$$

Hanc formulam integratam exhibet arcus hyperbolicus, ſi $x$ ſit intra limites $\pm x=a$, $\pm x=\infty$. Si $x$ poſita ſit intra limites $\pm x=\frac{aa}{\sqrt{aa+bb}}$, $\pm x=a$; ſine dubio formula eſt imaginaria. At ſi limites fuerint $\pm x=o$, $\pm x=\frac{aa}{\sqrt{aa+bb}}$ realis eſt, ſed nondum conſtat utrum ad rectificationem hyperbolæ reducatur. Fiat $x=cz$, & dividatur æquatio, per $\sqrt{e}$ ut oriatur $\frac{LN}{\sqrt{e}} = S\frac{dz\sqrt{-aa+cczz.\frac{aa+bb}{aa}}}{\sqrt{-\frac{aae}{cc}+ezz}}$. De qua formula idem dicendum eſt, ac de ſuperiore. Nam ſi limites $z$ ſint $\pm z=\frac{a}{c}$, $\pm z=\infty$ reducitur ad rectificationem hyperbolæ; ſi ſint $\pm z=\frac{aa}{c\sqrt{aa+bb}}$

$\pm z =$

$\pm z = \frac{a}{c}$ imaginaria eſt; demum ſtatutis limitibus $\pm z = o$, $\pm z = \frac{aa}{c\sqrt{aa+bb}}$ eſt realis, ſed de ejus integratione per arcus hyperbolicos nondum conſtat.

12. Integranda proponatur formula $\frac{dz\sqrt{-f+gzz}}{\sqrt{-p+qzz}}$. Fiat comparatio, & hi valores prodibunt $a = \sqrt{f}$, $e = q$, $c = \frac{\sqrt{fq}}{\sqrt{p}}$, $b = \frac{\sqrt{pg-qf}}{\sqrt{q}}$, qui valor ne imaginarius ſit, poſtulat, ut $pg > fq$. Poſita hac conditione deſcribatur hyperbola cujus ſemiaxis primarius $KL = \sqrt{f}$, ſecundarius $KM = \frac{\sqrt{gp-fq}}{\sqrt{q}}$, ſumatur in primario abſciſſa $KP = \frac{z\sqrt{fq}}{\sqrt{p}}$ & habebitur $\frac{LN}{\sqrt{q}} = S\frac{dz\sqrt{-p+gzz}}{\sqrt{-p+qzz}}$. Hæc formula ita conſtruitur per arcus hyperbolicos, ſi $z$ contineatur intra fines $\pm z = \frac{\sqrt{p}}{\sqrt{q}}$, $\pm z = \infty$. Sed $z$ conſtituta intra limites $\pm z = \frac{\sqrt{f}}{\sqrt{g}}$, $\pm z = \frac{\sqrt{p}}{\sqrt{q}}$ eſt imaginaria; demum ſi limites fuerint $\pm z = o$, $\pm z = \frac{\sqrt{f}}{\sqrt{g}}$ realis eſt, ſed adhuc dubiæ conſtructionis.

13. Ad habendam hyperbolam æquilateram oportet, ut $f = \frac{gp-fq}{q}$, ſive $q = \frac{gp}{2f}$, quo valore ſubſtituto obtinemus $\frac{LN}{\sqrt{g}} = S\frac{dz\sqrt{-f+gzz}}{\sqrt{-2f+gzz}}$.

14. Iiſdem poſitis abſciſſa $KQ = x$ ſumatur in ſecundo axe, & invenietur

$$LN = S\frac{dx\sqrt{bb+xx.\frac{aa+bb}{bb}}}{\sqrt{bb+xx}},$$

quæ formula, quicumque ſit valor $x$, ſemper rectificato arcu hyperbolico conſtruetur. Statue $x = cz$, & divide æquationem per $\sqrt{e}$, ut obtineas

$$\frac{LN}{\sqrt{e}} = \frac{dz\sqrt{bb+cczz.\frac{aa+bb}{bb}}}{\sqrt{\frac{bbe}{ce}+ezz}}.$$

15. Integranda ſit formula $\frac{dz\sqrt{f+gzz}}{\sqrt{p+qzz}}$, in qua $f$, $g$, $p$, $q$ ſunt poſitivæ: facta comparatione cum ſuperiore reperies $b = \sqrt{f}$, $e = q$, $c = \frac{\sqrt{fq}}{\sqrt{p}}$, $a = \frac{\sqrt{gp-fq}}{\sqrt{q}}$; ex quo valore eruitur eſſe debere $gp > fq$, ne primus axis proveniat imaginarius. Si in formula adſit hæc conditio deſcribe hyperbolam,

cujus ſemiaxis primus $KL = \frac{\sqrt{gp - fq}}{\sqrt{q}}$, alter $KM = \sqrt{f}$; tum ſume in ſecundo axe abſciſſam $KQ = \frac{z\sqrt{fq}}{\sqrt{p}}$ erit $\frac{LN}{\sqrt{q}} = S\frac{dz\sqrt{f + gzz}}{\sqrt{p + qzz}}$, quæ formula, quicumque ſit valor $z$, hanc conſtructionem admittit.

16. In hyperbola æquilatera fiet $f = \frac{gp - fq}{q}$, ſive $q = \frac{gp}{2f}$, quo valore ſubſtituto habebimus $\frac{LN}{\sqrt{g}} = S\frac{dz\sqrt{f + gzz}}{\sqrt{2f + gzz}}$.

17. Ut alias formulas, quæ non exhibent elementum arcus ſectionum conicarum ſimiliter integremus, proponimus formulam $\frac{dz\sqrt{f + gzz}}{\sqrt{p + qzz}}$, in qua $f, g, p, q$ negative, & poſitive accipi poſſunt. Ad eam transformandam utimur ſubſtitutione $x = \frac{\sqrt{f + gzz}}{\sqrt{p + qzz}}$, ex qua oritur $z = \frac{\sqrt{f - pxx}}{\sqrt{-g + qxx}}$. His poſitis manifeſtum eſt $D.\, xz = xdz + zdx$. Subſtituatur in primo membro pro $x$ ejus valor datus per $z$, & in ſecundo pro $z$, ejus valor datus per $x$, & oritur Lemma I. $Dxz = (A)\frac{dz\sqrt{f + gzz}}{\sqrt{p + qzz}} + (B)\frac{dx\sqrt{f - pxx}}{\sqrt{-g + qxx}}$, ex quo lemmate liquet formulam B integrari per rectificationem ſectionum conicarum, quoties integratur formula A.

18. Sint in lemmate omnes $f, g, p, q$ poſitivæ, quo in caſu, ut traditum eſt N. 15. formula A, ſi $gp > fq$, integratur per rectificationem hyperbolæ. Ergo etiam integrabitur formula B in qua ad imaginaria vitanda utile erit mutare ſigna, atque ita exponere $\frac{dx\sqrt{-f + pxx}}{\sqrt{g - qxx}}$; quæ proinde formula rectificata hyperbola conſtruitur ſi limites ſint $\pm x = \frac{\sqrt{f}}{\sqrt{p}}$, $\pm x = \frac{\sqrt{g}}{\sqrt{q}}$. Verum ſi fuerit intra limites $\pm x = 0$, $\pm x = \frac{\sqrt{f}}{\sqrt{p}}$ aut $\pm x = \frac{\sqrt{g}}{\sqrt{q}}$, $\pm x = \infty$, formula erit imaginaria.

19. Si $f, g, p$ poſitivæ ſint $q$ negativa Lemma ſuperius hanc æquationem præbebit $Dxz = (A)\frac{dz\sqrt{f + gzz}}{\sqrt{p - qzz}} + (B)\frac{dx\sqrt{-f + pxx}}{\sqrt{g + qxx}}$, exiſtente $x = \frac{\sqrt{f + gzz}}{\sqrt{p - qzz}}$, & $z = \frac{\sqrt{-f + pxx}}{\sqrt{g + qxx}}$. Formula A ut docuimus N. 9. ſemper pertinet ad rectificationem ellypſis, ſi limites $z$ fuerint $\pm z = 0$, $\pm z = \frac{\sqrt{p}}{\sqrt{q}}$; Ergo

etiam

etiam formula B ſi limites $x$ ſint $\pm x = \frac{\sqrt{f}}{\sqrt{p}}$, $\pm x = \infty$. Utraque autem formula extra eos limites imaginaria eſt.

20. Quum $f$ & $p$ poſitivæ ſunt, $g$, $q$ negativæ, aut viceverſa, ſi $fq > gp$ formula differentialis A integratur ellypſi rectificata ut habetur N. 6. dummodo $z$ media ſit intra limites $\pm z = 0$, $\pm z = \frac{\sqrt{g}}{\sqrt{p}}$; Ergo etiam altera formula B dummodo limites $x$ ſint $\pm x = \frac{\sqrt{f}}{\sqrt{p}}$, $\pm x = \infty$. In hac hypotheſi æquatio lemmatis formam hanc aſſumit $Dxz = (A)\frac{dz\sqrt{f-gzz}}{\sqrt{p-qzz}} + (B)\frac{dx\sqrt{f-pxx}}{\sqrt{g-qxx}}$ exiſtente $x = \frac{\sqrt{f-gzz}}{\sqrt{p-qzz}}$, & $z = \frac{\sqrt{f-pxx}}{\sqrt{g-qxx}}$. Quoniam formula B eamdem formam habet, ac A videtur conſtrui poſſe ex N. 6. eodem modo ac A; verum ſi adverteris limites indeterminatarum $z$, $x$, cognoſces utramque formulam non poſſe ex methodo N. 6 integrari, ſed dumtaxat A, adeoque erit $SB = xz - SA$; ex quo colligitur formulam B pertinere ad rectificationem ellypſis etiam ſi limites $x$ ſint $\pm x = \frac{\sqrt{f}}{\sqrt{p}}$ $\pm x = \infty$.

21. Quod ſi $gp > fq$ lemma hanc præbebit æquationem $Dxz = (A)\frac{dz\sqrt{-f+gzz}}{\sqrt{-p+qzz}} + (B)\frac{dx\sqrt{-f+pxx}}{\sqrt{-g+qxx}}$, exiſtente $x = \frac{\sqrt{-f+gzz}}{\sqrt{-p+qzz}}$, & $z = \frac{\sqrt{-f+pxx}}{\sqrt{-g+qxx}}$. Quoniam $x$ eſt intra limites $\pm x = \infty$, $\pm x = \frac{\sqrt{g}}{\sqrt{q}}$, ſi $z$ ſit intra limites $\pm z = \frac{\sqrt{p}}{\sqrt{q}}$, $\pm z = \infty$, tam formula A, quam formula B hiſce limitibus conſtitutis integrabitur eadem hyperbola rectificata per regulam traditam N. 12. Quapropter nulla nova formula ex noſtra æquatione integratur.

22. Caſus iſte quamquam ad rem noſtram videtur inutilis; tamen omittendus non eſt, quia præbet duos arcus hyperbolicos, quorum differentia algebraice rectificabilis eſt. Ut hoc eleganter demonſtremus, pone $f = \frac{a^6}{aa+bb}$, $g = p = a^2$, $q = 1$, ut formula in hanc convertatur $Dxz = (A)$

$$\frac{dz\sqrt{a^2z^2 - \frac{a^6}{a^2+b^2}}}{\sqrt{zz-a^2}} + (B)\frac{dx\sqrt{a^2x^2 - \frac{a^6}{a^2+b^2}}}{\sqrt{x^2-a^2}},$$

ſive multiplicando per $\frac{\sqrt{a^2+b^2}}{aa}$, $\frac{\sqrt{a^2+b^2}\,.\,Dxz}{aa} = (A)\frac{dz\sqrt{\frac{aa+bb}{aa}zz - aa}}{\sqrt{zz-aa}} + (B)$

$dx$

$\dfrac{dx\sqrt{\frac{aa+bb}{aa}.xx-aa}}{\sqrt{xx-aa}}$. Seimus ex num. 11, utramque formulam A, B exhibere elementum arcus hyperbolæ cujus primus ſemiaxis $=a$, ſecundus $=b$, ſumptis abſciſſis in axe primo ita, ut earum origo ſit centrum hyperbolæ. Quare ſit eadem hyperbola, quæ antea deſcripta eſt; ſume KP $=x$, duc ordinatam PN, (*Fig.* 2) erit arcus LN $=S\dfrac{dx\sqrt{\frac{aa+bb}{aa}xx-aa}}{\sqrt{xx-aa}}$. Similiter ſumpta KS $=z$, ordinataque SO, erit arcus LO $=S\dfrac{dz\sqrt{\frac{aa+bb}{aa}.zz-aa}}{\sqrt{zz-aa}}$: Subſtitutis autem valoribus $x$, $z$ invicem unam per aliam ita datas invenies

$$x=\frac{a\sqrt{z^2-\frac{a^4}{a^2+b^2}}}{\sqrt{z^2-a^2}},\quad z=\frac{a\sqrt{x^2-\frac{a^4}{a^2+b^2}}}{\sqrt{x^2-a^2}},$$ ex quibus conſtat $z$, $x$ reciprocari.

23. Hiſce ſtatutis formulam integremus, ut habeamus $\dfrac{\sqrt{aa+bb}\,.\,xz}{aa}=$ LN + LO + M. Quantitas M eſt quantitas addita in integratione. Ad hanc determinandam parum prodeſt ſupponere $x=a$, quia in hac hypotheſi $z$ evadit infinita ſimul cum arcu L O. Hunc in finem meliorem methodum ſequemur, ſi determinemus caſum ubi $z=x$. Erit itaque $x=\dfrac{a\sqrt{x^2-\frac{a^4}{a^2+b^2}}}{\sqrt{x^2-a^2}}$; ergo $x^4-a^2x^2=a^2x^2+\dfrac{a^6}{a^2+b^2}$, quæ reſoluta exhibet ſequentes radices $x=a\sqrt{1+\dfrac{b}{\sqrt{aa+bb}}}$, $x=a\sqrt{1-\dfrac{b}{\sqrt{a^2+b^2}}}$. Secunda in hyperbola locum non habet; poſcit enim $x<a$. Itaque ſeca K T $=a\sqrt{1+\dfrac{b}{\sqrt{aa+bb}}}$, & duc TV. In hac hypotheſi uterque arcus LN, LO evadit LV. Valoribus iſtis in æquatione inventa ſubſtitutis fiet

$$\frac{aa+\frac{baa}{\sqrt{aa+bb}}.\sqrt{aa+bb}}{aa}=\sqrt{aa+bb}+b=2\,\mathrm{LV}+\mathrm{M},$$ per quam obtinemus valorem M $=\sqrt{aa+bb}+b-2$LV. Poſito autem hoc in æquatione naſce-

naſcetur $LO - 2LV + LN = \frac{xz\sqrt{aa+bb}}{aa} - \sqrt{aa+bb} - b$, ſive $VO - VN = \frac{xz\sqrt{aa+bb}}{aa} - \sqrt{aa+bb} - b$. Hinc efformatur theorema. Abſciſſa $KT = a\sqrt{1 + \frac{b}{\sqrt{aa+bb}}}$, accipiatur $KP = x$, &

$$KS = z = \frac{a\sqrt{xx - \frac{a^4}{aa+bb}}}{\sqrt{xx-aa}},$$ erit differentia duorum arcuum VO, VN algebraica, & æqualis $\frac{xz\sqrt{aa+bb}}{aa} - \sqrt{aa+bb} - b$.

24. Aliam item ejuſdem formulæ transformationem obtinemus utentes hac methodo. Ponatur $x = \frac{\sqrt{p+qzz}}{\sqrt{f+gzz}}$, ex qua provenit $z = \frac{\sqrt{p-fxx}}{\sqrt{-q+gxx}}$; quadrata alterutra ex his formulis obtinebitur $gzzxx = p + qzz - fxx$, cujus differentia ſumenda eſt hoc modo $gzx\,D.\,zx = qzdz - fxdx$: dividatur per $zx$, & obtinebitur $g\,D.\,zx = \frac{qdz}{x} - \frac{fdx}{z}$, in prima collocetur valor $x$ datus per $z$, in ſecunda valor $z$ datus per $x$, & habebitur

Lemma II. $g\,D\,zx = (A)\,\frac{qdz\sqrt{f+gzz}}{\sqrt{p+qzz}} - (B)\,\frac{fdx\sqrt{-q+gxx}}{\sqrt{p-fxx}}$, quod lemma tradit integrationem formulæ B, quoties integratur formula A.

25. Si omnes $f$, $g$, $p$, $q$ ſunt poſitivæ formula A ex N. 15. integratur rectificata hyperbola, dummodo $gp > fq$; Ergo etiam formula B, ſi $x$ contineatur intra limites $\pm x = \frac{\sqrt{p}}{\sqrt{f}}$, $\pm x = \frac{\sqrt{q}}{\sqrt{g}}$ extra quos formula eſt imaginaria. Verum formula B iiſdem conditionibus prædita eſt ac illa, quæ integrata eſt N. 18.

26. Si $q$ tantum poſitiva ſit, reliquæ omnes negativæ, aut viceverſa lemma ſecundum hanc formam accipiet $g\,Dz = -(A)\,\frac{qdz\sqrt{f+gzz}}{\sqrt{p-qzz}} - (B)\,\frac{fdx\sqrt{q+gxx}}{\sqrt{p-fxx}}$ exiſtente $x = \frac{\sqrt{p-qzz}}{\sqrt{f+gzz}}$, & $z = \frac{\sqrt{p-fxx}}{\sqrt{q+gxx}}$. Formula B eſt prorſus ſimilis formulæ A, atque ſi $z$ media ſit intra limites $\pm z = o$, $\pm z = \frac{\sqrt{p}}{\sqrt{q}}$, $x$ poſita erit intra limites $\pm x = \frac{\sqrt{p}}{\sqrt{f}}$, $\pm x = o$, quare utraque conſtruitur per regulam traditam N. 9. Poſſem ex hoc caſu deducere duos arcus ellypticos habere differentiam algebraice rectificabilem, quo in caſu abſciſ-

ſæ

ſæ $x$, $z$ ſumendæ eſſent in ellypſis axe minore; ſed arcus hos paullo infra determinabo acceptis abſciſſis in axe majore.

27. Si $g$, $q$ fuerint negativæ, reliquæ poſitivæ vel viceverſa lemma hanc æquationem prebebit $g\,\mathrm{D}zx = (\mathrm{A})\,\frac{q\,dz\sqrt{f-gzz}}{\sqrt{p-qzz}} + (\mathrm{B})\,\frac{f\,dx\sqrt{q-gxx}}{\sqrt{p-fxx}}$. Si ponamus $fq > gp$ ex N. 6. formula A integratur ellypſi rectificata, ſi $z$ ſit intra limites $\pm z = 0$, $\pm z = \frac{\sqrt{p}}{\sqrt{q}}$. Hoc in caſu cum ſit $x = \frac{\sqrt{p-qzz}}{\sqrt{f-gzz}}$, & $z = \frac{\sqrt{p-fxx}}{\sqrt{q-gxx}}$, erit $x$ intra limites $\pm x = \frac{\sqrt{p}}{\sqrt{f}}$, $\pm x = 0$. Quare utraque formula ex N. 6. integratur per rectificationem Ellypſis.

28. Determinandi jam ſunt arcus, quarum differentia per algebraicam integrationem obtinetur. Formulam ita diſpono

$$g\,\mathrm{D}zx = \frac{dz\sqrt{fq-qgzz}}{\sqrt{\frac{p}{q}-zz}} + \frac{dx\sqrt{fq-fgxx}}{\sqrt{\frac{p}{f}-xx}}.$$

Statuo $g = \frac{a^2-b^2}{a^3}$, $p = a^3$, $f = q = a$, ut formula evadat

$$\frac{a^2-b^2}{a^3}\,\mathrm{D}xz = (\mathrm{A})\,\frac{dz\sqrt{aa-zz\frac{a^2-b^2}{a^2}}}{\sqrt{aa-zz}} + (\mathrm{B})\,\frac{dx\sqrt{aa-xx.\frac{aa-bb}{aa}}}{\sqrt{aa-xx}}.$$

Conſtat ex num. 5. utramque formulam A, B præbere elementum arcus ellyptici, cujus ſemiaxis major $= a$, minor $= b$, ſumptis abſciſſis $z$, $x$ in axe majore. Itaque ſectis in ellypſis BEA (*Fig.* 1.) axe majore CF $= x$, CI $= z$, ut valeant æquationes $z = \frac{a\sqrt{a^2-x^2}}{\sqrt{aa-x^2.\frac{aa-bb}{aa}}}$, $x = \frac{a\sqrt{aa-zz}}{\sqrt{aa-zz.\frac{a^2-b^2}{a^2}}}$, quæ oſtendunt $z$, $x$ reciprocari, ductiſque ordinatis FD, IE, erit

$$\mathrm{BD} = \mathrm{S}\frac{dx\sqrt{a^2-x^2.\frac{a^2-b^2}{a^2}}}{\sqrt{a^4-x^2}}, \;\&\; \mathrm{BE} = \mathrm{S}\frac{dz\sqrt{a^2-z^2.\frac{a^2-b^2}{a^2}}}{\sqrt{a^2-z^2}}.$$

29. Quamobrem integrando erit $\frac{a^2-b^2}{a^3}\,xz = \mathrm{BE} + \mathrm{BD} + \mathrm{M}$. Ut definias conſtantem M, adverte poſita $x = 0$, fore $z = a$, & BE $=$ quadranti ellyptico BEA: demum BD $= 0$; igitur BEA $+$ M $= 0$, & M $= -$ BEA; ergo

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ergo æquatio $\frac{a^2-b^2}{a^3}.xz = BE + BD - BEA$; ſed $BEA - BE = AE$;

ergo $\frac{a^2-b^2}{a^3}.xz = BD - AE$. Hinc ſequens theorema habeto. In quocumque quadrante ellyptico BEA abſciſſa $CF = x$, accipe $CI = z =$ $\frac{a\sqrt{aa-xx}}{\sqrt{aa-xx.\frac{aa-bb}{aa}}}$; differentia duorum arcuum BD, AE algebraice integratur, & invenitur æqualis $\frac{a^2-b^2}{a^3}.xz$. Si $a=b$, ut ellypſis convertatur in circulum, manifeſtum eſt $\frac{a^2-b^2}{a^3}.xz = o$; ergo differentia arcuum BD, AE nulla erit. Quæ veritas colligitur ex maxime obviis proprietatibus circuli. Nam quum in hypotheſi ſit $CI = z = \sqrt{aa-xx} = FD$, arcus BD, AE æquales ſint neceſſe eſt. Verum in ellypſi differentia arcuum BD, AE nulleſcit numquam exceptis caſibus, in quibus ſit aut $x=o$, aut $x=a$, in quorum primo uterque arcus nullus eſt, in altero coincidit cum quadrante ellyptico.

30. Ut duo puncta DE coincidant in unum punctum V, oportet eſſe

$$x = z = \frac{a\sqrt{aa-xx}}{\sqrt{aa-xx.\frac{aa-bb}{aa}}},$$ ex qua oritur æquatio

$$a^2x^2 - x^4.\frac{a^2-b^2}{a^2} = a^4 - a^2x^2,$$ quæ reſoluta ſufficit ſequentes radices

$x = \frac{a\sqrt{a}}{\sqrt{a-b}}$, $x = \frac{a\sqrt{a}}{\sqrt{a+b}}$. Prima non convenit ellypſi, quia poſtulat $x > a$. Itaque ſeca $CT = \frac{a\sqrt{a}}{\sqrt{a+b}}$, & age perpendicularem TV, differentia duorum arcuum BV, AV erit quantitas algebraica. Ut ejus valorem invenias, ſubſtitue in formulis valorem inventum $x$, ideſt $\frac{a\sqrt{a}}{\sqrt{a+b}}$, & invenies differentiam arcuum $= a - b$. Quoniam $BV - AV = a - b$

item $BD - AE = \frac{aa-bb}{a^3}.xz$, detracta hac ex prima æquatione fiet $VD - VE = a - b - \frac{xz.\overline{aa-bb}}{a^3} = \frac{\overline{a-b}.\overline{a^3-xz.\overline{a+b}}}{a^3}$.

31. In eadem hypotheſi $g, q$ negativarum, $f, p$ poſitivarum ſi ſupponatur

$gp > fq$, ita diſponatur æquatio $g D z x =$ (A) $\frac{q dz \sqrt{-f+gzz}}{\sqrt{-p+qzz}}$ + (B) $\frac{f dx \sqrt{-q+gxx}}{\sqrt{-p+fxx}}$. Quoniam $x = \frac{\sqrt{-p+qzz}}{\sqrt{-f+gzz}}$, & $z = \frac{\sqrt{-p+fxx}}{\sqrt{-q+gxx}}$, ſi $z$ conſtituatur intra limites $\pm z = \frac{\sqrt{p}}{\sqrt{q}}$, $\pm z = \infty$, poſita erit $x$ intra limites $\pm x = 0$, $\pm x = \frac{\sqrt{q}}{\sqrt{g}}$. Quapropter licet formulæ A & B habeant eamdem formam, tamen ſi A integratur ex methodo N. 12., B ex eadem methodo integrari non poteſt. Verumtamen etiam ſi limites $x$ ſint $\pm x = 0$, $\pm x = \frac{\sqrt{q}}{\sqrt{g}}$ æquatio noſtra demonſtrat formulam B, in qua ad vitanda imaginaria ſigna mutabimus, integrari rectificata hyperbola.

32. Quæ formulæ inventæ ſunt hactenus, dependent a rectificatione ſolius ellypſis, vel ſolius hyperbolæ. Ut eas detegamus, quæ per utriuſque curvæ rectificationem abſolvuntur, novum lemma eſt conſtituendum. Hanc ob rem formulam generalem ita diſpono

$$\frac{dz\sqrt{f+gzz}}{\sqrt{p+qzz}} = \frac{f dz + gzz dz}{\sqrt{f+gzz} . \sqrt{p+qzz}} = \frac{g dz . \overline{fq + gqzz}}{gq \sqrt{f+gzz} . \sqrt{p+qzz}}$$

$$= \frac{dz . \overline{fq - gp}}{q . \sqrt{f+gzz} . \sqrt{p+qzz}} + \frac{g dz . \overline{p+qzz}}{q . \sqrt{f+gzz} . \sqrt{p+qzz}} = \frac{dz . \overline{fq - gq}}{q . \sqrt{f+gzz} . \sqrt{p+qzz}}$$

$$+ \frac{g dz \sqrt{p+qzz}}{q . \sqrt{f+gzz}}.$$ Quapropter obtinemus

Lemma III. (A) $\frac{\overline{fq - gp} . dz}{\sqrt{f+gzz} . \sqrt{p+qzz}} =$ (B) $\frac{q dz \sqrt{f+gzz}}{\sqrt{p+qzz}}$ — (C) $\frac{g dz \sqrt{p+qzz}}{\sqrt{f+gzz}}$.

33. Ut utraque formula B, C ſit in poteſtate, aliam hypotheſim non inveniemus, quam ſupponere negativas $g$, $q$. In hac hypotheſi æquatio erit (A) $\frac{dz}{\sqrt{f-gzz} . \sqrt{p-qzz}} =$ + (B) $\frac{q dz \sqrt{f-gzz}}{\overline{fq-gp} \sqrt{p-qzz}}$ — (C) $\frac{g dz \sqrt{p-qzz}}{\overline{fq-gp} . \sqrt{f-gzz}}$. Supponamus $fq > gp$ formula B integratur per arcum ellypticum, formula C per hyperbolicum. Nam poſitis limitibus $\pm z = 0$, $\pm z = \frac{\sqrt{g}}{\sqrt{q}}$, integratur B ex N. 6., C ex N. 31; ſi limites ſint $\pm z = \frac{\sqrt{p}}{\sqrt{q}}$, $\pm z = \frac{\sqrt{f}}{\sqrt{g}}$, utraque imaginaria; ſi ſint $\pm z = \frac{\sqrt{f}}{\sqrt{z}}$, $\pm z = \infty$, integratur B ex N. 20., C ex N. 12.

34. Formulam $\frac{dz}{\sqrt{f-gzz} . \sqrt{p-qzz}}$, quæ mox integrata eſt, in aliam convertimus ope ſubſtitutionis $zz = M - xx$, in qua $M$ eſt quantitas determinanda

da in operationis progreſſu. Facta autem ſubſtitutione provenit $\frac{-x\,dx}{\sqrt{M-xx}\,.\,\sqrt{f-gM+gxx}\,.\,\sqrt{p-qM+qxx}}$. Duplici modo obtineri poteſt, ut una ex duabus radicibus extrahi poſſit, nempe ſi ponatur vel $M=\frac{f}{g}$, vel $M=\frac{p}{q}$. In prima ſuppoſitione formula hæc oritur

$$\frac{-dx}{\sqrt{f-gxx}\,.\,\sqrt{\frac{gp-fq}{g}+qxx}}.$$

Quando ponimus $fq>gp$, ſi fiat $fq-gp=gm$, quod trahit $fq>gm$, habebimus $\frac{-dx}{\sqrt{f-gxx}\,.\,\sqrt{-m+qxx}}$. Hæc realis erit, ſi $x$ media ſit inter limites $\pm x=\frac{\sqrt{f}}{\sqrt{g}}$, $\pm x=\frac{\sqrt{m}}{\sqrt{q}}$, qui reſpondent limitibus $\pm z=o$, $\pm z=\frac{\sqrt{p}}{\sqrt{q}}$; nam ſi $z$ conſtituatur intra limites $\pm z=\frac{\sqrt{f}}{\sqrt{g}}$, $\pm z=\infty$, indeterminata $x$ evadit imaginaria. In altera ſuppoſitione nempe $M=\frac{p}{q}$ naſcitur hæc formula

$$\frac{-dx}{\sqrt{p-qxx}\,.\,\sqrt{\frac{fq-gp}{q}+gxx}}.$$

Quoniam $fq>gp$ ſupponitur, ſi fiat $fq-gp=mq$, obtinetur $\frac{-dx}{\sqrt{p-qxx}\,.\,\sqrt{m+gxx}}$. Realis eſt formula, ſi $x$ conſiſtat intra limites $\pm x=\frac{\sqrt{p}}{\sqrt{q}}$, $\pm x=o$, qui reſpondent limitibus $\pm z=o$, $\pm z=\frac{\sqrt{p}}{\sqrt{q}}$; ſi enim $z$ ponatur intra fines $\pm z=\frac{\sqrt{f}}{\sqrt{g}}$, $\pm z=\infty$, fit $x$ imaginaria. Ambæ autem formulæ ſcilicet $\frac{-dx}{\sqrt{f-gxx}\,.\,\sqrt{-m+qxx}}$, & $\frac{-dx}{\sqrt{p-qxx}\,.\,\sqrt{m+gxx}}$ integrantur integrata formula $\frac{dz}{\sqrt{f-gzz}\,.\,\sqrt{p-qzz}}$, cui ſeorſim æquantur.

35. Nunc revertimus ad Lemma tertium, & ſupponentes in eo negativas $g$, $p$ nanciſcimur æquationem (A) $\frac{\overline{fq-gp}\,.\,dz}{\sqrt{f-gzz}\,.\,\sqrt{-p+qzz}}=$ (B) $\frac{q\,dz\,.\,\sqrt{f-gzz}}{\sqrt{-p+qzz}}+$ (C) $\frac{g\,dz\sqrt{-p+qzz}}{\sqrt{f-gzz}}$. Si ponatur $fq>gp$ formula $A$ integratur ex N. 34 per arcus ellypticos & hyperbolicos, formula, $C$ ex N. 18 & 25 per ſolam rectificationem hyperbolæ. Ergo per arcus ſectionum conicarum integrabitur etiam formula $B$, quæ realis erit, ſi $z$ media ſit inter fines $\pm z=\frac{\sqrt{p}}{\sqrt{q}}$, $\pm z=\frac{\sqrt{f}}{\sqrt{g}}$.

36. In lemmate tertio ſupponamus dumtaxat $q$ negativam, mutatiſque ſignis in omnibus terminis naſciſcemur (A) $\frac{fq+gp \cdot dz}{\sqrt{f+gzz} \cdot \sqrt{p-qzz}} = $ (B) $\frac{qdz\sqrt{f+gzz}}{\sqrt{p+qzz}}$ — (C) $\frac{gdz\sqrt{p-qzz}}{\sqrt{f+gzz}}$. Formula A ex N. 34 integratur rectificato arcu ellyptico, & hyperbolico, formula B rectificato arcu ellyptico ex N. 9: ergo formula C rectificatis ſectionibus conicis eſt in poteſtate. Si $z$ media ſit inter $\pm z = 0$, $\pm z = \frac{\sqrt{p}}{\sqrt{q}}$, formulæ tres reales ſunt, ſecus imaginariæ.

37. Determinatis his duabus formulis, quæ dependent tum ab ellypſeos, tum ab hyperbolæ rectificatione, revocamus lemma primum, in quo ſupponimus $g$, $p$ negativas, ut habeamus $Dxz = $ (A) $\frac{dz\sqrt{f-gzz}}{\sqrt{-p+qzz}} + $ (B) $\frac{dx\sqrt{f+pxx}}{\sqrt{g+qxx}}$ exiſtente $x = \frac{\sqrt{f-gzz}}{\sqrt{-p+qzz}}$ & $z = \frac{\sqrt{f+pxx}}{\sqrt{g+qxx}}$. Si $fq > gp$ formula A ex N. 35 integratur per rectificationem utriuſque ſectionis. Ergo etiam formula B, ſi $z$ conſtituatur inter limites $\pm z = \frac{\sqrt{p}}{\sqrt{q}}$, $\pm z = \frac{\sqrt{f}}{\sqrt{g}}$, quibus poſitis formula A poteſt eſſe realis, erit $x$ intra fines $\pm x = \infty$, $\pm x = 0$, quod indicat formulam B ſemper eſſe realem.

38. Si in lemmate primo ponas $g$ negativam, habebis $Dxz = $ (A) $\frac{dz\sqrt{f-gzz}}{\sqrt{p+qzz}} + $ (B) $\frac{dx\sqrt{f-pxx}}{\sqrt{g+qxx}}$ exiſtente $x = \frac{\sqrt{f-gzz}}{\sqrt{p+qzz}}$, $z = \frac{\sqrt{f-pxx}}{\sqrt{g+qxx}}$. Si $z$ conſtituatur inter limites $\pm z = 0$, $\pm z = \frac{\sqrt{f}}{\sqrt{g}}$, invenietur $x$ eſſe intra limites $\pm x = \frac{\sqrt{f}}{\sqrt{p}}$, $\pm x = 0$, extra quos limites utraque formula imaginaria eſt. Finibus hiſce ſancitis utraque formula integrationem recipit ex N. 36. Quocirca cognoſces plures arcus ellypticos, & hyperbolicos conjunctos per ſigna $+-$ eſſe algebraice rectificabiles. Verum re diligenter perpenſa comperies, nullos alios haberi arcus niſi eos, quorum differentiæ rectificabiles ſunt, de quibus antea loquuti ſumus.

39. Advocamus nunc lemma ſecundum, & ſupponentes negativas $f$, $q$, ſive quod idem eſt $g$, $p$ invenimus $gDzx = -$ (A) $\frac{qdz\sqrt{f-gzz}}{\sqrt{-p+qzz}}$ — (B) $\frac{fdx\sqrt{q+gxx}}{\sqrt{p+fxx}}$. Si $fq > pg$ formula A per arcus utriuſque ſectionis integratur ex N. 35. Ergo etiam formula B, quæ eamdem formam habet ac illa, quæ integrata eſt N. 37. Quare ejuſdem formulæ ex ultima noſtra æquatione nova poteſt obtineri conſtructio.

40. Denique in lemmate ſecundo ſupponamus $g$ negativam, ſive quod idem

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dem eſt $g$ poſitivam reliquas omnes negativas. Oritur Æquatio $g$ D $xz = -$ (A) $\frac{q\,dz\sqrt{f-gzz}}{\sqrt{p+qzz}}$ + (B) $\frac{f\,dx\sqrt{q+gxx}}{\sqrt{-p+fxx}}$ exiſtente $x = \frac{\sqrt{p+qzz}}{\sqrt{f-gzz}}$, & $z = \frac{\sqrt{-p+fxx}}{\sqrt{q+gxx}}$. Si $z$ conſtituatur intra limites $\pm z = o$, $\pm z = \frac{\sqrt{f}}{\sqrt{g}}$, habebit $x$ limites $\pm x = \frac{\sqrt{p}}{\sqrt{f}}$, $\pm x = \infty$. Atqui poſitis his limitibus formula $A$ ex N. 36. integratur. Ergo etiam formula B, quæ formula imaginaria eſt, ſi $x$ ex ſtatutis finibus egrediatur.

41. Poſt has demonſtrationes in aperto eſt formulam $\frac{dz\sqrt{f+gzz}}{\sqrt{p+qzz}}$, quæcumque ſint $f$, $g$, $p$, $q$ vel poſitivæ, vel negativæ ſemper integrari elypſi & hyperbola rectificatis. Ut autem facilius inveniri poſſit, quo in loco hujuſce diſquiſitionis quilibet caſus contineatur, ſequentem tabulam formavimus, quæ indicis locum tenere poteſt, in qua omnes caſus diſtincti ſunt & numeri expoſiti, ubi ſinguli continentur. Littera E indicabit formulam a rectificatione ellypſis, littera H a rectificatione hyperbolæ dependere. Ubi utraque littera conjungitur formula poſtulabit utriuſque ſectionis rectificationem.

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## TABULA.

$+f+g+p+q$; $-f-g-p-q$

Si $gp > fq$ . . . . . . . . . . H. 15
Si $fq > gp$ . . . . . . . . . . E. H. 37. 39

. . . . . . . . . . . . . . . .

$+f+g+p-q$, $-f-g-p+q$

Si limites ſint $\pm z = o$, $\pm z = \frac{\sqrt{p}}{\sqrt{q}}$ E. 9

extra hos limites imaginaria

. . . . . . . . . . . . . . . .

$+f+g-p+q$, $-f-g+p-q$

Si limites ſint $\pm z = o$, $\pm z = \frac{\sqrt{p}}{\sqrt{q}}$ imaginaria

$\pm z = \frac{\sqrt{p}}{\sqrt{q}}$, $\pm z = \infty$ E. H. 40

. . . . . . . . . . . . . . . .

$+f-g+p+q$; $-f+g-p-q$

Si limites ſint $\pm z = o$, $\pm z = \frac{\sqrt{f}}{\sqrt{g}}$ E. H. 36

$\pm z = \frac{\sqrt{f}}{\sqrt{g}}$, $\pm z = \infty$ imaginaria

. . . . . . . . . . . . . . . . $-f$

$-f+g+p+q;\quad +f-g-p-q$

Si limites ſint $\pm z = o,\ \pm z = \frac{\sqrt{f}}{\sqrt{g}}$ imaginaria

$\pm z = \frac{\sqrt{f}}{\sqrt{g}},\ \pm \zeta = \infty$ E. 19

\- - - - - - - - - - - - - - -

$+f+g-p-q;\quad -f-g+p+q$

ſemper imaginaria

\- - - - - - - - - - - - - - -

$+f-g+p-q;\quad -f+g-p+q$

Si li- $\pm z = o,\ \pm \zeta = \frac{\sqrt{p}}{\sqrt{q}}$ E. 6

Si $fq > gp$ mites $\pm \zeta = \frac{\sqrt{p}}{\sqrt{q}},\ \pm z = \frac{\sqrt{f}}{\sqrt{g}}$ imaginaria

ſint $\pm \zeta = \frac{\sqrt{f}}{\sqrt{g}},\ \pm \zeta = \infty$ E. 20

Si li- $\pm z = o,\ \pm \zeta = \frac{\sqrt{f}}{\sqrt{g}}$ H. 31

Si $gp > fq$ mites $\pm z = \frac{\sqrt{f}}{\sqrt{g}},\ \pm \zeta = \frac{\sqrt{p}}{\sqrt{q}}$ imaginaria

ſint $\pm \zeta = \frac{\sqrt{p}}{\sqrt{q}},\ \pm z = \infty$ H. 32

\- - - - - - - - - - - - - - -

$-f+g+p-q;\quad +f-g-p+q$

Si $gp > fq$ H. 18. 28

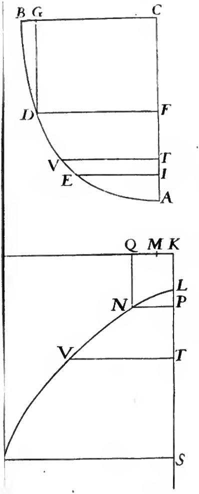
& limi- tes ſint $\pm z = \frac{\sqrt{f}}{\sqrt{g}},\ \pm \zeta = \frac{\sqrt{p}}{\sqrt{q}}$

Si $fq > gp$ E. H. 35

extra hos limites imaginaria.

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42. Quiſnam ſit uſus bujuſce tabulæ, opus non eſt, ut explicem. Nam propoſita aliqua formula inveniendus eſt in tabula caſus, ubi ſpecies $f, g, p, q$ eodem ſigno donatæ ſunt. Ibi invenies conditiones, in quibus formula aut imaginaria eſt, aut pertinet ad rectificationem alterutrius, vel utriuſque ſectionis conicæ, & numeros denotantes, quo in loco hoc ſit demonſtratum. Cauſa exempli ſit propoſita formula $\frac{dz\sqrt{f-gz^2}}{\sqrt{-p+q\zeta^2}}$. Fac invenias in tabula caſum, ubi $f, q$ ſigno $+$, $g, p$ ſigno $-$ affectæ ſint: atque hic ultimum in tabula locum tenet. Hic notatum reperies, formulam realem eſſe non poſse, niſi $z$ media ſit inter limi-


B G C
D F
V T
E I
A
Q M K
L
N P
V T
S

limites $\pm z = \frac{\sqrt{f}}{\sqrt{g}}$, $\pm z = \frac{\sqrt{p}}{\sqrt{q}}$. Præterea cognoſces, ſi $gp > fq$ formulam integrari hyperbola rectificata, atque hoc demonſtrari num. 18. aut num. 25. Contra ſi $fq > gp$, comperies formulam ad ſui integrationem indigere rectificatione utriuſque ſectionis, atque hoc probari num. 35. Ita de caſibus reliquis.

## *CAPUT DECIMUM TERTIUM.*

### De formulis, quorum integratio dependet a rectificatione ellypſis, & hyperbolæ.

1. IN ſuperiore capite, ubi per arcus ellypticos, & hyperbolicos conſtruendam curavimus formulam $\frac{dz\sqrt{f+gzz}}{\sqrt{p+gzz}}$, quam deinceps canonicam appellabimus, dictum eſt, formulas omnes, quæ ad hanc reduci poſſunt, rectificatis hyperbola, & ellypſi ſimiliter conſtrui. In hoc vero quænam ſint hujuſmodi formulæ inveſtigandum eſt. De theoria hac ingenioſe egit Alembertus formulam reducens non ad noſtram canonicam, ſed ad duas alias formulas. Poſt ipſum methodo ſæpe ſimplicitatem concilians atque elegantiam promovit theoriam addens nonnulla P. Riccatus opuſc. 2. Tom. 2. opuſcul., ex quo decerpta ſunt omnia, quæ in hoc capite tradentur.

2. His præmiſſis rem aggrediens adverto nos in cap. ſup. num. 32., ubi per arcus ellypticos & hyperbolicos integravimus formulam $\frac{dz\sqrt{f+gzz}}{\sqrt{p+qzz}}$, quam, ut diximus, appellabimus canonicam, ad hanc ipſam deduxiſſe formulam aliam $\frac{dz}{\sqrt{f+gzz}\cdot\sqrt{p+qzz}}$ uſos hac methodo. Formulam canonicam ita diſpone $\frac{dz\sqrt{f+gzz}}{\sqrt{p+qzz}} = \frac{fdz+gzzdz}{\sqrt{f+gzz}\cdot\sqrt{p+qzz}} = \frac{fqdz+gqz^2dz}{q\cdot\sqrt{f+gzz}\cdot\sqrt{p+qzz}}$

$= \frac{fq-gp\cdot dz}{q\sqrt{f+gzz}\cdot\sqrt{p+qzz}} + \frac{gdz\sqrt{p+qzz}}{q\sqrt{f+gzz}}$; Igitur $\frac{dz}{\sqrt{f+gzz}\cdot\sqrt{p+qzz}}$

$= \frac{qdz\sqrt{f+gzz}}{fq-gp\cdot\sqrt{p+gzz}} - \frac{gdz\sqrt{p+qzz}}{fq-gp\cdot\sqrt{f+gzz}}$. Quæ duæ formulæ quoniam conveniunt cum canonica, conſtat formulam $\frac{dz}{\sqrt{f+gzz}\cdot\sqrt{p+qzz}}$ ſemper reduci ad canonicam. Advertendum primo. Si una ex radicibus per conſtantem multiplicata alteram daret, tum $fq = gp$: quare inutilis eſt hujuſmodi præparatio. Verum in hoc caſu evidens eſt, formulam non indigere arcubus ellypticis, & hyperbolicis, ſed integrari ſuppoſitis dumtaxat quadraturis circuli, & hyperbo-

bolæ. Advertendum ſecundo, ſuperiori capite N. 34. tres formulas ſummatas exhibitas fuiſſe, quæ in præſenti continentur. Quare in illis caſibus utilius erit eam integrationem uſurpare.

3. Si radices duas formulæ ſuperioris ſimul multiplicemus, aliam obtinebimus hujus formæ $\frac{dz}{\sqrt{a+bzz+cz^4}}$, in qua poteſt etiam eſſe $b=o$. Si trinomium $a+bzz+cz^4$ ſit reſolubile in duo binomia realia hujus formæ $g+zz$, conſtat ex N. I, formulam integrari ſuppoſita rectificatione ellypſis, & hyperbolæ. Verum ſi in hujuſmodi binomia reſolubile non ſit, quod contingit, quum $a > \frac{bb}{4c}$, docendum eſt, quo pacto formula ſit tractanda. Primum ejiciendus eſt ex trinomio ſecundus terminus ope ſubſtitutionis $zz+\frac{b}{2c}=uu$, ex qua naſcitur $zz=uu-\frac{b}{2c}$, $dz=\frac{udu}{\sqrt{uu-\frac{b}{2c}}}$, $\sqrt{a+bzz+cz^4}=\sqrt{a-\frac{bb}{4c}+cu^4}$, ſive poſitis $\frac{b}{2c}=n$, $a-\frac{bb}{4c}=cm$, $zz=uu-n$, $dz=\frac{udu}{\sqrt{uu-n}}$, $\sqrt{a+bzz+cz^4}=\sqrt{c\cdot m+u^4}$. Transformata itaque formula habebimus $\frac{dz}{\sqrt{a+bzz+cz^4}}=\frac{udu}{\sqrt{u^2-n}\cdot\sqrt{c\cdot m+u^4}}$. Uſurpanda nunc eſt ſecunda ſubſtitutio $uu+\sqrt{m+u^4}=yy$, ex qua oritur $m+u^4=y^4-2u^2yy+u^4$, ſive $uu=\frac{y^4-m}{2yy}=\frac{y^2}{2}-\frac{m}{2y^2}$, & ſumptis differentiis $udu=\frac{ydy}{2}+\frac{mdy}{2y^3}=dy\cdot\frac{y^4+m}{2y^3}$. Præterea $uu-n=\frac{y^2}{2}-\frac{m}{2y^2}-n=\frac{y^4-2ny^2-m}{2y^2}$. Hoc trinomium $y^4-2ny^2-m$ reſolubile ſemper eſt in duo binomia realia, nempe $y^2-n+\sqrt{n^2+m}$, $y^2-n-\sqrt{n^2+m}$, ſive ſubſtitutis valoribus $m$, $n$, $y^2-\frac{b}{2c}+\sqrt{\frac{a}{c}}$, $y^2-\frac{b}{2c}-\sqrt{\frac{a}{c}}$, quæ duo binomia in noſtra hypotheſi ſunt ſemper realia, quia quum debeat eſſe $\frac{a}{c} > \frac{bb}{4cc}$, neceſſario $\frac{a}{c}$ erit poſitiva: Ergo

$$\sqrt{u^2-n}=\frac{\sqrt{yy-\frac{b}{2c}+\sqrt{\frac{a}{c}}}\cdot\sqrt{yy-\frac{b}{2c}-\sqrt{\frac{a}{c}}}}{y\sqrt{2}}.$$

De-

Demum $u^4+m=\frac{y^4}{4}-\frac{2m}{4}+\frac{m^2}{4y^4}+m=\frac{y^4}{4}+\frac{2m}{4}+\frac{m^2}{4y^4}$. Ergo $\sqrt{u^4+m}=\frac{y^2}{2}+\frac{m}{2y^2}=\frac{y^4+m}{2y^2}$. Itaque, peractis ſubſtitutionibus, fiet

$$\frac{dz}{\sqrt{a+bzz+cz^4}}=\frac{dy\sqrt{2}}{\sqrt{c}.\sqrt{yy-\frac{b}{2c}+\sqrt{\frac{a}{c}}}.\sqrt{y^2-\frac{b}{2c}-\sqrt{\frac{a}{c}}}}$$

, quæ ex N. 2 ad canonicam reducitur. Facto autem calculo invenies

$$\frac{dz}{\sqrt{a+bz^2+cz^4}}=\frac{dy\sqrt{\frac{-b}{2c}+\sqrt{\frac{a}{c}}+yy}}{\sqrt{2a}.\sqrt{\frac{-b}{2c}-\sqrt{\frac{a}{c}}+yy}}-\frac{dy\sqrt{\frac{-b}{2c}-\sqrt{\frac{a}{c}}+yy}}{\sqrt{2a}.\sqrt{\frac{-b}{2c}+\sqrt{\frac{a}{c}}+yy}}$$

, exiſtente $y=\sqrt{zz+\frac{b}{2c}+\sqrt{\frac{a+bz^2+cz^4}{c}}}$.

4. Ut ad canonicam redigam formulam $\frac{zzdz}{\sqrt{f+gzz}.\sqrt{p+qzz}}$, deduco formulam numeri 2 a canonica hoc modo

$\frac{dz\sqrt{f+gzz}}{\sqrt{p+qzz}}-\frac{fdz}{\sqrt{f+gzz}.\sqrt{p+qzz}}=\frac{gzzdz}{\sqrt{f+gzz}.\sqrt{p+qzz}}$: Igitur

$\frac{zzdz}{\sqrt{f+gzz}.\sqrt{p+qzz}}=\frac{dz\sqrt{f+gzz}}{g\sqrt{p+qzz}}-\frac{fdz}{g\sqrt{f+gzz}.\sqrt{p+qzz}}$: atqui ex N. 2 $\frac{fdz}{g\sqrt{f+gzz}.\sqrt{p+qzz}}=\frac{fqdz\sqrt{f+gzz}}{g.\overline{fq-gp}.\sqrt{p+qzz}}-\frac{fdz\sqrt{p+qzz}}{\overline{fq-gp}.\sqrt{f+gzz}}$; Ergo facta ſubtractione $\frac{zzdz}{\sqrt{f+gzz}.\sqrt{p+qzz}}=$ $\frac{-pdz\sqrt{f+gzz}}{\overline{fq-gp}.\sqrt{p+qzz}}+\frac{fdz\sqrt{p+qzz}}{\overline{fq-gp}.\sqrt{f+gzz}}$, quæ duæ coincidunt cum canonica. Inutilis eſt hæc præparatio, ſi $fq=gp$; ſed formula in hoc caſu ad ſui integrationem non indiget, niſi circuli, aut hyperbolæ quadratura.

5. Ut ad canonicam reducam formulam $\frac{zzdz}{\sqrt{a+bzz+cz^4}}$, quum trinomium $a+bzz+cz^4$ non poteſt reſolvi in duo binomia realia hujus formæ $f+gzz$, neceſse eſt, ut prius ad canonicam perducam formulam $\frac{dz\sqrt{f+gzz}.\sqrt{p+qzz}}{zz}$. Hanc ob rem utar methodo, quam deinceps latius patere oſtendam. Quantita-

tis $\frac{\sqrt{f+gzz}.\sqrt{p+qzz}}{z}$ accipio differentiam hoc modo

$$D\frac{\sqrt{f+gzz}.\sqrt{p+qzz}}{z}=\frac{qdz\sqrt{f+gzz}}{\sqrt{p+qzz}}+\frac{gdz\sqrt{p+qzz}}{\sqrt{f+gzz}}$$

$$-\frac{dz\sqrt{f+gzz}.\sqrt{p+qzz}}{zz}$$; Ergo facta tranſpoſitione

$$\frac{dz\sqrt{f+gzz}.\sqrt{p+qzz}}{zz}=-D\frac{\sqrt{f+gzz}.\sqrt{p+qzz}}{z}$$

$$+\frac{qdz\sqrt{f+gzz}}{\sqrt{p+qzz}}+\frac{gdz\sqrt{p+qzz}}{\sqrt{f+gzz}}$$. Quare propoſita formula inventa eſt æqualis duabus, quæ conveniunt cum canonica, dempta formula integrabili.

6. Ad reducendam formulam $\frac{zzdz}{\sqrt{a+bzz+cz^4}}$, quum $\frac{a}{c}>\frac{bb}{4cc}$, quo in caſu trinomium non eſt reſolubile in binomia realia, utor eadem methodo, iiſdemque ſubſtitutionibus, quibus uſus ſum N. 3. Effecta prima ſubſtitutione invenies $\frac{zzdz}{\sqrt{a+bzz+cz^4}}=\frac{udu\sqrt{uu-n}}{\sqrt{c.m+u^4}}$. Peracta ſecunda oritur

$$\frac{zzdz}{\sqrt{a+bzz+cz^4}}=\frac{dy\sqrt{\frac{-b}{2c}+\sqrt{\frac{a}{c}}+yy}.\sqrt{\frac{-b}{2c}-\sqrt{\frac{a}{c}}+yy}}{\sqrt{2c}.yy}$$; atqui

ex N. 5. $$\frac{dy\sqrt{\frac{-b}{2c}+\sqrt{\frac{a}{c}}+yy}.\sqrt{\frac{-b}{2c}-\sqrt{\frac{a}{c}}+yy}}{yy}=$$

$$-D\frac{\sqrt{\frac{-b}{2c}+\sqrt{\frac{a}{c}}+yy}.\sqrt{\frac{-b}{2c}-\sqrt{\frac{a}{c}}+yy}}{y}$$

$$+\frac{dy\sqrt{\frac{-b}{2c}+\sqrt{\frac{a}{c}}+yy}}{\sqrt{\frac{-b}{2c}-\sqrt{\frac{a}{c}}+yy}}+\frac{dy\sqrt{\frac{-b}{2c}-\sqrt{\frac{a}{c}}+yy}}{\sqrt{\frac{-b}{2c}+\sqrt{\frac{a}{c}}+yy}}$$; ergo facta ſubſtitutione provenict

$$\frac{zzdz}{\sqrt{a+bzz+cz^4}}=-D\frac{\sqrt{\frac{-b}{2c}+\sqrt{\frac{a}{c}}+yy}.\sqrt{\frac{-b}{2c}-\sqrt{\frac{a}{c}}+yy}}{y\sqrt{2c}}$$

$+dy$

$$\frac{+dy\sqrt{\frac{-b}{2c}+\sqrt{\frac{a}{c}}+yy}}{\sqrt{2c}\,.\sqrt{\frac{-b}{2c}-\sqrt{\frac{a}{c}}+yy}}+\frac{dy\sqrt{\frac{-b}{2c}-\sqrt{\frac{a}{c}}+yy}}{\sqrt{2c}\,.\sqrt{\frac{-b}{2c}+\sqrt{\frac{a}{c}}+yy}}$$ Q. E. Inv.

7. Ex his formulas alias magis late patentes nanciſcor, & ad canonicam perduco. Hanc ob rem ſumo differentiam formulæ $z^r\sqrt{a+bzz+cz^4}$ hoc modo $Dz^r\sqrt{a+bzz+cz^4}=rz^{r-1}dz\sqrt{a+bzz+cz^4}$ $\frac{+bz^{r+1}dz+2cz^{r+3}dz}{\sqrt{a+bzz+cz^4}}$. Formulas duas ad eamdem denominationem reduco, ut fiat $Dz^r\sqrt{a+bzz+cz^4}=\frac{raz^{r-1}dz+\overline{r+1}.bz^{r+1}dz+\overline{r+2}.cz^{r+3}dz}{\sqrt{a+bzz+cz^4}}$.

8. Æquationem hanc primum ita diſpono (A) $\frac{z^{r+3}dz}{\sqrt{a+bzz+cz^4}}=Dz^r\frac{\sqrt{a+bzz+cz^4}}{\overline{r+2}.c}$ $-$(B) $\frac{\overline{r+1}.bz^{r+1}dz}{\overline{r+2}.c\sqrt{a+bzz+cz^4}}$ $-$(C) $\frac{raz^{r-1}dz}{\overline{r+2}.c\sqrt{a+bzz+cz^4}}$. Si $r=1$, formulæ $B$, $C$ reducuntur ad canonicam vel ex N. 4, & 2, ſi trinomium reſolvi poſſit in duo binomia realia, vel ex N. 6, & 3, ſi reſolutio binomii in trinomia realia fieri non poſſit. Si $r=3$, formnla B ex caſu ſuperiore, formula $C$ ex N. 4, aut 6 ad canonicam perducitur. Si $r=5$, formulæ $B$, $C$ ex duobus caſibus ſuperioribus reſolvuntur, atque ita deinceps in infinitum: ergo formula $\frac{z^{r+3}dz}{\sqrt{a+bzz+cz^4}}$ exiſtente $r$ numero poſitivo, & impari ſemper reducitur ad canonicam, atque integratur per arcus ellypticos, & hyperbolicos. Notum eſt omnibus analyſtis, formulas duas $\frac{zdz}{\sqrt{a+bzz+cz^4}}$, $\frac{dz}{z\sqrt{a+bzz+cz^4}}$ ad ſui integrationem ad ſummum requirere quadraturas circuli, & hyperbolæ. Colligemus proinde, formulam ſuperiorem $\frac{z^{r+3}dz}{\sqrt{a+bzz+cz^4}}$, exiſtente $r$ numero pari, & poſitivo inregrari quadratis circulo, & hyperbola. Nam poſito $r=o$, apparet $\frac{z^3dz}{\sqrt{a+bzz+cz^4}}$ dependere a duabus ſuprapoſitis; facto $r=2$, formula $\frac{z^5dz}{\sqrt{a+bzz+cz^4}}$ dependet ab integratione duarum $\frac{z^3dz}{\sqrt{a+bzz+cz^4}}$,

$\frac{z\,dz}{\sqrt{a+bzz+cz^4}}$; atque ita deinceps progreſſu ſatis manifeſto.

9. Eamdem æquationem N. 7 nova hac ratione diſpono

$$(A)\ \frac{z^{r-1}dz}{\sqrt{a+bzz+cz^4}}=D\ \frac{z^r\sqrt{a+bzz+cz^4}}{ra}$$

$$-(B)\ \frac{\overline{r+1}.bz^{r+1}dz}{ra\sqrt{a+bzz+cz^4}}-(C)\frac{\overline{r+2}.cz^{r+3}dz}{ra\sqrt{a+bzz+cz^4}}.$$ Si $r=-1$, formula $B$ evaneſcit; formula autem $C$ ex N. 4, aut 6 ad canonicam reducitur. Si $r=-3$, formula $B$ ex caſu ſuperiore, formula $C$ ex N. 4, aut 6 integratur. Si $r=-5$, ex duobus caſibus ſuperioribus formulæ $B$, $C$ ſunt in poteſtate, atque ita deinceps: Ergo formula $\frac{z^{r-1}dz}{\sqrt{a+bzz+cz^4}}$ eſt in poteſtate, ſi $r$ eſt numerus impar, & negativus. Si ſtatuas $r$ æqualem numero pari, & negativo, patebit formulam $\frac{z^{r-1}dz}{\sqrt{a+bzz+cz^4}}$ ad ſui integrationem non indigere, niſi quadraturis circuli, & hyperbolæ. Namque facto $r=-2$, $\frac{dz}{z^3\sqrt{a+bzz+cz^4}}$ pendebit a duabus $\frac{dz}{z\sqrt{a+bz^2+cz^4}}$, $\frac{z\,dz}{\sqrt{a+bzz+cz^4}}$; facto $r=-4$, $\frac{dz}{z^5\sqrt{a+bzz+cz^4}}$ pendebit a $\frac{dz}{z^3\sqrt{a+bzz+cz^4}}$, $\frac{dz}{z\sqrt{a+bzz+cz^4}}$; atque ita deinceps progreſſu ſatis manifeſto.

10. Ex his, quæ hactenus demonſtrata ſunt, manifeſtum eſt, formulam $\frac{z^r dz}{\sqrt{a+bzz+cz^4}}$ reduci ad canonicam, atque adeo integrari per arcus ellypticos & hyperbolicos, quotieſcumque $r$ ſit numerus par vel poſitivus, vel negativus. Si vero $r$ ſit impar, non indiget niſi quadraturis circuli, & hyperbolæ.

11. Tranſeo ad formulam $\frac{z^t dz\sqrt{f+gzz}}{\sqrt{p+qzz}}$, quæ nullo negotio reducitur ad ſuperiorem. Nam ſecta multiplicatione, & diviſione per $\sqrt{f+gzz}$ oritur

$$\frac{z^t dz\sqrt{f+gzz}}{\sqrt{p+qzz}}=\frac{fz^t dz}{\sqrt{f+gzz}.\sqrt{p+qzz}}+\frac{gz^{t+2}dz}{\sqrt{f+gzz}.\sqrt{p+qzz}},$$

quæ

quarum utraque in N. 10. continetur. Itaque ſi $s$ ſit numerus par vel affirmativus, vel negativus integratur ſuppoſita rectificatione ellypſis, & hyperbolæ; ſi $s$ ſit impar, contenta eſt quadraturis circuli, & hyperbolæ.

12. Idem dicendum eſt de formula $z^s dz \sqrt{a+bzz+cz^4}$. Fiat enim multiplicatio, & diviſio per $\sqrt{a+bzz+cz^4}$; atque hæc obtinetur æquatio

$$z^s dz \sqrt{a+bbzz+cz^4} = \frac{az^s dz}{\sqrt{a+bzz+cz^4}} + \frac{bz^{s+2} dz}{\sqrt{a+bzz+cz^4}} + \frac{cz^{s+4} dz}{\sqrt{a+bzz+cz^4}},$$

quæ tres formulæ ſpectant ad N. 10. Igitur ſi $s$ ſit numerus par vel poſitivus, vel negativus, $z^s dz \sqrt{a+bzz+cz^4}$ integratur per arcus ellypticos, & hyperbolicos; ſi $s$ ſit impar, per ſolas quadraturas circularem, & hyperbolicam.

13. Luce clarius eſt, formulam $z^s . \overline{a+bzz+cz^4}^{\,e} . dz \sqrt{a+bzz+cz^4}$ ad ſuperiorem pertinere, ſi $e$ ſit numerus integer, & affirmativus; nam elato trinomio ad poteſtatem $e$, plures formulæ exorientur, quæ ad ſuperiorem pertinebunt. Atqui formula expoſita huic æquivalet $z^s dz . \overline{a+bzz+cz^4}^{\,\frac{2e+1}{2}}$, in qua $2e+1$ erit numerus impar, & poſitivus; ergo exiſtente $r$ numero impari, & poſitivo, formula $z^s dz . \overline{a+bzz+cz^4}^{\,\frac{r}{2}}$, integratur, ſi $s$ ſit par, per arcus ellypticos, & hyperbolicos; ſi $s$ ſit impar, per quadraturas circuli, & hyperbolæ. Alio modo idem oſtenditur. Formula $\frac{z^s . \overline{a+bzz+cz^4}^{\,e} . dz}{\sqrt{a+bzz+cz^4}}$ convertitur in plures formulas, quæ ex N 10. integrantur, elevato trinomio ad poteſtatem $e$ poſitivam, & integram: ſed hæc æquivalet $z^s dz . \overline{a+bzz+cz^4}^{\,\frac{2e-1}{2}}$; ergo facto $2e-1=r$, qui neceſſario erit numerus impar, & poſitivus, habebimus formulam $z^s dz . \overline{a+bzz+cz^4}^{\,\frac{r}{2}}$, quæ integrabitur per arcus ellypticos & hyperbolicos, ſi $s$ ſit numerus par; per quadraturas circuli, & hyperbolæ, ſi $s$ ſit impar.

14. Simili ratione $\frac{z^s . \overline{f+gzz}^{\,e} . dz \sqrt{f+gzz}}{\sqrt{p+qzz}}$ exiſtente $e$ numero integro, & poſitivo, elevato binomio ad poteſtatem $e$, convertitur in plures formulas, quæ integrantur eſt N. 11: ſed expoſita formula huic æquivalet

$z^s dz$.

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$\frac{z^t dz.\overline{f+gzz}^{\frac{2e+1}{2}}}{\sqrt{p+qzz}}$, in qua $2e+1$ neceſſario eſt numerus impar; Ergo exiſtente $r$ numero impari, & poſitivo, formula $\frac{z^t dz.\overline{f+gzz}^{\frac{r}{2}}}{\sqrt{p+qzz}}$ obtinetur rectificatis ellypſi, & hyperbola, ſi $t$ ſit numerus par vel poſitivus, vel negativus, quadratis circulo, & hyperbola, ſi $t$ ſit impar.

15. Formula item $\frac{z^t dz.\overline{p+qzz}^e.\overline{f+gzz}^{\frac{r}{2}}}{\sqrt{p+qzz}}$ poſitis $t$, $e$, $r$ numeris, qui ſupra definiti ſunt, elevato binomio ad poteſtatem $e$, in plures mutatur, quæ integrationem recipiunt ex N. 14: atqui ea æqualis eſt $z^t dz.\overline{p+qzz}^{\frac{2e-1}{2}}.\overline{f+gzz}^{\frac{r}{2}}$, in qua $2e-1$, eſt numerus impar, & poſitivus, qui ſiat $=s$: Ergo $z^t dz.\overline{f+gzz}^{\frac{r}{2}}.\overline{p+qzz}^{\frac{s}{2}}$ rectificatis ellypſi & hyperbola integratur, ſi $r$, $s$ ſint numeri impares, & poſitivi, & $t$ numerus par vel poſitivus, vel negativus. Si vero $t$ ſit impar vel poſitivus, vel negativus, formula per quadraturas circuli, & hyperbolæ abſolvitur.

16. Abſolvam nunc formulam $\frac{z^t dz}{\overline{a+bzz+cz^4}^{\frac{r}{2}}}$, vel trinomium $a+bzz+cz^4$ poſſit reſolvi in binomia realia, vel ſecus, exiſtente $t$ pari aut poſitivo aut negatio, & $r$ impari, & poſitivo, quæ formula aliquanto eſt difficilior. Conſidero primum caſus, in quibus aut $t>r+1$, aut $t=r+1$, aut $t=r-1$. Hanc ob rem formulam ita diſpono $\frac{z^{t-r-1}.zdz.z^r}{\overline{a+bzz+cz^4}^{\frac{r}{2}}}$. Statuo

$\frac{a+bzz+cz^4}{zz}=uu$, ut formula evadat $\frac{z^{t-r-1}.zdz}{u^r}$; atqui

$z^4+\frac{b-uu}{c}zz=\frac{-a}{c}$; Igitur $z^2=\frac{-b+uu}{2c}+\sqrt{\frac{\overline{b-u^2}^2}{4cc}-\frac{a}{c}}$; Ergo

$zdz=\frac{udu}{2c}-\frac{udu.\overline{b-uu}}{4cc\sqrt{\frac{\overline{b-u^2}^2}{4cc}-\frac{a}{c}}}$. Demum

$z^t dz$

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$$\frac{z^s dz}{(a+bzz+cz^4)^{\frac{r}{2}}} = \frac{du}{2cu^{r-1}} - \frac{du.\overline{b-u^2}}{4cc\sqrt{\frac{\overline{b-uu}^2}{4cc}-\frac{a}{c}}.u^{r-1}}$$ ducta in

$$\left(\frac{-b+u^2}{2c}+\sqrt{\frac{\overline{b-u^2}^2}{4cc}-\frac{a}{c}}\right)^{\frac{s-r-1}{2}}$$

. Quoniam tam $s$, quam $r+1$ ſupponitur par, evidens eſt numerum $s-r-1$ eſſe parem; Ergo $\frac{s-r-1}{2}$ erit numerus integer. Itaque ſi ſit $s$ aut $<$, aut $=r+1$; elevato binomio ad poteſtatem integram $\frac{s-r-1}{2}$ factaque multiplicatione, plures exorientur formulæ, quæ aut erunt integrabiles, aut pertinebunt ad N. 10, & 12, & indigebunt ſectionum conicarum rectificatione. Si $s=r-1$ noſtra formula in hanc mutatur

$$\frac{z^{r-1}dz}{(a+bzz+cz^4)^{\frac{r}{2}}} = \frac{du}{2cu^{r-1}\sqrt{\frac{\overline{b-uu}^2}{4cc}-\frac{a}{c}}}$$ quæ ex N. IX per rectificationem ſectionum integratur.

17. Ad reliquos caſus abſolvendos pone $z=\frac{1}{u}$, & habebis

$$\frac{z^s dz}{(a+bz^2+cz^4)^{\frac{r}{2}}} = \frac{-u^{-s+2r-2}du}{(au^4+bu^2+c)^{\frac{r}{2}}}$$. Hæc formula ex numero ſuperiore integratur ſi $-s+2r-2$ ſit aut $>$, aut $=r+1$; ſive ſi $s$ ſit aut $<$, aut $=r-3$. quæ hypotheſis includit etiam $s$ negativam: ſed hi caſus unice reliqui erant ad pleniſſime integrandam formulam. Si ponerem $-s+2r-2=r-1$, qui caſus item ex numero ſuperiori abſolvitur, iterum provenit $s=r-1$, qui pariter eodem loco abſolutus eſt.

18. Ex duobus numeris ſuperioribus conſtat, formulam

$$\frac{z^s dz}{(f+gzz^2)^{\frac{r}{2}}.(p+qzz^2)^{\frac{r}{2}}}$$ egere rectificatione conicarum ſectionum, ſi $r$ ſit numerus impar, & $s$ par vel poſitivus, vel negativus. Ergo

$$\frac{(f+gzz)^e.z^s dz}{(f+gz^2)^{\frac{r}{2}}.(p+qzz)^{\frac{r}{2}}}$$ ſimiliter integrabitur, ſi $e$ ſit numerus integer, & poſitivus. Nam elevato binomio ad poteſtatem $e$ mutatur formula in plures, quæ

ad

ad ſuperiorem ſpectant. Atqui formula æquivalet huic $\frac{\overline{f+gzz}^{\frac{2e-r}{2}}\, z^t dz}{\overline{p+qzz}^{\frac{r}{2}}}$, in qua $2e-r$ eſt ſemper numerus impar. Pone $2e-r=s$, ſi $2e>r$; ſi $2e<r$, fac $2e-r=-s$. Provenient formulæ duæ $\frac{z^t dz.\overline{f+gzz}^{\frac{s}{2}}}{\overline{p+qzz}^{\frac{r}{2}}}$, $\frac{z^t dz}{\overline{f+gzz}^{\frac{s}{2}}.\overline{p+qzz}^{\frac{r}{2}}}$, quæ rectificatis ſectionibus conicis integrantur.

19. Omnes formulæ, quæ hactenus rectificatis ſectionibus conicis integratæ ſunt, ad has duas reducuntur $z^t dz.\overline{a+bzz+cz^4}^{\frac{r}{2}}$, $z^t dz.\overline{f+gzz}^{\frac{s}{2}}.\overline{p+qzz}^{\frac{r}{2}}$, in quibus $t$ eſt numerus par vel poſitivus, vel negativus; $r$, $s$ ſunt impares pariter vel negativi, vel poſitivi. Formulæ iſtæ duæ, quæ complectuntur omnes illas, quæ tractatæ ſunt a N. 13 uſque ad N. 18, ſi vel alteruter, vel uterque ex numeris $r$, $s$ ſit par, aut $t$ ſit impar, ſectionum conicarum rectificatione non indigebunt, ſed integrabiles erunt, vel algebraice, vel per notas quadraturas circuli, & hyperbolæ.

20. Ex numero ſuperiore facile deducimus, formulas $x^{\frac{u}{2}} dx.\overline{a+bx+cx^2}^{\frac{r}{2}}$, $x^{\frac{u}{2}} dx.\overline{f+gx}^{\frac{s}{2}}.\overline{p+qx}^{\frac{r}{2}}$ integrari rectificatis ſectionibus conicis, ſi numeri $u$, $s$, $r$ ſint omnes impares vel poſitivi, vel negativi. Nam fiat $x=zz$, & orientur $2z^{u+1}dz.\overline{a+bzz+zz^4}^{\frac{r}{2}}$, $2z^{u+1}dz.\overline{f+gzz}^{\frac{s}{2}}.\overline{p+qzz}^{\frac{r}{2}}$, in quibus $u+1$ eſt par: Ergo iſtæ coincidunt cum formulis numeri ſuperioris.

21. Formulæ duæ $x^t dx.\overline{k+bx}^{\frac{u}{2}}.\overline{a+bx+cx^2}^{\frac{r}{2}}$. $x^t dx.\overline{k+bx}^{\frac{u}{2}}.\overline{f+gx}^{\frac{s}{2}}.\overline{p+qx}^{\frac{r}{2}}$, ſi $t$ ſit numerus integer, & poſitivus; $u$, $s$, $r$ numeri impares vel poſitivi, vel negativi, obtinentur ſuppoſitis rectificationibus ellypſis & hyperbolæ. Fiat enim $k+bx=by$ & orietur

$b^{\frac{u}{2}}$:

$$b^{\frac{u}{2}} . \overline{y - \frac{k}{b}}^{\,s} . y^{\frac{u}{2}} dy . \overline{\begin{matrix} a & + by & + cy^2 \\ -\frac{bk}{b} & -\frac{2ck}{b} y & \\ +\frac{ck^2}{b^2} & & \end{matrix}}^{\frac{r}{2}}$$

$$b^{\frac{u}{2}} . \overline{y - \frac{k}{b}}^{\,s} . y^{\frac{u}{2}} dy . \overline{\begin{matrix} f & +gy^2 \\ -\frac{qk}{b} & \end{matrix}}^{\frac{t}{2}} . \overline{\begin{matrix} p & +gy \\ -\frac{qk}{b} & \end{matrix}}^{\,y}$$ , quæ elevato binomio ad poteſtatem integram, & poſitivam $t$, factaque multiplicatione plures orientur termini, qui omnes integrabuntur per numerum ſuperiorem. Adverte, tria binomia $k + bx$, $f + gx$, $p + qx$ inæqualia eſſe oportere; ſecus formula non egeret rectificatione ſectionum conicarum. Si $u = s = r$, formula hanc formam indueret $x^s dx . \overline{a + bx + cx^2 + ex^3}^{\frac{r}{2}}$, quæ hanc continet $x^s dx \sqrt{a + bx + cx^2 + ex^3}$; nam quadrinomium $a + bx + cx^2 + ex^3$ ſemper habet factorem realem hujus formæ $k + bx$.

22. Quamquam, exiſtente $s$ numero poſitivo, de formula $\frac{dx}{x^s \sqrt{a + bx + cx^2 + ex^3}}$ nihil poſſumus pronunciare; tamen, ſi ambæ $b$, $c = o$, facile ſtatuemus, quandonam formula contenta ſit notis quadraturis circuli, & hyperbolæ, quandonam poſcat rectificationes hyperbolæ, & ellypſis. Hanc ob rem præmitto primo, formulam $\frac{dx}{x \sqrt{a + ee^3}}$ integrari per quadraturam circuli, aut hyperbolæ, quod tibi conſtabit, ſi utaris ſubſtitutione $\sqrt{e + ex} = y$. Præmitto deinde, formulas $\frac{x dx}{\sqrt{a + ex^3}}$, $\frac{dx}{\sqrt{a + ex^3}}$ poſtulare rectificationem ellypſis, & hyperbolæ, ut probatum eſt numero ſuperiore. His præmiſſis ſumo differentiale formulæ $\frac{\sqrt{a + ex^3}}{x^q}$, quod eſt hujuſmodi

$$D \frac{\sqrt{a + ex^3}}{x^2} = \frac{-q dx \sqrt{a + ex^3}}{x^{q+1}} + \frac{3e dx}{2x^{q-2} \sqrt{a + ex^3}}$$ ſive facta opportu-

ex reductione D $\frac{\sqrt{a+ex^3}}{x^q} = \frac{-qadx}{x^{q+1}\sqrt{a+ex^3}} - \frac{\overline{-2q+3}.edx}{2x^{q-2}\sqrt{a+ex^3}}$.

23. Formulam inventam ita diſpono

$$(A)\ \frac{dx}{x^{q+1}\sqrt{a+ex^3}} = -D\frac{\sqrt{a+ex^3}}{qax^q}\ (B)\ \frac{\overline{-2q+3}.edx}{2qax^{q-2}\sqrt{a+ex^3}},$$ ex qua

conſtat formulam A dependere a B. Si $q=1$, formula B poſcit rectificationes; ſi $q=4$, B integratur ex caſu primo; ſi $q=7$, ex ſecundo; atque ita deinceps: Ergo ſi $q$ ſit numerus ex hac ſerie 1, 4, 7, 10, 13 &c. formula A poſcit rectificationes conicarum ſectionum. Si $q$ ſit $=2$, formula B pariter indiget rectificatione ſectionum conicarum: Ergo etiam formula A; ſi $q=5$, formula B integratur ex primo caſu; ſi $q=8$, ex ſecundo; atque ita deinceps. Ergo formula A integratur per rectificationem ſectionum conicarum, ſi $q$ ſit numerus ex ſerie 2, 5, 8, 11, 14 &c. Demum ſi $q=3$, formula B ſolas quadraturas requirit; Ergo etiam formula A; ſi $q=6$, formula B integratur ex primo caſu; ſi $q=9$, ex ſecundo; atque ita deinceps; Ergo per quadraturas circuli, & hyperbolæ integratur formula A, ſi $q$ in hac ſerie contineatur 3, 6, 9, 12, 15 &c. Quare diviſis omnibus numeris in tres ſeries

1 4 7 10 13 &c. cui eſt terminus generalis $=3n-2$
2 5 8 11 14 &c. cujus terminus generalis $=3n-1$
3 6 9 12 15 &c. cui convenit terminus generalis $=3n$

formula $\frac{dx}{x^r\sqrt{a+ex^3}}$ per ſolas quadraturas circuli & hyperbolæ integratur, ſi $r$ contineantur in prima ſerie, ſive poſito $n$ numero integro ſit $=3n-2$; per rectificationem hyperbolæ & ellypſis, ſi $r$ contineatur aut in ſecunda, aut in tertia ſerie, ſive ſit aut $r=3n-1$, aut $r=3n$.

24. Ex ſuperiore determinatur formula $\frac{dx\sqrt{a+ex^3}}{x^r}$; nam facta multiplicatione per $\sqrt{a+ex^3}$, oritur $\frac{dx\sqrt{a+ex^3}}{x^r} = \frac{dx.\overline{a+ex^3}}{x^r\sqrt{a+ex^3}} = \frac{adx}{x^r\sqrt{a+ex^3}} + \frac{edx}{x^{r-3}\sqrt{a+ex^3}}$, quarum utraque eſt in poteſtate. Si exiſtente $n$ quolibet numero integro, & poſitivo, fuerit aut $r=3n$, aut $r=3n-1$, utraque formula requirit rectificationem ellypſis, & hyperbolæ; ſi vero $=3n-2$, contenta eſt notis quadraturis circuli, & hyperbolæ. Alio modo. Si æquationem inventam N. 22 opportune diſponas, hanc obtinebis

$$\frac{dx\sqrt{a+ex^3}}{x^{q+1}} = -D\frac{\sqrt{a+ex^3}}{qx^q} + \frac{3edx}{2qx^{q-2}\sqrt{a+ex^3}}$$ ſive facta $q+1=r$

dx

$$\frac{dx\sqrt{a+ex^3}}{x^s} = -D\frac{\sqrt{a+ex^3}}{\overline{s-1}\,.\,x^{s-1}} + \frac{3e\,dx}{2.\overline{s-1}\,.\,x^{s-3}\sqrt{a+ex^3}},$$ quarum ultima ex ſuperiori numero eſt in poteſtate. Excipias tamen velim caſum, in quo $s=1$: nam formulæ duæ dividuntur per $o$. Verum formula $\frac{dx\sqrt{a+ex^3}}{x}$ integratur per quadraturam circuli, & hyperbolæ, ut cognoſces, ſi utaris ſuſtitutione $\sqrt{a+ex^3}=y$.

25. Hinc colligimus integrationem formulæ $\frac{dx.\,\overline{a+ex^3}^{\frac{r}{2}}}{x^s}$, exiſtente $r$ numero impari, & poſitivo, non ſuperare rectificationem ſectionum conicarum. Namque ea in hanc formam mutari poteſt $\frac{\overline{a+ex^3}^{\frac{r+1}{2}}dx}{x^s\sqrt{a+ex^3}}$, in qua $r+1$ debet eſſe numerus par, & $\frac{r+1}{2}$ numerus integer; Ergo elevato binomio ad poteſtatem poſitivam, & integram, formula in plures dividetur, quæ ex N. 21, & 23 ſunt in poteſtate.

26. Nihil reliquum eſt, niſi ut integremus formulam $\frac{dx}{x^s.\,\overline{a+ex^3}^{\frac{r}{2}}}$.

Hanc ob rem ſumo differentiale quantitatis $\frac{1}{x^{s+2}.\,\overline{a+ex^3}^{\frac{r-2}{2}}}$, & invenio

$$D\frac{1}{x^{s+2}.\,\overline{a+ex^3}^{\frac{r-2}{2}}} = \frac{-\overline{s-2}.dx}{x^{s+3}.\overline{a+ex^3}^{\frac{r-2}{2}}} - \frac{3.\overline{r-2}.dx}{2x^s.\overline{a+ex^3}^{\frac{r}{2}}},$$ ſive

$$(A)\frac{dx}{x^s.\overline{a+ex^3}^{\frac{r}{2}}} = \frac{-2}{3.\overline{r-2}}.D\frac{1}{x^{s+1}.\overline{a+ex^3}^{\frac{r-2}{2}}} - (B)\frac{2.\overline{s+2}.dx}{3.\overline{r-2}.x^{s+3}.\overline{a+ex^3}^{\frac{r-2}{2}}},$$

ex qua æquatione conſtat, formulam A dependere a B. Si $r=3$, formula B eſt in poteſtate; ergo etiam formula A; ſi $r=5$, formula B ex caſu primo nota eſt, adeoque fiet nota formula A; ſi $r=5$, B ex caſu ſecundo innoteſcit, igitur & formula A. Quare progreſſu in infinitum producto conſtat, formulam A eſſe in poteſtate, neque ad ſui integrationem requirere plus quam ſectionum conicarum rectificationes.

27. Quapropter generatim formula $x^s dx.\,\overline{a+ex^3}^{\frac{r}{2}}$, ſi $s$ ſit nume-

 rus

rus integer, $r$ præterea impar, & uterque vel poſitivus, vel negativus, vel eſt algebraice integrabilis, vel pendet a quadratura circuli, & hyperbolæ, vel certe per rectificationem hyperbolæ & ellypſis integratur. Quæ dicta ſunt, ſatis oſtendunt, formulam non indigere rectificatione ſectionum conicarum, ſi $s = 3n + 1$, etiamſi $n$ poſitiva ſit: immo in hac hypotheſi formula eſt algebraice integrabilis, quod palam faciet ſubſtitutio $\overline{a + ex^3}^{\frac{r}{2}} = y$.

28. Si $s$ ſit numerus integer & poſitivus, $u$, $s$, $r$ numeri integri, & impares vel poſitivi vel negativi, formulæ $\dfrac{dx.\overline{k+bx}^{\frac{u}{2}}.\overline{a+bx+cx^2}^{\frac{r}{2}}}{x^{s+2+\frac{u+2r}{2}}}$

$\dfrac{dx.\overline{k+bx}^{\frac{u}{2}}.\overline{f+gx}^{\frac{s}{2}}.\overline{p+qx}^{\frac{r}{2}}}{x^{s+2+\frac{u+s+r}{2}}}$ facilime reducuntur ad N. 21. Fiat enim $x = \frac{1}{y}$, & orientur $-y^s dy.\overline{ky+b}^{\frac{u}{2}}.\overline{ay^2+by+c}^{\frac{r}{2}}$

$-y^s dy.\overline{ky+b}^{\frac{u}{2}}.\overline{fy+g}^{\frac{s}{2}}.\overline{py+q}^{\frac{r}{2}}$, quæ ex N. 21 integrationem accipiunt. In formulis ſuperioribus, quum $s$ debeat eſſe poſitivus numerus, & integer, evidens eſt $2s+4$, debere eſſe numerum parem, neque poſſe eſſe $< 4$; cæterum $s$ foret negativus. Quare ſi deinceps per ſpeciem $s$ intelligamus numerum poſitivum parem, neque minorem quam 4, formulæ ſequentes erunt in poteſtate, exiſtentibus $u$, $r$, $s$ imparibus vel poſitivis, vel negativis.

$$\frac{dx.\overline{k+bx}^{\frac{u}{2}}.\overline{a+bx+cx^2}^{\frac{r}{2}}}{x^{\frac{s+u+2r}{2}}}, \quad \frac{dx.\overline{k+bx}^{\frac{u}{2}}.\overline{f+gx}^{\frac{s}{2}}.\overline{p+qx}^{\frac{r}{2}}}{x^{\frac{s+u+s+r}{2}}}.$$

29. Iiſdem ſuppoſitis

$$\frac{dx.\overline{k+bx}^{\frac{u}{2}}.\overline{a+bx+cx^2}^{\frac{r}{2}}}{\overline{m+nx}^{\frac{s+u+2r}{2}}}, \quad \frac{dx.\overline{k+bx}^{\frac{u}{2}}.\overline{f+gx}^{\frac{s}{2}}.\overline{p+qx}^{\frac{r}{2}}}{\overline{m+nx}^{\frac{s+u+s+r}{2}}}$$

nullo negotio reducuntur ad ſuperiores facta ſubſtitutione $m + nx = ny$.

30. Si $s$ ſit numerus integer vel poſitivus, vel negativus, $r$ ut ſupra, formula $\dfrac{dx.\overline{a+ex^3}^{\frac{r}{2}}}{x^{s+2+\frac{3r}{2}}}$ per ſubſtitutionem $x = \frac{1}{y}$ in hanc mutatur $-y^s dy.\overline{ay^3+e}^{\frac{r}{2}}$,

quæ

quæ ex numero 27 ſemper integratur vel abſolute, vel per notas quadraturas, vel ſectionum conicarum rectificationes. Quoniam $2\varepsilon+4$ in formula ſuperiore debet eſſe numerus par vel poſitivus, vel negativus, ſi deinceps per $\varepsilon$ intelligamus hunc numerum, formula $\dfrac{dx\,.\,\overline{a+ex^3}^{\frac{r}{2}}}{x^{\frac{\varepsilon+3r}{2}}}$ ſemper erit in poteſtate.

31. Quare iiſdem poſitis, formula $\dfrac{dx\,.\,\overline{a+ex^3}^{\frac{r}{2}}}{\overline{m+nx}^{\frac{\varepsilon+r}{2}}}$ eſt in poteſtate; quia nullo negotio reducitur ad ſuperiorem ope ſubſtitutionis $m+nx=ny$. Aliqua adnotanda ſunt maximi momenti in formulis, quæ continentur N. 28, 29. Generatim formulæ $dx\,.\,\overline{m+nx^2}^{\frac{\mu}{2}}\,.\,\overline{k+bx^2}^{\frac{\nu}{2}}\,.\,\overline{a+bx+cx^2}^{\frac{r}{2}}$ $dx\,.\,\overline{m+nx^2}^{\frac{\mu}{2}}\,.\,\overline{k+bx^2}^{\frac{\nu}{2}}\,.\,\overline{f+gx^2}^{\frac{s}{2}}\,.\,\overline{p+qx^2}^{\frac{r}{2}}$, in quibus poteſt etiam $m=0$, & $n=1$, integrari non poſſunt etiam advocatis rectificationibus ſectionum conicarum, ſi $\mu$, $\nu$, $s$, $r$ ſint numeri impares, & poſitivi. Namque ſecunda formula collata cum formulis N. 28, aut 29 daret $\varepsilon+\nu+r+s=-\mu$. Ergo $\varepsilon=-\mu-\nu-r-s$, atque adeo $\varepsilon$ eſſet negativa: idem dic, ſi faciamus comparationem in prima formula, atqui exiſtente $\varepsilon$ negativa formulæ non ſunt in poteſtate: ergo neque propoſitæ. Verum formula $dx\,.\,\overline{m+nx^2}^{\frac{\mu}{2}}\,.\,\overline{a+ex^3}^{\frac{r}{2}}$ integratur ex N. præſenti, nam ea formula, licet $\varepsilon$ ſit negativus numerus, dummodo integer, abſolvitur. Ut formula $\dfrac{dx\,.\,\overline{k+bx^2}^{\frac{\nu}{2}}\,.\,\overline{f+gx^2}^{\frac{s}{2}}\,.\,\overline{p+qx^2}^{\frac{r}{2}}}{\overline{m+nx^2}^{\frac{\mu}{2}}}$ ſit in poteſtate, quum exponentes omnes ſunt poſitivi, neceſſe eſt, ut $\mu$ ſuperet ſummam aliorum exponentium $\nu$, $s$, $r$ ſaltem per 4 unitates; nam erit $\mu=\varepsilon+\nu+s+r$: ergo $\mu-\nu-s-r=\varepsilon$, ſed $\varepsilon$ non debet $<4$; ergo &c. Idem dicas de alia formula $\dfrac{dx\,.\,\overline{k+bx^2}^{\frac{\nu}{2}}\,.\,\overline{a+bx+cx^2}^{\frac{r}{2}}}{\overline{m+nx^2}^{\frac{\mu}{2}}}$. Verum hæc conditio non

requi-

requiritur in formula $\frac{dx.\overline{a+cx^3}^{\frac{r}{2}}}{\overline{m+nx^2}^{\frac{\mu}{2}}}$, nam hæc ſemper ad integrationem perducitur ſaltem per rectificationem conicarum ſectionum. De formula

$\frac{dx.\overline{f+gx^2}^{\frac{s}{2}}.\overline{p+qx^2}^{\frac{r}{2}}}{\overline{k+bx^2}^{\frac{\nu}{2}}.\overline{m+nx^2}^{\frac{\mu}{2}}}$ pronuncia, eam eſſe in poteſtate ſi $\nu+\mu-s-r$ non ſit $<4$. Idem dicas de formula $\frac{dx.\overline{a+bx+cx^2}^{\frac{r}{2}}}{\overline{k+bx^2}^{\frac{\nu}{2}}.\overline{m+nx^2}^{\frac{\mu}{2}}}$ facta $s=r$. Immo etiam de formula $\frac{dx.\overline{k+bx^2}^{\frac{\nu}{2}}.\overline{m+nx^2}^{\frac{\mu}{2}}}{\overline{a+bx+cx^2}^{\frac{r}{2}}}$, ſi $2r$ ſuperet $\nu+\mu$ ſaltem per quatuor unitates. Itaque $\frac{dx.\overline{a+bx+cx^2}^{\frac{r}{2}}}{\overline{f+gx+bx^2}^{\frac{s}{2}}}$, ſi $2s$ ſuperet $2r$ ſaltem per quatuor unitates, ſive $s$ ſuperet $r$ per unitates duas, abſolvitur, quotieſcumque alterutrum ex trinomiis ſit reſolubile in factores reales. Verum ſi neutrum hanc reſolutionem admittat, nondum liquet, qua ratione formula integrari poſſit.

Item formula $\frac{dx.\overline{k+bx^2}^{\frac{\nu}{2}}}{\overline{m+nx^2}^{\frac{\mu}{2}}.\overline{f+gx^2}^{\frac{s}{2}}.\overline{p+qx^2}^{\frac{r}{2}}}$ erit in poteſtate ſi $\mu+s+r$ ſuperet $\nu$ ſaltem per quatuor unitates; quod dicendum eſt etiamſi $r=s$, & habeatur trinomium non reſolubile in factores reales. Formula autem

$\frac{dx.\overline{m+nx^2}^{\frac{\mu}{2}}}{\overline{a+cx^3}^{\frac{r}{2}}}$ ſemper abſolvitur. Demum formula

$\frac{dx}{\overline{m+nx^2}^{\frac{\mu}{2}}.\overline{k+bx^2}^{\frac{\nu}{2}}.\overline{f+gx^2}^{\frac{s}{2}}.\overline{p+qx^2}^{\frac{r}{2}}}$ ſemper abſolvitur. Verum formula

dx

$\frac{dx}{(a+bx+cx^2)^{\frac{r}{2}} \cdot (f+gx+hx^2)^{\frac{s}{2}}}$ conficietur, ſi alterutrum ex trinomiis ſit reſolubile in factores reales. Quod ſi neutrum reſolubile ſit, nondum conſtat, quo pacto formula integretur. Quæ dicta ſunt hactenus, hanc regulam ſuppeditant œcumenicam. Quotieſcumque in differentiali formula adſint tantum quatuor binomia primi ordinis elata ad poteſtatem imparem diviſam per 2, & omiſſo hoc diviſore a ſumma exponentium binomiorum, quæ ſunt in diviſore, ſuperet ſaltem quatuor unitatibus ſummam ſimilium exponentium in numeratore, methodus ſuppetit, qua ſaltem per rectificationem ſectionum conicarum ad integrationem perducatur. Si deſit hæc conditio, de integratione nihil licet pronunciare. Idem dicas, ſi adſit trinomium ſecundi ordinis, quod tamquam duo binomia ſpectandum erit, adeoque ejus exponens bis erit accipiendum. Verum ſi duo trinomia adſint, quorum neutrum in binomia realia reſolvi poſſit, nondum conſtat, quo pacto formula poſſit integrari. Quod ſi adſint duo binomia unum primi ordinis, alterum tertii, ſemper formula aliqua ratione ad integrationem perducitur.

32. Reducamus nunc ad arcus ellypticos, & hyperbolicos formulam $\frac{dx}{(a+bx+cx^2)^t \cdot \left(g+\frac{h}{a+bx+cx^2}\right)^{\frac{s}{2}}}$, in qua $t$ eſt numerus integer, $s$ præterea impar. Utere ſubſtitutione $\frac{a+bx+cx^2}{h}=\frac{z}{A}$; quantitas A determinabitur in progreſſu, prout libuerit. Formula ſtatim in hanc mutatur $\frac{A^t dx}{h^t z^{t-\frac{s}{2}} \cdot (gz+h)^{\frac{s}{2}}}$. Ex æquatione ſubſtitutionis habebimus $x^2+\frac{bx}{c}=\frac{hz}{cA}-\frac{a}{c}$; ergo $x=\frac{-b}{2c}+\sqrt{\frac{hz}{cA}+\frac{bb}{4cc}-\frac{a}{c}}$, & facta $A=\frac{h}{c}$ erit $x=\frac{-b}{2c}+\sqrt{z+\frac{bb}{4cc}-\frac{a}{c}}$; igitur $dx=\frac{dz}{2\sqrt{z+\frac{bb}{4cc}-\frac{a}{c}}}$; quo valore ſubſtituto in formula $\frac{dx}{c^t \cdot z^{\frac{2t-s}{2}} \cdot (gz+h)^{\frac{s}{2}}}$, quæ reſultat ex ea, quæ paullo ante inventa eſt, ſubſtituto valore A, habebimus $\frac{dz}{2c^t \cdot z^{\frac{2t-s}{2}} \cdot \sqrt{z+\frac{bb}{4cc}-\frac{a}{c}} \cdot (gz+h)^{\frac{s}{2}}}$, quæ ex N. 20, ſuppoſitis rectificationibus ſectionum conicarum, ſemper integratur.

33. Methodus ſuperioris numeri palam docet, etiam formulam $x^m dx$

$$\frac{x^m dx}{\overline{a+bx+cx^2}^{\frac{r}{2}} \cdot \overline{g+\frac{b}{a+bx+cx^2}}^{\frac{s}{2}}}$$, exiſtente $m$ numero poſitivo & integro, per arcus ſectionum conicarum integrari. Nam factis iiſdem ſubſtitutionibus orietur

$$\frac{dz \cdot \overline{\frac{-b}{2c}+\sqrt{z+\frac{bb}{4cc}-\frac{a}{c}}}^{m}}{2c^{r} \cdot z^{\frac{2r-s}{2}} \cdot \overline{z+\frac{bb}{4cc}-\frac{a}{c}}^{\frac{r}{2}} \cdot \overline{gz+b}^{\frac{s}{2}}}$$

. Si elevetur binomium radicale ad poteſtatem integram, & poſitivam $m$, formula in plures dividetur, quarum aliquæ non indigebunt rectificatione ellypſis, & hyperbolæ, aliæ per hanc rectificationem ex N. 20 integrabuntur. Si $b=o$, etiamſi $m$ ſit numerus negativus, dummodo integer, formula continetur N. 20.

34. Formulæ ſuperiores inſerviunt integrandis formulis

$$\frac{dx}{\overline{a+bx+cx^2}^{\frac{r}{2}} \cdot \overline{f+gx+bx^2}^{\frac{s}{2}}}, \quad \frac{x^m dx}{\overline{a+bx+cx^2}^{\frac{r}{2}} \cdot \overline{f+gx+bx^2}^{\frac{s}{2}}},$$

exiſtente $m$ numero integro, & poſitivo, & $r$, $s$ numeris imparibus vel poſitivis, vel negativis, quando $bb=cg$. Nam ſecunda formula, quæ in primam deſinit, quum $m=o$, ita diſponi poteſt

$$\frac{x^m dx}{\overline{a+bx+cx^2}^{\frac{r+s}{2}}} \cdot \left(\frac{f+gx+bx^2}{a+bx+cx^2}\right)^{\frac{s}{2}}$$

ſive

$$\frac{x^m dx}{\overline{a+bx+cx^2}^{\frac{r+s}{2}} \cdot \overline{\frac{b}{c}+\frac{g-\frac{bb}{c} \cdot x+f-\frac{ab}{c}}{a+bx+cx^2}}^{\frac{s}{2}}}$$

, quæ, ſi $gc=bb$, ex duobus numeris ſuperioribus recepit integrationem. Protuli hujuſmodi formulæ reductionem ad methodum, qua deinceps utemur, indicandam. Nam formula multo facilius reducitur. Ita enim diſponatur

$$\frac{x^m dx}{c^{\frac{r}{2}} \cdot \overline{\frac{a}{c}+\frac{b}{c}x+x^2}^{\frac{r}{2}} \cdot b^{\frac{s}{2}} \cdot \overline{\frac{f}{b}+\frac{g}{b}x+x^2}^{\frac{s}{2}}}$$

. Fiat $x+\frac{b}{2c}=x+\frac{g}{2b}=z$.

Igitur

Igitur formula in hanc mutabitur

$$\frac{dz . \overline{z - \frac{b}{2c}}^{m}}{c^{\frac{r}{2}} . \overline{\frac{a}{c} - \frac{bb}{4cc} + zz}^{\frac{r}{2}} . b^{\frac{s}{2}} . \overline{\frac{f}{b} - \frac{gg}{4bb} + zz}^{\frac{s}{2}}}\text{; ſive}$$

$$\frac{dz . \overline{z - \frac{b}{2c}}^{m}}{\overline{a - \frac{bb}{4c} + czz}^{\frac{r}{2}} . \overline{f - \frac{gg}{4b} + bzz}^{\frac{s}{2}}}$$

, quæ, elevato binomio ad poteſtatem $m$, plures formulas ſuppeditat pertinentes ad N. 15.

35. Progredior ad formulam magis compoſitam, & difficilem

$$\frac{dx}{\overline{a + bx + cx^2}^{r} . \overline{g + \frac{kx + b}{a + bx + cx^2}}^{\frac{s}{2}}}$$

, in qua $r$ eſt numerus integer, $s$ præterea impar. Utar ſubſtitutione $\frac{a + bx + cx^2}{kx + b} = \frac{z}{A}$, in qua $A$ eſt quantitas determinanda in operationis progreſſu. Formula autem in hanc mutatur

$$\frac{A^s dx}{x^{\frac{2r-s}{2}} . \overline{kx + b}^{r} . \overline{gz + A}^{\frac{s}{2}}}$$

. Ab hac formula ut $x$ arceamus, æquationem ſubſtitutionis ita diſtribue $x^2 + \overline{\frac{b}{c} - \frac{kz}{cA}} . x = \frac{-a}{c} + \frac{bz}{cA}$, & reſolutione effecta

$$x = \frac{-b}{2c} + \frac{kz}{2cA} + \sqrt{\frac{bb}{4cc} - \frac{2bkz}{4ccA} + \frac{k^2 zz}{4ccA^2} - \frac{a}{c} + \frac{bz}{A}}.$$

Ut formula fiat ſimplicior, pone $A = \frac{k}{2c}$, & orietur

$$x = \frac{-b}{2c} + z + \sqrt{\frac{bb}{4cc} - \frac{bz}{c} + zz - \frac{a}{c} + \frac{2b}{k} z}.$$

Ad faciliorem efficiendam analyſim ſtatuo $\frac{bb}{4cc} - \frac{a}{c} = m$, $\frac{2b}{k} - \frac{b}{c} = 2n$. Igitur $x = \frac{-b}{2c} + z + \sqrt{m + 2nz + zz}$; ergo ſumptis differentiis $dx = dz + \frac{ndz + zdz}{\sqrt{m + 2nz + zz}}$,

$= \dfrac{\overline{n+z+\sqrt{m+2nz+zz}}\,.\,dz}{\sqrt{m+2nz+zz}}$. Item $kx+b = \dfrac{-kb}{2c}+b+kz+k\,.\,\sqrt{m+2nz+zz}$; atqui $\dfrac{-kb}{2c}+b=kn$; igitur

$$\frac{dx}{\overline{kx+b}^{t}} = \frac{dz}{k^{t}\,.\,\sqrt{m+2nz+zz}\,.\,\overline{n+z+\sqrt{m+2nz+zz}}^{\,t-1}}.$$ Quapropter propoſita formula in hanc vertitur

$$\frac{dz}{2^{t}c^{t}\,.\,z^{\frac{2t-s}{2}}\,.\,\overline{gz+\frac{k}{2c}}^{\frac{s}{2}}\,.\,\sqrt{m+2nz+zz}\,.\,\overline{n+z+\sqrt{m+2nz+zz}}^{\,t-1}}.$$

36. Si $t=1$, formula ſuperior evadit multo ſimplicior; nempe

$\dfrac{dz}{2c\,.\,z^{\frac{2-s}{2}}\,.\,\overline{gz+\frac{k}{2c}}^{\frac{s}{2}}\,.\,\sqrt{m+2nz+zz}}$. Hæc, quicumque ſit numerus $s$ vel poſitivus, vel negativus, continetur ſemper in N. 28; atque adeo integratur per rectificationes conicarum ſectionum.

37. Si ponas $t=o$, formula in hanc mutabitur

$\dfrac{z^{\frac{s}{2}}\,dz\,.\,\overline{n+z+\sqrt{m+2nz+zz}}}{\overline{gz+\frac{k}{2c}}^{\frac{s}{2}}\,.\,\sqrt{m+2nz+zz}}$, quæ facta multiplicatione in tres haſce transformatur

$$\frac{nz^{\frac{s}{2}}\,dz}{\overline{gz+\frac{k}{2c}}^{\frac{s}{2}}\,.\,\sqrt{m+2nz+zz}} + \frac{z^{\frac{2+s}{2}}\,dz}{\overline{gz+\frac{k}{2c}}^{\frac{s}{2}}\,.\,\sqrt{m+2nz+zz}} +$$

$\dfrac{z^{\frac{s}{2}}\,dz}{\overline{gz+\frac{k}{2c}}^{\frac{s}{2}}}$. Harum tertia non indiget rectificationibus ſectionum conicarum, ſed duæ primæ ne his quidem conceſſis integrari poſſunt. Nam, neglecto diviſore 2, ſumma exponentium in denominatore non eſt major ſaltem quatuor unitatibus ſumma exponentium in numeratore. Idem prorſus reperies, ſi $t$ ſit numerus integer quidem, ſed negativus: nam tranſlatis factoribus, prout opus eſt, in numeratorem, ut exponentes poſitivi fiant, elevatoque multinomio ad poteſtatem poſitivam $-t+1$, diviſaque formula in plures, aliquæ exorientur for-

formulæ, in quibus ſumma exponentium denominatoris, omiſſo diviſore 2, non ſuperat per quatuor unitates ſummam exponentium numeratoris: quæ formulæ quomodo integrentur, adhuc ignotum eſt.

38. Si vero $s$ fuerit numerus integer poſitivus, & unitate major, præparare oportet formulam, eam multiplicando, ac dividendo per $\overline{n+z-\sqrt{m+2nz+zz}}^{\,s-1}$, ut hanc formam accipiat

$$\frac{dz\cdot\overline{n+z-\sqrt{m+2nz+zz}}^{\,s-1}}{2^s\cdot c^s\cdot\overline{nn-m}^{\,s-1}\cdot z^{\frac{2s-s}{2}}\cdot\overline{gz+\frac{k}{2c}}^{\frac{s}{2}}\cdot\sqrt{m+2nz+zz}}.$$ Sit primo $s=2$, & formula in has tres tribuetur

$$\frac{n\,dz}{4cc\cdot\overline{nn-m}\cdot z^{\frac{4-s}{2}}\cdot\overline{gz+\frac{k}{2c}}^{\frac{s}{2}}\cdot\sqrt{m+2nz+zz}}$$

$$\frac{+dz}{4cc\cdot\overline{nn-m}\cdot z^{\frac{2-s}{2}}\cdot\overline{gz+\frac{k}{2c}}^{\frac{s}{2}}\cdot\sqrt{m+2nz+zz}}$$

$$\frac{-dz}{4cc\cdot\overline{nn-m}\cdot z^{\frac{4-s}{2}}\cdot\overline{gz+\frac{k}{2c}}^{\frac{s}{2}}}.$$ Tertia ex his formulis integratur ſine auxilio ſectionum conicarnm. Prima, & ſecunda ex N. 28. per arcus ellypticos, & hyperbolicos integrationem accipiunt. Idem ſemper invenies, ſi $s$ ſit numerus quilibet integer & poſitivus. Etenim diviſa formula in plures, aliæ per ſolas quadraturas circuli, & hyperbolæ conſtruentur, alias N. 28 complectetur.

39. Quæ hactenus explicata ſunt, oſtendunt, quando nam conſtrui poſſit per arcus ſectionum conicarum formula $$\frac{dx}{\overline{a+bx+cx^2}^{\frac{r}{2}}\cdot\overline{f+gx+bx^2}^{\frac{s}{2}}}.$$ Etenim hæc ita poteſt exponi $$\frac{dx}{\overline{a+bx+cxx}^{\frac{r+s}{2}}\cdot\overline{\dfrac{b}{c}+\dfrac{\overline{g-\frac{bb}{c}}\cdot x+\overline{f-\frac{ab}{c}}}{a+bx+cx^2}}^{\frac{s}{2}}}.$$

Quum ſupponamus $r$, $s$ eſſe numeros impares, manifeſtum eſt, fore $\frac{r+s}{2}$ numerum integrum, qui fieri poteſt $=t$. Supra demonſtravimus formulam ſem-

per eſſe in poteſtate, ſi $r = 1$. Hoc autem contingit primo ſi $s = r = 1$; ſecundo ſi $r$ ſit poſitiva, $s$ negativa, ita ut $r + s = 2$, ut ſi $r = 7$, $s = -5$; tertio ſi $r$ negativa ſit, $s$ poſitiva, ita ut $s + r = 2$, ut ſi $s = 5$, $r = -3$. In his omnibus caſibus formula conſtruitur per rect.ficationcm ſectionum conicarum.

40. Præterea oſtendimus Ignotum eſſe, quo pacto formula conſtruatur, ſi aut $s = o$, aut $s$ ſit negativa. Hoc autem accidit, ſi ex duabus ſpeciebus $r$, $s$ una negativa ſit, altera poſitiva, & vel ſint æquales, vel negativa ſuperet poſitivam, aut utraque ſit negativa.

41. Demum demonſtravimus, arcus ſect:onum conicarum ſemper præbere integrationem formulæ, quotieſcumque $s$ ſit poſitiva. Quod obtinebis, ſi aut utraque $s$, $r$ poſitiva ſit, aut exiſtente una poſitiva, & altera negativa, poſitiva excedat negativam.

42. Si in formula $x^s dx . \overline{a + bx + cxx}^{\frac{r}{3}}$, ſit numerus integer, & poſitivus, $r$ vero vel poſitivus, vel negativus, quem tamen non metiatur ternarius, arcus ſectionum conicarum ejus integrationi ſufficient. Uſurpanda eſt ſubſtitutio $a + bx + cxx = cz^3$; ergo $x^2 + \frac{b}{c} x = z^3 - \frac{a}{c}$, ſive

$$x = \frac{-b}{2c} + \sqrt{z^3 - \frac{a}{c} + \frac{bb}{4cc}}, \quad \& \quad dx = \frac{3z^2 dz}{2\sqrt{z^3 - \frac{a}{c} + \frac{bb}{4cc}}}.$$ Quapropter

factis ſubſtitutionibus formula in hanc mutatur

$$\frac{\overline{\frac{-b}{2c} + \sqrt{z^3 - \frac{a}{c} + \frac{bb}{4cc}}}^{s} . 3c^{\frac{r}{3}} z^{r+2} dz}{2\sqrt{z^3 - \frac{a}{c} + \frac{bb}{4cc}}}.$$ Quum $s$ ſit numerus integer, & poſitivus, elevato binomio ad poteſtatem $s$, factaque multiplicatione, plures formulæ exorientur, quarum aliquæ erunt integrabiles, aliæ ſemper conſtruentur per arcus ellypticus, & hyperbolicos ex N. 27.

43. Superior formula, poſita $b = o$, reduceretur ad arcus ſectionum conicarum, tametſi $s$ foret numerus integer, & negativus ex N. 27, quotieſcumque per notas quadraturas non haberetur.

44. Progredior ad formulam $x^s dx . \overline{a + bx + cx^2}^{\frac{r}{4}}$, in qua $s$ eſt numerus integer, & poſitivus, $r$ vero impar vel poſitivus vel negativus. Pono $a + bx + cx^2 = cz^2$; ergo $x^2 + \frac{b}{c} x = zz - \frac{a}{c}$, ſive

$$x = -\frac{b}{2c} + \sqrt{zz - \frac{a}{c} + \frac{bb}{4cc}}, \quad \& \quad dx = \frac{z\,dz}{\sqrt{zz - \frac{a}{c} + \frac{bb}{4cc}}}.$$ Quare

for-

formula in hanc mutabitur $\overline{\frac{-b}{2c}+\sqrt{zz-\frac{a}{c}+\frac{bb}{4cc}}}^{\,s} \cdot \frac{c^{\frac{r}{4}} z^{\frac{r+2}{2}} dz}{\sqrt{zz-\frac{a}{c}+\frac{bb}{4cc}}}$, quæ elevato binomio ad poteſtatem integram $s$, in plures formulas convertitur: atque hæ omnes, vel ſunt integrabiles, vel reducuntur ad N. 20.

45. poſito $b=0$, etiamſi $s$ ſit negativus, formula, niſi integretur aut abſolute, aut per notas quadraturas, in ſummam colligitur per arcus ellypticos, & hyperbolicos ex N. 20.

46. Reliquum eſt, ut verba faciam de formula $x^s dx . \overline{a+bx+cxx}^{\frac{r}{6}}$, in qua $s$ eſt numerus integer, & poſitivus, $r$ vel poſitivus vel negativus, qui tamen neque ſit par, neque diviſibilis per 3, ſecus enim formula in numeris ſuperioribus contineretur. Uſurpetur hæc ſubſtitutio $a+bx+cx^2=cz^3$, ex qua fit $x^2+\frac{b}{c}x=z^3-\frac{a}{c}$; ergo $x=\frac{-b}{2c}+\sqrt{z^3-\frac{a}{c}+\frac{bb}{4cc}}$, & ſumptis differentiis $dx=\frac{3z^2dz}{2\sqrt{z^3-\frac{a}{c}+\frac{bb}{4cc}}}$. Itaque formula in hanc mutabitur

$$\overline{\frac{-b}{2c}+\sqrt{z^3-\frac{a}{c}+\frac{bb}{4cc}}}^{\,s} \cdot \frac{3c^{\frac{r}{6}} z^{2+\frac{r}{2}} dz}{2\sqrt{z^3-\frac{a}{c}+\frac{bb}{4cc}}}.$$

Si binomium elevetur ad poteſtatem integram $s$, & fiat multiplicatio, plures formulæ orientur, quarum aliquæ abſolute integrabiles erunt, nempe ſecunda, quarta, ſexta, aliæque, quæ tenent ſedes pares; aliæ, nimirum prima, tertia, quinta, & reliquæ poſitæ in ſedibus imparibus conſtruentur ex N. 30.

47. Si in præmiſſa formula, eſſet $b=0$, facta ſubſtitutione orietur formula $\frac{3c^{\frac{r}{6}}}{2} z^{2+\frac{r}{2}} dz . \overline{z^3-\frac{a}{c}}^{\frac{s-1}{2}}$, quæ eſt in noſtra poteſtate, licet $s$ ſit numerus negativus. Namque ſi $s$ eſt impar aut integrabilis erit algebraice, aut per notas quadraturas circuli, & hyperbolæ; ſi vero $s$ ſit par, continetur in eodem N. 30.

## CAPUT DECIMUM QUARTUM

### De quadrandis ſuperficiebus, & de cubandis ſolidis, quæ ex rotatione gignuntur.

De quadraturis, & rectificationibus curvarum ſupra diximus ſatis, formulasque tradidimus, per quas peracta integratione obtinentur. Nunc theoria amplificanda eſt, atque extendenda ad ſuperficies, & ſolida, quæ ex curvarum rotatione producuntur.

1. Sit curva AB, (*Fig.* 1.) cujus coordinatæ orthogonales $AB = x$, $BD = y$. Rotetur planum ABD circa axem AB; quæritur ſolidum genitum a ſuperficie ABD. Augeatur AB elemento infiniteſimo $Bb = dx$, & ex puncto b agatur ordinata bd infinite proxima BD; demum agatur Dm parallela Bb. Manifeſtum eſt, ſolidum ortum ex rotatione infiniteſimæ ſuperficiei BDdb eſſe elementum ſolidi orti a rotatione ſpatij ADB: atqui ſolidum productum a ſuperficie BDdb non differt niſi per quantitatem reſpective infiniteſimam a ſolido, quod producit rectangulum DBbm, quod conſtat eſſe cylindrum, cujus baſis eſt circulus radij BD, altitudo autem eſt Bb; ergo hic cylinder eſt elementum ſolidi requiſiti orti a ſuperficie ABD. Inveniendus eſt itaque hic cylinder. Sit ratio radij ad circumferentiam ut $r:p$. Fiat ut $r:p::y:\frac{p}{r}y$, quæ deſignabit circumferentiam deſcriptam a puncto D. Dimidium producti circumferentiæ $\frac{p}{r}.y$ in radium $y$ exhibebit aream circularem $= \frac{p}{2r}.y^2$. Si hæc multiplicetur per $Bb = dx$, prodibit cylinder genitus a rectangulo BDmb, nempe $\frac{p}{2r}y^2dx$ elementum ſolidi geniti ab area ABD. Quapropter formula integrata exhibebit ſolidum ortum a rotatione areæ $ABD = \frac{p}{2r}Sy^2dx$. Ductis AC, DC parallelis AB, BD, retentiſque iiſdem denominationibus nempe $AB = DC = x$, $BD = AC = y$, eodem modo demonſtrabitur ſolidum ortum ex rotatione areæ ADC circa axem AC fore $= \frac{p}{2r}Sx^2dy$.

2. Afferamus exempla facilem theoriam reddentia faciliorem. Sit AD (*Fig.* 2.) linea recta ſecans axem AB in A, & cum eo faciens angulum $= 1$. Ut conus a triangulo rectangulo ADB deſcriptus determinetur, adverte eſſe $Cc.1 : Sc.1 :: x : y$; ergo $y = \frac{Sc.1}{Cc.1}.x$; ergo conus deſcriptus a rotatione circa

$$AB = \frac{p}{2r}.S\frac{\overline{Sc.1}^2}{\overline{Cc.1}^2}x^2dx = \frac{p.\overline{Sc.1}^2}{2r.\overline{Cc.1}^2}.\frac{x^3}{3} = \frac{p}{2r}.\frac{xy^2}{3},$$ quem conſtat eſſe

tertiam partem cylindri ejuſdem baſis & altitudinis. Eadem methodo non mutatis denominationibus invenies conum deſcriptum a triangulo ADC rotato circa

circa axem AC eſſe $= \frac{p}{2r} \cdot \frac{x^2 y}{3}$; ergo primus conus erit ad alterum $:: y : x ::$ $Sc.s : Cc.s$. Hinc facilis evadit ſolutio problematis. Invenire duos conos, in quibus altitudo unius ſit æqualis radio baſis alterius, ut ſint in ratione data. Nam angulus rectus CAB ita dividatur, ut ſinus anguli BAD ad ſinum anguli CAD ſit in ratione data: actis ex quolibet puncto D ordinatis DB, DC, coni duo, quorum unus gignitur a triangulo ADB circa AB, alter a triangulo ADC circa AC, præditi ſunt requiſitis conditionibus.

3. Expoſita coni recti ſoliditate liceat mihi formulas ad ſolida pertinentes ex hoc quoque ſonte deducere. Chorda minima Dd (*Fig.* 1.) producatur donec ſecet lineam abſciſſarum in P. Vocetur PB $= s$; ergo Pb $= s + dx$, & bd $= y + dy$. Conus in rotatione deſcriptus a triangulo Pdb conſtat $=$

$$\frac{p}{2r} \cdot \frac{\overline{s+dx} \cdot \overline{y+dy}^2}{3} = \frac{p}{2r} \cdot \frac{sy^2 + 2sydy + sdy^2 + dxdy^2 + y^2dx + 2ydxdy}{3}$$

Item conus deſcriptus a triangulo PBD $= \frac{p}{2r} \cdot \frac{sy^2}{3}$; ergo facta deductione fruſtum conicum deſcriptum a trapeſio BDdb $= \frac{p}{2r} \cdot \frac{2sydy + y^2dx}{3}$; reliques enim terminos omittimus, quia horum reſpectu ſunt infiniteſimi: atqui fruſtum conicum non diſcrepat ab elemento ſolidi geniti ab area ADB; ergo hoc elementum $= \frac{p}{2r} \cdot \frac{2sydy + y^2dx}{3}$. Quum autem propter ſimilitudinem triangulorum PBD, Dmd ſit $s : y :: dx : dy$, inveniemus elementum ſolidi requiſiti æquale tum $\frac{p}{2r} \cdot y^2dx$, tum $\frac{p}{2r} \cdot sydy$; ergo ſolidum quæſitum $= \frac{p}{2r} \cdot Sy^2dx = \frac{p}{2r} Ssydy$. Prima ex his formulis eadem eſt ac illa, quæ habetur num. 1. In ſecunda obſervandum ſpeciem $s$ indicare ſubtangentem curvæ ſumptam in axe rotationis, quod libro ſequenti clarius apparebit. Quare in illis curvis, in quibus datur ſubtangens, magnam poteſt utilitatem afferre. Simili methodo iiſdem conſervatis denominationibus invenies ſolidum productum a ſpatio ADC rotato circa axem AC $= \frac{p}{2r} Sx^2dy = \frac{p}{2r} Ssxdx$, qua in ultima formula $s$ exprimit QC ſubtangentem in axe rotationis acceptam.

4. Sit AD (*Fig.* 1) parabola, cujus æquatio $ax = y^2$. Itaque ſolidum ortum a ſpatio ADB circa AB invenietur eſſe $= \frac{p}{2r} Sy^2dx = \frac{p}{2r} Saxdx = \frac{p}{2r} \cdot \frac{ax^2}{2}$. Similiter ſolidum ortum a ſpatio ADC circa AC invenietur eſſe $= \frac{p}{2r} Sx^2dy = \frac{p}{2r} S\frac{y^4dy}{a^2} = \frac{p}{2r} \cdot \frac{y^5}{5a^2} = \frac{x^2y}{5}$. Igitur primum ſolidum ad

ad alterum erit ut $\frac{a}{2} : \frac{y}{5}$. Quare ſi $y = \frac{5a}{2}$, ſolida invenientur æqualia, ſi $y < \frac{5a}{2}$, ſolidum productum ab area ADC erit minus alio, erit majus, ſi $y > \frac{5a}{2}$.

5. Sit hyperbola æquilatera MGN (*Fig.* 3) inter aſymptota AM, AN, in qua, poſitis $AF = FG = a$, valet æquatio $aa = xy$; quæritur ſolidum genitum ab area GFDB rotata circa axem AB. Quum hoc ſolidum $= \frac{p}{2r} S\, y^2 dx$, invenietur $= \frac{p}{2r} S \frac{a^4 dx}{x^2} = \frac{p}{2r} . A - \frac{a^4}{x}$. Addita eſt conſtans, quæ omitti non poteſt. Ut hæc determinetur, ponamus ſolidum nullum eſſe, quum $x = a$; ergo $A = a^3$; ergo ſolidi valor erit $= \frac{p}{2r} . a^3 - \frac{a^4}{x} = \frac{p}{2r} . a^3 . \frac{x - a}{x}$. Ex hac formula colliges, ſi abſciſſæ creſcant in ratione arithmetica, ut fiant $a$, $2a$, $3a$, $4a$ &c. ſolida fient $\frac{p}{2r} a^3$ ducta in $0$, $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$ &c. Si vero ponatur $x$ infinita, ſolidum evadet $= \frac{p}{2r} a^3$, hoc eſt æquale cylindro genito in eadem rotatione a rectangulo AFGC. Si accipiatur $x < a$, ſolidum evadit negativum, & infinitum eſt, ſi fiat $x = 0$. Quare ſolidum infinite longum genitum a ſpatio GFN finitum eſt, infinitum autem, quod producitur a ſpatio AFGM, facta rotatione circa aſymptotum AN.

6. Sit quadrans ellypticus GDFA, (*Fig.* 4) cujus axis major $AF = a$, minor $AG = b$; quæritur proportio ſolidorum, quæ generantur ab eadem area AGF in duabus rotationibus circa AF, & circa AG. Vocatis $AB = CD = x$, $DB = AC = y$. Conſtat fore $y^2 = b^2 - \frac{b^2 x^3}{aa}$, & $xx = a^2 - \frac{a^2 y^2}{b^2}$. Quare ſolidum ortum ex rotatione circa AF, quum inventum fuerit $= \frac{p}{2r} S y^2 dx$, fiet $= \frac{p}{2r} . S\, b^2 dx - \frac{b^2 x^2 dx}{a^2} = \frac{p}{2r} . b^2 x - \frac{b^2 x^3}{3a^2}$, & facta $x = a$ evadet ſolidum genitum a quadrante $\frac{p}{2r} . \frac{2ab^2}{3}$. Similiter quum ſolidum, quod producit rotatio circa AG determinatum fit $= \frac{p}{2r} S x^2 dy$, evadet $= \frac{p}{2r} S\, a^2 dy = \frac{a^2 y^2 dy}{b^2} = \frac{p}{2r} . a^2 y - \frac{a^2 y^3}{3b^2}$, & facta $y = b$, ſolidum, quod gignit rotatio circa AG erit $= \frac{p}{2r} . \frac{2a^2 b}{3}$; ergo primum ſolidum ad alterum

eſt

eſt ut $b:a$, hoc eſt in ratione reciproca axium, circa quos fit rotatio. Si $a=b$, ut habeatur quadrans circuli, gignetur hemiſpherium $=\frac{p}{2r}\cdot\frac{2a^3}{3}$, hoc eſt duplum coni, in quo non minus altitudo, quam radius baſis $=x$, & ad cylindrum ejuſdem baſis & altitudinis erit ut $2:3$. Quod ſi fiat $a=r$, ſoliditas hemiſphærii $=\frac{pr^2}{3}$.

7. Quæ dicta ſunt hactenus inſerviunt reſolvendis aliis quæſtionibus, quæ in ſolidis ortis ex rotatione proponi poſſunt. Aliquas ad exemplum proponam. Sit curva A D, (*Fig.* 5) cujus coordinatæ orthogonales $AB=x$, $BD=y$, quæ rotetur non circa axem A B, ſed circa axem K H parallelum A B, qui diſtet per $AH=c$, ut $KD=y+c$: quæritur ſolidum deſcriptum ab area A B D. Manifeſtum eſt ex ſuperioribus, ſolidum genitum ab area A D K H fore $=\frac{p}{2r}S\,\overline{y+c}^2.dx$; item cylindrum genitum a rectangulo A H K B $=\frac{p}{2r}c^2x=\frac{p}{2r}Sc^2dx$; igitur detrahendo cylindrum a ſolido orto ex area A H K D, remanebit ſolidum productum a ſpatio A B D rotato circa H K $=\frac{p}{2r}Sy^2dx+2cydx$. Q. E. I.

8. Sit hyperbola æquilatera A D, (*Fig.* 6) cujus primus ſemiaxis $AH=c$: ducta tangente A B, quæritur ſolidum productum a trilineo A D B rotato circa ſecundum axem H K. Vocatis $AB=x$, $BD=y$, ſcimus curvæ æquationem eſſe $2cy+yy=xx$: ergo ſolidum quæſitum $=\frac{p}{2r}.\frac{x^3}{3}$; hoc eſt æquat conum, in quo & altitudo, & radius baſis $=x$. Sit quadrans circuli A G D F (*Fig.* 7) rotandus circa axem H K parallelum radio A B. Formulam generalem inventam num. ſuperiore divido in duas hoc modo $\frac{p}{2r}Sy^2dx+\frac{pc}{r}Sydx$. Prima exprimit ſolidum, quod producit area G A B D rotata circa A B, quod ſolidum, facta $AB=AF$, evadit emiſpherium radij A F. Ut alteram inveniamus, notandum $\frac{pc}{r}$ eſſe circumferentiam deſcriptam radio H A, præterea $Sydx$ æquat ſpatium G A B D, quod facta $AB=AF$ evadit quadrans circuli radij A F. Igitur ſolidum genitum a quadrante A G F rotato circa H K ſuperat hemiſphærium radij A F per ſolidum, quod fit multiplicando quadrantem radij A F per circumferentiam deſcriptam radio H A.

9. Spectavi curvam, quæ poſita eſt poſt axem rotationis, & lineam abſciſſarum; verum ſi ipſa media fuerit, ut in Fig. 8, ut inquiratur ſolidum productum a ſpatio A B D rotato circa axem H K, evidens eſt, a cylindro orto a rectangulo A H K B demendum eſſe ſolidum genitum a ſpatio A H K D. Itaque ſolidum genitum a ſpatio A D B rotato circa H K $=\frac{p}{2r}c^2x-\frac{p}{2r}S\,\overline{c-y}^2.dx=\frac{p}{2r}S\,\overline{2cy-yy}.dx$. Sit quadrans radij $AH=c$, cu-

cujus tangens ſit A B, ( *Fig.* 9 ) ut vocata A B $=x$, B D $=y$ æquatio ſit $2cy-yy=xx$; ergo ſolidum ortum a trilineo A D B rotato circa H K erit $\frac{p}{2r}\cdot S\,x^2dx=\frac{p}{2r}\cdot\frac{x^3}{3}$, & facta $x=c$, evadet $=\frac{p}{2r}\cdot\frac{c^3}{3}$, hoc eſt cono, cujus altitudo, & ſemidiameter baſis $=c$, qui conus gignitur a triangulo A H F rotato circa H F. Sit quadrans A F E I ( *Fig.* 7 ) rotandus circa axem H K, quæritur ſolidum productum a quadrante A F I. Agatur quælibet B E K, & ſit A B $=x$, B E $=y$. Formulam inventam in duas divido $\frac{pc}{r}Sy\,dx-\frac{p}{2r}Sy^2dx$. Prima dat circumferentiam radij H A ductam in aream A E I B, quæ facta A B $=$ A F deſinit in quadrantem; altera dat ſolidum genitum ab area A I E B rotata circa A B, quod ſolidum facta A B $=$ A F deſinit in hemiſphærium; ergo ſolidum ortum a quadrante A I F circa H K æquat ſolidum, quod prodit, ſi quadrans radij A F ducatur in circumferentiam radij A H, dempto hemiſphærio. Q. E. I. Quum numero ſuperiore inventum fuerit ſolidum genitum a quadrante A F G circa H K, ſi hoc addatur ſolido genito a quadrante A F I, invenies ſolidum productum a ſemicirculo G F I circa H K æquale duplici quadranti, ſeu eidem ſemicirculo ducto in circumferentiam radij A H; ſi vero ſecundum ſolidum a primo detrahas invenies differentiam æqualem duplici hemiſphærio, ſeu integræ ſphæræ.

10. Exemplum ultimum aperit methodum, qua invenienda ſunt ſolida genita in rotatione ab area, quæ inter duas curvas, aut inter duos ejuſdem curvæ ramos continentur. Sint duæ curvæ A D, A E ( *Fig.* 10 ) relatæ ad eaſdem abſciſſas A B, quæ rotentur circa axem K H, quæritur ſolidum genitum ab area A D E. Vocata A B $=x$, ſit B D $=y$, B E $=\overset{1}{y}$; ergo K D $=c+y$, K E $=c+\overset{1}{y}$. Itaque ex ſuperioribus ſolidum genitum ab area A D K H $=\frac{p}{2r}S\,\overline{c+y}^2\cdot dx$, & quod producitur ab area A E K H $=\frac{p}{2r}\cdot S\,\overline{c+\overset{1}{y}}^2\cdot dx$: ergo ſi hoc ab illo ſubtrahatur, habebitur ſolidum ortum a ſpatio A D E $=\frac{p}{2r}\cdot$ $S\,\overline{2c\cdot\overline{y-\overset{1}{y}}\cdot dx+\overline{y^2-\overset{1}{y}{}^2}\cdot dx}$. Quod erat inveniendum. Formula. $S\,\overline{y-\overset{1}{y}}\cdot dx$ nihil eſt aliud qnam area A D E, tum $Sy^2dx$ eſt productnm ab area A D B rotata circa A B, & $S\overset{1}{y}{}^2dx$ eſt ſolidnm ortum ab area A E B rotata circa camdem A B. Igitur ſolidum productum ab area A D E circa H K æquat $\frac{pc}{r}$. A E D addita differentia ſolidorum, quæ prodncunt areæ A D B, A E B circa A B. Exemplum nnicum affero. Super diametro N O deſcriptus ſit ſemicitculus N F O, ( *Fig.* 11 ) in quo ducta qnacumque chorda M I, parallela N O alius ſemicirculus deſcribatur M G I; quæritur ſolidum productum a lunula M G I F rotata circa N O. Ex centro H agatur normalis H G omnia bifariam dividens. Vocetur H O $=a$, A I $=b$, A H $=c=\sqrt{aa-bb}$. Ordina K D. Sit A B $=x$, B D $=y$, B E $=\dot{y}$, ut æquationes curvarum G I, F I ſint

$bb - xx = yy$, $aa - xx = \overline{c + \overset{1}{y}}^{2}$. Liquet $S\,\overline{y - \overset{1}{y}}\,.\,dx = GFED$; ergo
$\frac{p}{2r}\,S\,2c.\overline{y - \overset{1}{y}}.dx = \frac{pc}{r}.\,GFED$. Præterea
$y^2 - \overset{1}{y}{}^2 = \begin{array}{l} b^2 - x^2 + 2c\sqrt{aa - x^2} \\ -aa + xx \\ -cc \end{array} = 2c.\,\overline{\sqrt{aa - xx} - c}$; ergo
$\overline{y^2 - \overset{1}{y}{}^2}.dx = 2c.dx\sqrt{aa - xx} - c\,dx$, & integrando
$S\,\overline{y^2 - \overset{1}{y}{}^2}.\,dx = 2c.\,\overline{HFEK - cx}$. Sed $cx = AHKB$; ergo
$S\,\overline{y^2 - \overset{1}{y}{}^2}\,.dx = 2c.\,\overline{HFEK - HKB} = 2c.AFEB$. Quapropter ſolidum ortum ex $GFED$ circa $HK = \frac{pc}{r}.\,GFED + \frac{pc}{r}.\,AFEB = \frac{pc}{r}.\,GABD$; ergo ſolidum ortum ex ſemilunula $GIF = \frac{pc}{r}.GAI$. Quadrans $GAI = \frac{pb^2}{8r}$; ergo ſolidum genitum a ſemilunula $GFI = \frac{p^2}{8r^2}.b^2c$, adeoque quod gignitur a lunula integra $= \frac{p^2}{4r^2}.b^2c$.

11. Si curva ordinatarum $\overset{1}{y}$ ad oppoſitam partem jaceret, ut negativa ſpectanda eſſet. Quare mutato ejus ſigno, ut evadat poſitiva, formula proveniet $\frac{p}{2r}\,S\,2c.\overline{y + \overset{1}{y}}.dx + \overline{y^2 - \overset{1}{y}{}^2}.dx$. Quod ſi $\overset{1}{y} = y$, brevior fiet nempe $\frac{2pc}{r}\,S\,y\,dx$. Si $GFI$ (*Fig.* 7) ſit ſemicirculus rotandus circa $HK$, ducto radio $ABF$ parallelo $HK$ ſolidum productum a ſpatio $GDEI = \frac{pc}{r}.\,GDEI$, ut antea inventum eſt.

12. Spectavi hactenus curvam & ſpatium integrum, a quo gignitur ſolidum, poſitum eſſe ad unam, eamdemque partem axis rotationis. Videndum eſt breviter, quid accidat formulis, ſi curva ab axe rotationis ſecetur. Sit curva $GFD$, (*Fig.* 12) cujus abſciſſæ $AB = x$, ordinatæ $BD = y$, quæ rotetur circa lineam abſciſſarum $AB$ ſecantem curvam in $F$. Ducta ordinata $GA$ ſpatium $AFG$ gignet ſolidum rotundum, & aliud producet ſpatium $FBD$. Formula præbens valorem ſolidi geniti ſcilicet $\frac{p}{2r}\,S\,y^2dx$, præbet ne ſummam, an differentiam duorum ſolidorum? Agatur in ſpatio ſuperiori ordinata $2B\,2D$, quæ certe tamquam negativa ſpectanda eſt, exiſtente $A\,2B = x$, ejuſque elemento $= dx$ poſitivo, qua de re area $AG\,2D\,2B$ tamquam negativa ſpectanda erit. Verum quum quadratum quantitatis negativæ poſitivum ſit, $y^2$ poſitivum erit, licet $y$ ſit negativa; ergo etiam $S\,y^2dx$. Quamobrem quum duplex ſolidum ortum ab utraque area poſitivum ſit, patet a formula $\frac{p}{2r}\,S\,y^2dx$ ita in-

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tegrata, ut evaneſcat facta $x=o$, exhiberi ſummam utriuſque ſolidi. Hoc exemplo ſimpliciſſimo confirmemus. Sit GFD linea recta faciens cum linea abſciſſarum angulum ſemirectum. Sit AF = AG = $a$, erit æquatio $x-a=y$; ergo $dx=dy$; igitur $\frac{p}{2r} S y^2dx = \frac{p}{2r}. S y^2dy = \frac{p}{2r}.\frac{y^3+a^3}{3}$. Ea conſtans addita eſt, quæ efficit, ut poſita $y=-a$, quantitas evaneſcat. Atqui $\frac{p}{2r}.\frac{y^3}{3}$ eſt conus genitus a triangulo FDB, & $\frac{p}{2r}.\frac{a^3}{3}$ eſt conus productus a triangulo AFG; ergo formula præbet utriuſque ſolidi ſummam. Quod ſi velles duorum ſolidorum differentiam oportet utrumque ſeparatim invenire, & alterum ab altero deducere.

13. Si in hypotheſi num. 5, ubi quæritur ſolidum genitum ab area ADB (*Fig.* 5) rotante circa HK, intelligatur HK accedere motu parallelo ad AB, tum tranſire ad plagam oppoſitam in 2H2K; perſpicuum eſt AH $=c$ evadere negativam; quare mutato ſigno ſpeciei $c$, ut ex negativa tranſeat in poſitivam formula proveniet $\frac{p}{2r} S y^2dx - 2cy\,dx$, in qua A2H $=c$. Hæc formula ita integrata, ut evaneſcat facta $x=o$, quodnam ſolidum ſufficiet? Ajo, eam ſufficere differentiam duorum ſolidorum, quæ oriuntur ab areis DF2K, AF2KB. Hoc clare cognoſces, ſi ſequaris methodum uſurpatam num. 5. Primum inveniendum eſt ſolidum ortum ex area DA2H2KD circa 2H2K. Hoc ex num. ſuperiore eſt ſumma ſolidorum, quæ oriuntur a duabus areis DF2K, AF2H; deinde demendus eſt cylinder ortus ex rotatione rectanguli A2H2KB, qui, etiamſi $c$ evadat negativa, eſt poſitivus. Facta autem hac detractione manifeſtum eſt, remanere differentiam duorum ſolidorum, quæ oriuntur ab areis DF2K, AF2KB. Ut hoc patefaciamus in exemplo maxime ſimplici, ſit AFD linea recta faciens angulum ſemirectum cum AB, ut ſit $y=x$. Itaque erit

$$\frac{p}{2r} S \overline{y^2dx - 2cy\,dx} = \frac{p}{2r} S \overline{x^2dx - 2cx\,dx} = \frac{p}{2r}.\frac{x^3}{3} - cx^2.$$

Hæc ita ſcribatur $\frac{p}{2r}.\frac{\overline{x-c}^3}{3} - c^2x + \frac{c^3}{3}$; atqui $\frac{p}{2r}.\frac{\overline{x-c}^3}{3}$ eſt conus genitus a triangulo FD2K; $\frac{p}{2r}c^2x$ eſt cylinder genitus a rectangulo A2H2KB, demum $\frac{p}{2r}.\frac{c^3}{3}$ eſt conus ortus a triangulo AF2H; ergo ſolidum ortum ab area AF2KB $=$ $\frac{p}{2r}.\overline{c^2x - \frac{c^3}{3}}$. Igitur noſtra formula exhibet differentiam duorum ſolidorum, quæ generantur ab areis FD2K, AF2KB.

14. Simili modo dum num. 10. inventum eſt ſolidum genitum ab area ADE (*Fig.* 10) inter duas curvas comprehenſa, rotata circa HK, ſi operationes effectas ſequaris, determinabis quid indicet formula, dum axis rotationis tranſit in 2H2K, & ſpecies $c$ evadit negativa. Mutato ſigno ſpeciei $c$ formula

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la hæc prodit $\frac{p}{2r}S - \overline{2c.\overline{y-\overset{1}{y}}.dx + \overline{y^2 - \overset{1}{y}{}^2}.dx}$. Hæc formula oritur, ſi primum inveniatur ſolidum ortum a ſpatio AD2K2H, quod æquat ſummam ſolidorum, quæ oriuntur a trilineis FD2K, FA2H. Deinde inveniendum ſolidum oriens ex ſpatio AE2K2H, quod æquat ſummam ſolidorum, quæ oriuntur a trilineis GE2K. GA2H: tum hoc alterum ſolidum a primo detrahendum eſt. Facta porro detractione remanet ſolidum, quod ortum eſt ex FGED, dempto ſolido orto ex AFG; ergo noſtra formula exprimit differentiam duorum ſolidorum, quæ producuntur ab areis FGED, AFG. Oſtendamus hunc quoque caſum exemplo facillimo. Sint AFD, AGE duæ lineæ rectæ. Sit $AB = x : BD = y :: 1 : m$; ergo $y = mx$. Similiter $x : BE = \overset{1}{y} :: 1 : n$; ergo $\overset{1}{y} = nx$. Noſtra igitur formula fiet

$$\frac{p}{2r}S - \overline{2c.\overline{m-n}.x\,dx + \overline{m^2-n^2}.x^2dx} = \frac{p}{2r}.\overline{m^2-n^2}.\frac{x^3}{3} - (\overline{m-n}.cx^2.$$

Videamus utrum hæc formula præbeat differentiam duorum ſolidorum, quæ producuntur ab areis FGED, AFG, quemadmodum definitum eſt. Facile invenies $F2H = \frac{c}{m}$, $G2H = \frac{c}{n}$; ergo $F2K = x - \frac{c}{m}$, $G2H = x - \frac{c}{n}$; tum $2KD = mx - c$, $2KE = nx - c$; ergo conus genitus a triangulo F2KD circa $F2K = \frac{p}{2r}.\frac{\overline{mx-c}^2.\overline{x-\frac{c}{m}}}{3} = \frac{p}{2r}.\overline{\frac{m^2x^3}{3} - mcx^2 + c^2x - \frac{c^3}{3m}}$. Similiter conus ortus a triangulo G2KE erit $= \frac{p}{2r}.\frac{n^2x^3}{3} - ncx^2 + c^2x - \frac{c^3}{3n}$; ergo ſi conus alter a primo auferatur remanebit ſolidum productum ab area $FGED = \frac{p}{2r}.\overline{m^2-n^2}.\frac{x^3}{3} - (\overline{m-n}.cx^2 + \frac{\overline{m-n}.c^3}{3mn}$. Conus productus a triangulo GA2H eſt $= \frac{p}{2r}.\frac{c^3}{3n}$, productus vero a triangulo FA2H eſt $\frac{p}{2r}.\frac{c^3}{3m}$. Ablato ſecundo cono a primo remanet ſolidum productum a triangulo $AFG = \frac{p}{2r}.\frac{\overline{m-n}.c^3}{3mn}$. Hoc ſi dematur a ſuperiore prodit formula paullo ante per integrationem inventa, quæ proinde exhibet differentiam duorum ſolidorum, quæ gignuntur ab areis FGED, FGA. Hæc ſatis ſint de ſolidis, & de cautione, qua uſurpandæ ſunt formulæ, dum axis rotationis tranſit ab una ad alteram partem lineæ abſciſſarum.

15. Nunc venio ad ſuperficies, quæ in rotatione a curvis generantur. Curva AD, ( *Fig.* 1 ) cujus abſciſſæ $AB = x$, $BD = y$ rotetur circa AB; quæritur ſuperficies genita a curva AD in rotatione. Agatur ordinata bd infinite proxima BD, & minima chorda Dd. Superficies fruſti conici producta in rotatione a recta Dd minor eſt ſuperficie genita ab arcu Dd quantitate ad ipſas relate infiniteſima; ergo ſuperficies genita a recta Dd eſt elementum ſuperficiei geni-

genitæ a curva A D, quam proinde per opportunam integrationem obtinebis, ſi quæ generatur a recta D d, cognoſcatur. Hanc ob rem produc D d, donec concurrat cum A B in P. Voca curvam $AD = s$, elementum $Dd = ds$, $DP = t$; ergo $PDd = t + ds$. Circumſerentia circuli deſcripti radio B D eſt $= \frac{p}{r} . y$; quæ deſcribitur radio $bd = \frac{p}{r} . \overline{y + dy}$; ergo ſuperficies conica deſcripta a recta $PDd = \frac{p}{2r} . \overline{t + ds} . \overline{y + dy} = ty + tdy + yds + dsdy$; & ſuperficies deſcripta a P D erit $= \frac{p}{2r} . ty$. Dematur hæc a priore, & fiat ſuperficies fruſti conici deſcripti a $Dd = \frac{p}{2r} . \overline{tdy + yds + dsy}$; deleatur ultimus terminus infiniteſimus reſpectu reliquorum, & eadem ſuperficies, quæ eſt elementum ſuperficiei deſcriptæ a curva A D, fiet $= \frac{p}{2r} . \overline{tdy + yds}$; atqui $t : y :: ds : dy$; ergo $tdy = yds$: ergo elementum ſuperficiei genitæ a curva $AD = \frac{p}{r} . yds = \frac{p}{r} . tdy$. Hæc ſecunda formula utilis eſſe poteſt, quum data eſt tangens. Simili methodo vocatis ut antea $AB = DC = x$, $BD = AC = y$ invenies elementum ſuperficiei genitæ ab eadem curva A D circa $AC = \frac{p}{r} . xds = \frac{p}{r} . tdx$, dummodo per ſpeciem $t$ intelligas rectam D Q.

16. Sit quadrans circuli A D H, (*Fig.* 13) qui rotetur circa radium $KA = r$ quæritur ſuperficies genita a quadrante A D H. Vocato de more $AB = x$, $BD = y$, ex num. ſuperiore conſtat, ſuperficiem eſſe $= \frac{p}{r} Syds$; atqui ex natura circuli $r : y :: ds : dx$: ergo $yds = rdx$; ergo ſuperficies prodit $\frac{p}{r} Srdx = px$; non additur conſtans in integratione, quia hoc modo ſuperficies evaneſcit, evaneſcente $x$. Si fiat $x = r$ naſcetur hemiſphærium $= pr$, hoc eſt duplum integri circuli genitoris; ergo tota ſphæra eſt quadrupla circuli maximi. Si quæratur ſuperficies genita a quadrante A D H circa tangentem A C; conſtat ex num. ſuperiore, eam eſſe $= \frac{p}{r} Sxds$; ſed ex natura circuli $r : r - x :: ds : dy$; ergo $rdy = rds - xds$, & $xds = rds - rdy$: ergo ſuperficies quæſita $= \frac{p}{r} S\overline{rds - rdy} = ps - py$. Quare facto $s = \frac{p}{4}$, & $y = r$, fiet $\frac{pp}{4} - pr$. Si ponas $s = \frac{p}{2}$, $y = o$, ſuperficies genita a demicircumferentia A H a erit $= \frac{pp}{2}$; ergo quæ producitur a tota circumferentia $= pp$. Igitur hæc ſuperficies ſe habebit ad ſuperficiem ſphæricam ut $pp : 2pr :: p : 2r$, ſcilicet ut circumferentia ad diametrum.

17. Inquiritur ſuperficies genita a quadrante ellyptico G D F (*Fig.* 14) tum in rotatione circa majorem axem A F, tum circa minorem A G. Vocetur axis major $AF = a$, minor $AG = b$, ordinatis D B, D C vocentur de more $AB = x$, $AC = y$. Notum eſt, vocato arcu $GD = s$, fore $ds = \dfrac{dx \sqrt{aa - xx . \frac{aa - bb}{aa}}}{\sqrt{aa - xx}}$;

ergo

ergo $y\,ds = \frac{y\,dx . \sqrt{aa - xx . \frac{aa-bb}{aa}}}{\sqrt{aa-xx}}$; atqui $y = \frac{b}{a}\sqrt{aa-xx}$; ergo $y\,ds = \frac{b}{a}\,dx \sqrt{aa-xx . \frac{aa-bb}{aa}}$; igitur $\frac{p}{r}\,S\,y\,ds = \frac{pb}{ra} S\,dx \sqrt{aa-xx . \frac{aa-bb}{aa}}$. Producatur AF in P, donec $AP = \frac{aa}{\sqrt{aa-bb}}$, & producatur GA in H donec $AH = AG$; deinde cum duobus ſemiaxibus AP, AH deſcribatur ellypſis HMP. Facile eſt invenire ordinatam $BM = \frac{b}{a}\sqrt{aa-xx . \frac{aa-bb}{aa}}$; ergo $S\,\frac{b}{a}\,dx \sqrt{aa-xx . \frac{aa-bb}{aa}} = AHMB$; ergo $\frac{p}{r}\,S\,y\,ds$, ſeu ſuperficies genita ab arcu GD rotante circa AF eſt $= \frac{p}{r} . AHMB$. Quare acta ordinata FI ſuperficies genita a toto quadrante GDF erit $= \frac{p}{r} . AHIF$. Similimodo ſi vocetur arcus $FD = s$, ſcimus fore $ds = \frac{dy\sqrt{bb+yy . \frac{aa-bb}{bb}}}{\sqrt{bb-yy}}$; ergo $x\,ds = \frac{x\,dy\sqrt{bb+yy . \frac{aa-bb}{bb}}}{\sqrt{bb-yy}}$; atqui $x = \frac{a}{b}\sqrt{bb-yy}$; ergo $x\,ds = \frac{a}{b}\,dy\sqrt{bb+yy . \frac{aa-bb}{bb}}$, & integrando $S\,x\,ds = \frac{a}{b} . S\,dy\sqrt{bb+yy . \frac{aa-bb}{bb}}$. Jam vero ſume $AQ = \frac{b^4}{aa-bb}$, & $AK = a$, & ſemiaxibus AK, AQ deſcribe hyperbolam KL, quam ſecet DC producta in N. Facile invenies $CN = \frac{a}{b}\sqrt{bb+yy . \frac{aa-bb}{bb}}$; ergo $\frac{a}{b}\,S\,dy\sqrt{bb+yy . \frac{aa-bb}{bb}} = KACN$. Igitur $\frac{p}{r}\,S\,x\,ds$, hoc eſt ſuperficies genita ab arcu DF circa $AG = \frac{p}{r} . KACN$, & ſuperficies ab integro quadrante $FDG = \frac{p}{r} KAGL$. Quare ſuperficies producta a quadrante circa axem majorem ad illam, quæ gignitur in rotatione circa axem minorem, ſe habet, ut area ellyptica AHIF ad hyperbolicam AKLG.

18. Sit FD (*Fig.* 15) tractoria, cujus æquatio vocata $FD = s$, $DB = y$ eſt $ds = \frac{-a\,dy}{y}$; ergo $y\,ds = -a\,dy$. & integrando, ut evaneſcat poſita

$y = a$

$y = a$ erit $S\,y\,ds = aa - ay$. igitur $\frac{p}{r} S y ds$ ſcilicet ſuperficies orta ex rotatione arcus FD circa AB $= \frac{pa}{r} . \overline{a - y}$; ſed $\frac{pa}{r}$ eſt integra circumferentia circuli deſcripti radio $= a$, & ordinata DC eſt FC $= a - y$; ergo ſuperficies deſcripta ab arcu FD eſt rectangulum ex circumferentia circuli radio $= a$ & recta FC, ſeu æqualis ſuperficiei cylindri, cujus baſis eſt circulus radij $= a$, altitudo eſt FC, adeoque ſuperficies genita a curva infinita FDG eſt æqualis ſuperficiei cylindricæ ejuſdem baſis, & altitudinis FA $= a$. Sit nunc logarithmica FDG (*Fig.* 16) æquationis $dx = \frac{-a\,dy}{y}$, & accepta ordinata FA $= a$ quæratur ſuperficies genita ab arcu FD rotato circa aſymptotum HG. Ab æquatione curvæ invenies $ds = \frac{-dy}{y} \sqrt{a^2 + y^2}$; ergo $y\,ds = -dy \sqrt{a^2 + y^2}$, & $\frac{p}{r} S y ds$ hoc eſt ſuperficies orta ab arcu FD $= \frac{p}{r} S - dy \sqrt{a^2 + y^2}$, quæ ſummatoria dependens a quadratura hyperbolæ ita accipienda eſt, ut fiat $= o$ poſita $y = o$. Producta BA in H, ut HA $=$ FA $= a$, centro A ſemiaxe primo AH deſcribatur hyperbola æquilatera HI, cui ex punctis F, C agantur ordinatæ FI, CM, erit ſuperficies deſcripta ab arcu FD $= \frac{p}{r}$ IFCM. Eodem ſumpto arcu F 2 D ad alteram partem puncti F invenies, ſuperficiem genitam ab arcu F 2 D eſſe $= \frac{p}{r}$. IF 2 C 2 M.

19. Si poſitis curvæ coordinatis AB $= x$, BD $= y$ fiat rotatio circa axem KH, (*Fig.* 5) ſatis erit, vocata AH $= c$, ſcribere in formula $y + c$ pro $y$, atque hæc proveniet $\frac{p}{r} S\,\overline{y + c}.ds$. Hæc ob ſuimet facilitatem novis exemplis non indiget. Quod ſi axis rotationis tranſeat ad alteram partem abſciſſarum in 2 H 2 K, ut $c$ evadat negativa, tum mutato ſigno ſpeciei $c$, formula evadet $\frac{p}{r} S\,\overline{y - c}.ds$. Notetur punctum F, ubi axis rotationis curvam ſecat. Ajo, formulam exhibere differentiam ſuperficierum, quæ oriuntur ab arcubus FD, FA. Nam quum in arcu AF ſit $y - c$ negativa, & $ds$ ſit poſitiva, ſuperficies genita ab hoc arcu circa 2 H 2 K proveniet negativa, adeoque demenda erit ex ſuperficie genita ab arcu FD, quæ poſitiva eſt. Oſtendamus hoc de more in exemplo maxime ſimplici. Sit linea AD linea recta faciens cum AB angulum ſemirectum. Erit $x = y$, $dy = dx$ & $ds = dy\sqrt{2}$; igitur ſuperficies genita a recta AFD ex formula erit

$$\frac{p}{r} S\,\overline{y\,dy\sqrt{2} - c\,dy\sqrt{2}} = \frac{p}{r} . \frac{y^2}{\sqrt{2}} - \frac{p}{r} . cy\sqrt{2}.$$

Hæc æqualis eſt ſuperficiei conicæ genitæ ex recta FD circa F 2 H dempta ſuperficie conica genita ab AF circa A 2 H. Nam prima ſuperficies $= \frac{p}{2r} . \overline{y - c}.\sqrt{2}.\overline{y - c} =$

$\frac{p}{r} . \overline{\frac{yy}{\sqrt{2}} - cy\sqrt{2} + \frac{cc}{\sqrt{2}}}$; altera $= \frac{p}{2r} . cc\sqrt{2} = \frac{p}{r} . \frac{cc}{\sqrt{2}}$. Si hæc a ſuperiore de-

dematur, redit formula inventa. Ab his collige tamquam negativam ſpectandam eſſe ſuperficiem, quæ generetur ab arcu ſito infra axem rotationis.

20. Hactenus ſpectavimus coordinaras orthogonales, quod ſi eæ obliquæ fuerint, qua ratione invenienda ſint ſolida, & ſuperficies ortæ ex rotatione, breviter ex ſuperioribus colligemus. Sit curva A D, ( *Fig.* 17. ) cujus coordinatæ $AB = x$, $BD = y$ faciant angulum $= \mu$. Si ea rotetur circa axem H K parallelum A B: quæritur primum ſolidum ortum a ſuperficie A B D. Sit $AH = c$ parallela D B K. Ex D demittatur D L O perpendicularis A B, quemadmodum & A I. Quoniam $DL = \frac{Sc.\mu}{r}.y$, $LO = \frac{Sc.\mu}{r}.c$, $DO = \frac{Sc.\mu}{r}.\overline{y+c}$, $BL = \frac{Cc.\mu}{r}.y$, & $AL = x - \frac{Cc.\mu}{r}y$, ſi in formula inventa num. 7. pro $x$ ſcribatur $x - \frac{Cc.\mu}{r}.y$, & pro $y+c$ ponatur $\frac{Sc.\mu}{r}.\overline{y+c}$ habebitur ſolidum genitum a ſpatio A D L. Fiat autem

$$\frac{p}{2r}.\frac{\overline{Sc.\mu}^2}{r^2}.S\overline{y^2+2cy}.\overline{dx - \frac{Cc.\mu}{r}dy}.$$

Huic ad habendum ſolidum ortum ab area A D B addendum eſt ſolidum ortum a triangulo D L B, quod facile invenies

$$= \frac{p}{2r}.\frac{Cc.\mu.\overline{Sc.\mu}^2}{r^3}.\overline{\frac{y^3}{3}+cy^2} = \frac{p}{2r}.\frac{\overline{Sc.\mu}^2}{r^2}S\frac{Cc.\mu}{r}.dy.\overline{y^2+2cy},$$

quæ addita ſuperiori formulæ dabit ſolidum requiſitum $= \frac{p.\overline{Sc.\mu}^2}{2r^3}S\overline{y^2+2cy}.dx$.

Si in hac formula ponas $c = o$, habebis ſolidum ortum a rotatione circa axem A B. Quod ſi axis tranſeat ad alteram partem lineæ A B, ut in 2 H 2 K evadet $c$ negativa, quare huic mutato ſigno orietur formula, quæ exprimet ſolidum, quod naſcitur ex rotatione circa 2 K 2 H. Verum in hac quoad quantitates poſitivas, & negativas eadem advertenda ſunt, quæ ſupra diximus.

21. Quoad ſuperficiem productam ab arcu A D, nihil eſt facilius quam formulam invenire. Namque ſatis eſt in formula num 19. pro $y+c$ ſcribere $\frac{Sc.\mu}{r}.\overline{y+c}$; quo facto proveniet ſuperficies ab arcu A D genita $= \frac{p.Sc.\mu}{rr}S\overline{y+c}.ds$; ſed advertendum eſt, in hac $ds$ non æquare $\sqrt{dy^2+dx^2}$, ſed $\sqrt{dx^2 - \frac{2Cc.\mu}{r}dydx + dx^2}$. Si axis rotationis ſit A B, erit $c = o$, & formula ſimplicior evadet. Si axis rotationis ad alteram partem tranſeat, fiet $c$ negativa; quare mutato ſigno proveniet formula $\frac{p.Sc.\mu}{rr}S\overline{y-c}.ds$, in qua de negativis eadem valent, quæ antea notata ſunt.

22. Antequam deferam curvas ad abſciſſas relatas, placet non omittere methodum, quæ per multiplicem integrarionem formulas deducir ex parallelepipedorum dimenſione. Hæc enim methodus in caſibus difficilioribus magnam affert urilitatem. Sit curva A D d, ( *Fig.* 18) cujus ordinatæ $DB = y$ faciant cum abſciſ-

abſciſſis AB $= x$ angulum quemcumque $= \mu$. Ea rotetur circa axem HK. Agantur duæ ordinatæ infinite proximæ DBK, dbk, & ſit DLO perpendicularis axi rotationis. Supponatur fieri rotatio per arcum infiniteſimum, ut arcus Dp deſcriptus a puncto D infiniteſimus ſit. Vocetur $dp$ arcus ſimilis arcui Dp, qui deſcribitur radio conſtante $= r$. Superiores omnes denominationes retinentur. Quæratur primum ſolidum minimum deſcriptum in hoc motu ab area BDdb. Accipe abſciſſam BM $= z$; ejuſque elementum Mm $= dz$. Parallelæ AB agantur MN, mn. Ducta Mi perpendiculari MN, conſtat fore Mi $= \frac{Sc.\mu}{r} dz$; ergo parallelogrammum MmnN $= \frac{Sc.\mu}{r}.dz.dx$. Perpendicularis HK agatur MS, quæ erit $= \overline{c+z}.\frac{Sc.\mu}{r}$. Sit Mq arcus in rotatione deſcriptus a puncto M, qui ut inveniatur, fiat $r : dp :: \frac{Sc.\mu}{r}.\overline{c+z} : \text{M}q = \frac{Sc.\mu.dp}{rr}.\overline{c+z}$: ergo parallelepipedum deſcriptum ab area infinities infiniteſima MmnN erit $= \frac{\overline{Sc.\mu}^2.dp}{r^3}.dx.\overline{z+c}.dz$. Ut inveniatur ſolidum deſcriptum ab area BMNb, ſpectandæ ſunt tamquam conſtantes $dx$, $dp$, & ut ſola variabilis $z$; ergo in hac ſuppoſitione facta integratione habebimus ſolidum deſcriptum a rectangulo BMNb $= \frac{\overline{Sc.\mu}^2.dp}{2r^3}.dx.\overline{z^2+2cz}$. In hac ponamus $z = y$, & obtinebimus ſolidum deſcriptum ab area BDdb nimirum $\frac{\overline{Sc.\mu}^2.dp}{2r^3}.\overline{y^2+2cy}.dx$. Hac obtenta ad inveniendum ſolidum elementare, quod eadem area deſcribit in integra revolutione, poſita $y$, & $dx$ conſtante integretur formula inventa, & fiet $\frac{\overline{Sc.\mu}^2.p}{2r^3}.\overline{y^2+2cy}.dx$. Species $p$ autem nunc indicat integram circumferentiam. Habemus itaque elementum ſolidi deſcripti in integra revolutione ab area ADB. Spectantes jam tamquam fluentes $x$, $y$ integremus, & habebimus ſolidum requiſitum $= \frac{\overline{Sc.\mu}^2.p}{2r^3} S\overline{y^2+2cy}.dx$, quæ cum ſuperiore apprime cohæret.

23. Ad habendam ſuperficiem ortam ab arcu AD, primum inveniamus formulam exprimentem rectangulum Dp. Dd. Fiat ut $r : dp :: \frac{Sc.\mu}{r}.\overline{y+c}$: Dp $= \frac{Sc.\mu.dp}{rr}.\overline{y+c}$. Hanc integremus ſuppoſita $y$ conſtante, ut fiat $\frac{Sc.\mu.p}{rr}.\overline{y+c}.ds$ repræſentans zonam elementarem deſcriptam ab Dd, dummodo $p$ integram circumferentiam deſignet. Integremus jam ſuppoſita $y$, & $s$ variabili, & fiat ſuperficies quæſita $= \frac{Sc.\mu.p}{rr} S\overline{y+c}.ds$, quæ pariter eadem eſt

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eſt cum ſuperiore. Hæc de curvis ad abſciſſas relatis dicta ſint ſatis.

24. Tranſeamus ad curvas relatas ad focum. A facilioribus incipientes ſpectemus curvam A D ( *Fig.* 19 ) relatam ad focum F, quæ rotetur circa axem A F tranſeuntem per punctum F. Quæritur ſolidum productum in rotatione ab area A F D. Ducatur F d faciens angulum infiniteſimum cum F D. Fiat rotatio per infiniteſimum angulum, ut punctum D deſcribat minimum arcum D p. Solidum genitum a ſpatio F D d eſt pyramis, cujus vertex eſt F, baſis vero rectangulum genitum ab D d; cujus baſis eſt D d, altitudo arcus minimus D p. Hæc piramis dupla eſt ejus, cujus vertex idem F, baſis vero eſt dimidium rectanguli producti ab D d; ſed hæc ex archimedeis eſt tertia pars producti ex area F D d in arcum D p: ergo pyramis deſcripta ab area F D d æquabit duas tertias partes producti ex area F D d in arcum D p: igitur integratione facta ſolidum genitum in integra revolutione ab elementari triangulo F D d æquabit duas tertias partes producti ex eadem area F D d in circumſerentiam circuli deſcripti a puncto D, quod ſolidum conſtat eſſe elementum ſolidi geniti ab area A D F. His poſitis elementum idem algebraice exponendum eſt. Ducatur D B normalis axi, & centro F, radio F D deſcribatur arcus minimus D r. Vocetur $FD = y$, $Dr = dx$, arcus I L deſcriptus radio $FI = r$ ſit $= \mu$, ejus elementum $LI = d\mu$; erit $BD = \frac{Sc.\mu}{r}.y$, area $FDd = \frac{y\,dx}{2}$. Ut inveniatur circumferentia a puncto D deſcripta fiat ut $r:p::BD = \frac{Sc.\mu}{r}.y : \frac{pSc.\mu}{r^2}.y$, quæ erit circumferentia deſcripta a puncto D; ergo elementum ſolidi requiſiti $= \frac{pSc.\mu}{3r^2}.y^2dx$; atque adeo ſolidum genitum ab $FAD = \frac{p}{3r}.Sy^2.\frac{Sc.\mu}{r}.dx$. Tres variabiles in hac formula reperiuntur nempe $y$, $\mu$, $dx$; ſed facile ad duas reducentur eliminata $dx$, eſt enim $r:d\mu::y:dx = \frac{y\,d\mu}{r}$; ergo ſolidum idem erit $= \frac{p}{3r}Sy^3.\frac{d\mu.Sc.\mu}{r^2}$.

25. Hanc eamdem formulam licet invenire per methodum indicatam num. 22. Nam ſumpta qualibet $FM = z$, deſcribatur arcus M N, cui ſit infinite proximus m n, ut $Mm = dz$. Quum autem ſit $r:d\mu::z:MN$, erit $MN = \frac{z\,d\mu}{r}$; ergo areola $MmnN = \frac{z^2d\mu.dz}{r}$. Ducatur M S normalis axi rotationis. Ut inveniatur arcus deſcriptus in rotatione minima a puncto M, is qui deſcribitur ab L vocetur $= dq$; tum fiat ut O L : M S, ſeu ut L F : F M, ideſt $r:z::dq:\frac{z\,dq}{r}$, qui eſt arcus deſcriptus a puncto M; ergo ſolidum genitum ab areola $MmnN$, $= \frac{z^2d\mu.dq}{r^2}dz$; quæ integrata in ſuppoſitione $d\mu$, & $dq$ conſtantis dabit ſolidum ortum ab area $FMN = \frac{z^3d\mu\,dq}{3r^2}$; in qua ſi pona-

tur $z = y$, habebimus ſolidum ortum ab area elementari FDd $= \frac{y^3 d\mu dq}{3r^2}$. Hæc iterum integretur in ſuppoſitione $d\mu$ conſtantis, ut habeatur ſolidum ab eadem area FDd ortum in integra revolutione $= \frac{y^3 q d\mu}{3r^2}$, denotante $q$ circumferentiam integram radio OL deſcriptam. Ut ſpecies $q$ dependens a quantitate $\mu$ eliminetur, fiat, ut OL $= Sc.\mu : q :: r : p$, vocata $p$ circumferentia conſtante deſcripta radio FL; ergo $q = \frac{p.Sc.\mu}{r}$; ergo ſolidum elementare ortum ab area FDd $= \frac{py^3.d\mu.Sc\mu}{3r^3}$; & integrando ſolidum ortum ab AFD $= \frac{p}{3r} S \frac{y^3.d\mu.Sc.\mu}{rr}$; ut antea. Hæc formula ſimplicior reddi poteſt, advertendo $\frac{d\mu.Sc.\mu}{r} = -dCc.\mu$; quare ſolidum ortum ab AFD $= \frac{p}{3r} S - y^3.\frac{dCc.\mu}{r}$, quæ formula, ubi $y$ datur per coſinum anguli AFD, rem expeditiſſime aperit.

26. Ad exemplum ſit primo AD circulus, cujus radius FA $= r$, erit $y = r$, ergo ſolidum genitum a ſectore FAD $= \frac{p}{3r}.r^3 - r^2.Cc.\mu$. Ea conſtans addita eſt, ut poſito $Cc.\mu = r$, ſolidum evaneſcat; atqui $r - Cc.\mu$ æquat ſinum verſum AB: ergo idem ſolidum $= \frac{rp.AB}{3}$. Sit $y = mCc.\mu$, quæ eſt æquatio circuli radii $= mr$, poſito foco in circumferentia. Differentiando fit $\frac{dy}{m} = dCc.\mu$; ergo ſolidum $\frac{p}{3r} S - \frac{y^3 dy}{mr} = \frac{p}{3r}.\frac{m^4r^4 - y^4}{4mr}$. Ea conſtans poſita eſt, quæ nullum reddit ſolidum, ſi $y = mr$. Quare facta $y = o$, habebitur emiſphærium $= \frac{p}{3r}.\frac{m^3r^3}{4}$. Si generatim ponas $y = \frac{m.\overline{Sc.\mu}^p.\overline{Cc.\mu}^q}{r^{p+q-1}}$, formula ſolidum exhibens hæc proveniet $\frac{p}{3r} S \frac{m^3.\overline{Sc.\mu}^{3p+1}.\overline{Cc.\mu}^{3q}.d\mu}{r^{3p+3q-1}}$. Summatoria hæc, ut antea fuſius probavimus, modo algebraice, modo ſuppoſitis ſolis quadraturis circuli, & hyperbolæ obtinetur. Quare ſolida ex his orta, aut ſolam quadraturam circuli poſtulant, aut prætereà quadraturam hyperbolæ.

27. Sit ſpyralis archimedea, quæ deſcribitur motu æquabili a puncto F (*Fig.* 10) gradiente per radium FI, interim dum hic circumvolvitur per quadrantem IB. Ducta qualibet FDL, quæratur ſolidum genitum a ſpatio FD, quod inter curvam, chordamque continetur. Vocato radio IF $= r$, & quadrante IB $= q$, cæteriſque retentis denominationibus æquatio ſpiralis hæc eſt $r : q :: y : \mu$; ergo $y = \frac{r}{q}.\mu$. Quo valore ſubſtituto, ſolidi formulam provenit $\frac{p}{3r}$.

S —

$S - \frac{r^3}{q^3} . \frac{\mu^3 . dCc.\mu}{r} = \frac{p}{3} \; S - r . \frac{\mu^3}{q^3} . dCc.\mu$, quæ ſummatoria dependet a quadratura curvæ, quæ deſcribitur ſuppoſita circuli rectificatione. Poſito eodem circulo IBK, (*Fig.* 21) ex quolibet puncto L ordinetur LP; tum ſumatur PQ ita ut FI : PQ ſit in ratione triplicata IB : IL; punctum Q erit in curva, cujus abſciſſæ $FP = Cc.\mu$, ordinatæ autem $PQ = \frac{r . \mu^3}{q^3}$; ergo area $QPI = S - \frac{r . \mu^3}{q^3} dCc.\mu$; ergo ſolidum deſcriptum a ſpatio FD, (*Fig.* 20) quod inventum eſt $= \frac{p}{3} S - \frac{r . \mu^3}{q^3} dCc.\mu$, obtinebitur, ſi area QPI (*Fig.* 21) ducatur in tertiam partem circumferentiæ circuli IBK. Curva IQ intra parallelas IN, KO habet infinitos ramos IM, MN, NO &c. Primi areæ IMK inſervit inveniendo ſolido genito a prima ſemiſpyra FbC; ſecundi area NMKI determinabit ſolidum ortum a ſecunda ſemiſpyra CEG. Quare ad invenienda ſolida duabus ſemiſpyris genita, oportebit ſimul accipere ſummas utriuſque areæ NMKI, MKI; ad inveniendam vero eorumdem ſolidorum differentiam, earumdem arearum opus eſt uti differentia, hoc eſt area NMI, quæ continetur inter duos ramos MN, MI. Area a tertio ramo terminata nempe ONIK dabit ſolidum productum a tertia ſemiſpyra; atque ita deinceps.

28. Nunc conſideremus ſolidum genitum in rotatione circa axem EG, (*Fig.* 22) qui non tranſit per focum F, ad quem curva refertur. Recta AF ponatur parallela axi rotationis EG; & areæ AFD ſumatur, ut antea, elementum DFd. Fiat rotatio per minimum angulum, ut puncta D, F deſcribant minimos arcus DP, Ff, quorum radij ſunt DE, FG normales axi rotationis. In hoc motu triangulum elementare FDd deſcribet ſolidum clauſum a duobus triangulis, & a tribus quadrilateris, quod repræſentat Fig. 23. Qui hujuſce ſolidi quantitatem exprimet, inveniet elementum ſolidi, quod in integra rotatione generat triangulum elementare FDd. Per punctum f ducatur planum tranſiens per Dd, quod dividet ſolidum in duas pyramides alteram triangularem fDdF, alteram quadrilateram fDdpP. Prima habetur, ſi triangulum FDd ducatur in tertiam partem Ff; ſecunda eſt productum areæ fPp, ſeu FDd ductæ in duas tertias partes DP: ergo totum ſolidum $= FDd . \frac{Ff + 2 DP}{3}$. Itaque integratione facta habebitur ſolidum in integra revolutione generatum a triangulo elementari FDd; fietque ſolidum $= \frac{FDd}{3}$ in duas circumferentias deſcriptas a puncto D, addita circumferentia deſcripta a puncto F: quod eſt elementum ſolidi geniti ab area ADd. Retentis ſuperioribus denominationibus vocetur præterea conſtans $BE = c$. Fiat $r : p :: c + \frac{Sc.\mu}{r} . y : \frac{cp}{r} + \frac{pSc.\mu}{rr} . y$, quæ eſt circumferentia deſcripta a puncto D. Item $r : p :: c : \frac{pc}{r}$, quæ erit circumferentia deſcripta a puncto F; ergo ſolidi quæſiti elementum invenitur $= \frac{ydx}{2} . \frac{cp}{r} + \frac{2p . Sc.\mu}{3rr} . y$;

ergo

ergo ſolidum requiſitum genitum ab area AFD $= \frac{p}{3r} S \overline{\frac{3cydx}{2} + \frac{Sc.\mu}{r}.y^2 dx}$.

Quoniam autem eſt $y:dx::r:d\mu$, erit $dx = \frac{yd\mu}{r}$; ergo ſolidum $= \frac{p}{3r}$ $S \frac{3cy^2 d\mu}{2r} + \frac{y^3 d\mu . Sc.\mu}{r^2}$.

29. Methodus uſurpata num. 25. huic quoque caſui applicatur. Namque ut ſupra invenietur areola MmnN $= \frac{z dz d\mu}{r}$. Vocetur $= dq$ arcus minimus, qui in rotatione deſcribitur a puncto $z$. Fiat ut LH:MK, ſive $c + Sc\mu : c + \frac{zSc.\mu}{r} :: dq : \frac{c + z\frac{Sc.\mu}{r}}{c + Sc.\mu}\, dq$, qui erit arcus deſcriptus a puncto M. Hic autem ductus in areolam MmnN dabit ſolidum ab hac areola genitum $= \frac{czdzd\mu + \frac{Sc.\mu}{r} z^2 dzd\mu}{r.\overline{c + Sc.\mu}} . dq$; quæ in hypotheſi $d\mu$ & $dq$ conſtantis integretur, ut habeatur $\frac{\overline{\frac{cz^2}{2} + \frac{Sc.\mu}{3r} z^3}}{r.\overline{c + Sc.\mu}} . d\mu dq$, in qua ponatur $z = y$; ut habeatur ſolidum ortum in minima revolutione a triangulo elementari FDd $= \frac{\frac{1}{2}cy^2 + \frac{Sc.\mu}{3r}.y^3}{r.\overline{c + Sc.\mu}} . d\mu dq$. Hæc iterum integretur ſuppoſita conſtante $d\mu$, & $y$, ut ſolidum exurgat ab eodem triangulo FDd genitum in integra revolutione $\frac{\frac{1}{2}cy^2 + \frac{Sc.\mu}{3r}.y^3}{r.\overline{c + Sc.\mu}} . qd\mu$, denotante $q$ circumferentiam deſcriptam a puncto L, radio KL. Ut $q$ determinetur per $\mu$ fiet, ut HL $= c + Sc.\mu : q :: r : p$; ergo $\frac{q}{c + Sc.\mu} = \frac{p}{r}$; ergo habebitur $\frac{p}{3r} . \frac{3}{2r} cy^2 + \frac{Sc.\mu}{rr} y^3 . d\mu$, quæ cum ſuperiore prorſus cohæret. In hac ſi ponas $\frac{d\mu Sc.\mu}{r} = -dCc.\mu$ facta integratione habebis $\frac{p}{3r^2} S \frac{3}{2} cy^2 d\mu - y^3 dCc.\mu$.

30. Formulæ inventæ applicatæ ſunt caſui, ubi curva jacet poſt axem rotationis, & huic parallelam tranſeuntem per focum. Quod ſi curva inter has medium locum teneat, quid fiet noſtris formulis? Evidens eſt tam arcum $\mu$, quam

$S\,c.\mu$ evadere negativum. Quare formulæ fient $\frac{p}{3r}\mathrm{S}-\frac{3c}{2r}y^2 d\mu+y^3.\frac{d\mu S\,c.\mu}{rr}$, $\frac{p}{3r^2}\mathrm{S}-\frac{3}{2}cy^2 d\mu-y^3 d\,C\,c.\mu$, quæ præbebunt negativum ſolidum ortum ex rotatione. Quare mutentur ſigna, ut poſitivum evadat, $\frac{p}{3r}\mathrm{S}\,\overline{\frac{3c}{2r}y^2 d\mu-y^3.\frac{d\mu S\,c.\mu}{rr}}$, $\frac{p}{3r^2}\mathrm{S}\,\frac{3}{2}cy^2 d\mu+y^3 d\,C\,c.\mu$. Reapſe ad has formulas devenies, ſi utramque methodum, qua uſi ſumus, huic caſui accomodes.

31. Quod ſi curva A D, (*Fig.* 24) cujus focus F ſecetur ab axe rotationis E G, tum in noſtris formulis ſpecies $c$ evadet negativa. Quare mutato hujus ſpeciei ſigno fiet ſolidum ortum a triangulo elementari F D d $=$

$$\frac{p}{3rr}.-\frac{3}{2}cy^2 d\mu+\frac{y^3 d\mu S\,c.\mu}{r}=\frac{p}{3r^2}.\overline{-\frac{3}{2}cy^2 d\mu-y^3 d\,C\,c.\mu}.$$

Quæ formulæ indicant differentiam inter ſolida duo, quorum unum producitur a quadrilatero x X D d, alterum a triangulo F X x. Ut hoc clare appareat, determinemus per duas methodos, quibus ſupra uſi ſumus, utriuſque ſolidi quantitatem. Quantum pertinet ad triangulum F X x, ſolidum ab eo genitum eſt pyramis, in minima revolutione, cujus baſis eſt idem F X x, altitudo vero lineola deſcripta a puncto F; ergo hoc ſolidum eſt tertia pars trianguli F X x in hanc lineolam, & facta integratione in ſuppoſitione conſtantis trianguli F X x, ſolidum ortum ex integra revolutione ejuſdem trianguli eſt tertia pars trianguli F X x in circumferentiam deſcriptam a puncto F. Ut inveniamus F X x fiat $\mathrm{FD}^2:\mathrm{FX}^2::$ triang. F D d : F X x, ſive retentis ſuperioribus denominationibus $y^2:\frac{r^2c^2}{\overline{S\,c.\mu}^2}::\frac{y\,dx}{2}:\frac{r^2c^2dx}{2y\overline{S\,c.\mu}^2}$, quæ exprimit triangulum X F x; circumferentia vero deſcripta a puncto X $=\frac{p}{r}.c$; ergo ſolidum productum in revolutione integra a triangulo X F x $=\frac{rpc^3dx}{6y\overline{S\,c.\mu}^2}$; atqui $r:y::d\mu:dx=\frac{y\,d\mu}{r}$; ergo idem ſolidum $=\frac{pc^3d\mu}{6\overline{S\,c.\mu}^2}$, quæ integrationem admittit, ejuſque integrale $=\frac{-pc^3C\,c.\mu}{6rS\,c.\mu}$, addita opportuna conſtante, quam hic non quærimus.

32 Altera methodus eamdem formulam invenit. Nam ſolidum deſcriptum ab elemento M m n N in minima revolutione, eſt idem elementum ductum in minimum arcum deſcriptum a puncto M. Vocata autem F M $=z$ elementum M m n N $=\frac{z\,dz\,d\mu}{r}$, in arcum minimum deſcriptum a puncto M. Hic autem eſt ad arcum deſcriptum a puncto F, quem voco $=dq::$ X M : X F $::\frac{rc}{S\,c.\mu}-z:\frac{rc}{S\,c.\mu}$; ergo arcus deſcriptus a puncto M $=dq-\frac{z\,dq.S\,c.\mu}{rc}$; ergo

go ſolidum deſcriptum ab elemento M m n N $= \frac{z\,dz\,d\mu\,dq}{r} - \frac{z^2\,dz\,d\mu\,dq.Sc.\mu}{r^2 c}$, quod integratum in ſuppoſitione $\mu$, $d\mu$, & $dq$ conſtantis dabit ſolidum genitum ab FMN $= \frac{z^2\,d\mu\,dq}{2r} - \frac{z^3\,d\mu\,dq\,Sc.\mu}{3r^2c}$. Poſita autem $z = $ x F $= \frac{rc}{Sc.\mu}$, habebimus ſolidum ortum a triangulo F X x $= \frac{rc^2\,d\mu\,dq}{2\,\overline{Sc.\mu}^2} - \frac{rc^2\,d\mu\,dq}{3\,\overline{Sc.\mu}^2} = \frac{rc^2\,d\mu\,dq}{6\,\overline{Sc.\mu}^2}$, & iterum integrando ſuppoſita $\mu$, & $d\mu$ conſtante, habebimus ſolidum in integra revolutione ab eodem triangulo genitum $= \frac{rc^2\,q\,d\mu}{6\,\overline{Sc.\mu}^2}$; denotante $q$ circumferentiam deſcriptam a puncto F. Atqui $r:p::c:q = \frac{pc}{r}$; ergo idem ſolidum $= \frac{pc^3\,d\mu}{6\,\overline{Sc.\mu}^2}$ prorſus ut antea.

32. Accedo ad ſolidum in minima revolutione deſcriptum a trapetio X x d D. Hoc ſolidum exprimitur in fig. 25. in qua DP, dp ſunt arcus minimi deſcripti a punctis D, d. Ducto plano tranſeunte per tria puncta x, P, D, ſolidum dividitur in pyramidem quadrangularem x P D d p, & triangularem x P D x. Hæc eſt tertia pars producti trianguli x D X in D P; illa, ut conſtat ex ſuperioribus exæquat duas tertias partes producti trianguli x D d in D P. Quare hoc unice curandum eſt, ut exprimantur triangula x D X, x D d. In fig. 24. agatur X D, & centro F intelligatur deſcriptus arcus x t. Quoniam eſt F D : F X :: D t : x t, erit analytice $y : \frac{rc}{Sc.\mu} :: dx : \frac{rc\,dx}{y\,Sc.\mu} = $ x t; ergo triangulum F X x $= \frac{r^2c^2\,dx}{2y\,\overline{Sc.\mu}^2}$; atqui F X : X D :: triang. F X x : X x D; ergo $\frac{rc}{Sc.\mu} : y - \frac{rc}{Sc.\mu} :: \frac{r^2c^2\,dx}{2y\,\overline{Sc.\mu}^2}$ : X x D $= \frac{rc\,dx}{2\,Sc.\mu} - \frac{r^2c^2\,dx}{2y\,\overline{Sc.\mu}^2} = \frac{rc\,dx}{2y\,\overline{Sc.\mu}^2}.\,y\,\overline{Sc.\mu - rc}$. Simili modo invenitur triangulum x D d. Namque F d : x d :: triang. F D d : x D d, ſeu analytice $y : \frac{y\,Sc.\mu - rc}{Sc.\mu} :: \frac{y\,dx}{2}$ : x D d $= \frac{y\,Sc.\mu - rc}{2\,Sc.\mu}.dx$. His triangulis inventis vocato $= dq$ minimo arcu deſcripto a puncto D, pyramis triangularis erit $= \frac{rc\,dx\,dq.\overline{y\,Sc.\mu - rc}}{6y\,\overline{Sc.\mu}^2}$; pyramis vero quadrangularis $- \frac{dx\,dq.\overline{y\,Sc.\mu - rc}}{3\,Sc.\mu}$.

Igitur totum ſolidum deſcriptum in minima revolutione ab area X x d D, fiet

=

$= \frac{rcdxdq}{6y\overline{Sc.\mu}^2} + \frac{dxdq}{3Sc.\mu}.\overline{ySc.\mu - rc}$; & integrando ſuppoſitis conſtantibus $y$, $\mu$, $dx$, fiet ſolidum genitum ab eadem area in integra revolutione $\frac{rcqdx}{6y\overline{Sc.\mu}^2} + \frac{qdx}{3Sc.\mu}.\overline{ySc.\mu - rc} = \frac{qdx}{6y\overline{Sc.\mu}^2}.\overline{2ySc.\mu + rc}.\overline{ySc.\mu - rc}$.

Ut eliminemus $q$ dependentem a quantitate $\mu$, fiat $r:p::\frac{\overline{ySc.\mu - rc}}{r}:q = \frac{p.\overline{ySc.\mu - rc}}{rr}$: ergo ſolidum ortum a revolutione trapetii XxdD erit $= \frac{pdx}{6r^2y\overline{Sc.\mu}^2}.\overline{2ySc.\mu + rc}.\overline{ySc.\mu - rc}^2$; atqui $\frac{dx}{y} = \frac{d\mu}{r}$; ergo idem $= \frac{pd\mu}{6r^3.\overline{Sc.\mu}^2}.\overline{2ySc.\mu + rc}.\overline{ySc.\mu - rc}^2 = \frac{pd\mu}{6r^3.\overline{Sc.\mu}^2}.\overline{2y^3.\overline{Sc.\mu}^3 - 3cry^2.\overline{Sc.\mu}^2 + r^3c^3}$. Q. E. I.

34. Ad eamdem formulam ſecunda, qua uſi ſumus, methodus nos perducet. Nam centro F deſcriptis minimis arcubus $M'N'$, $m'n'$ infinite proximis, vocetur $XM' = z$. Quoniam eſt $r:d\mu::FM = \frac{rc + zSc.\mu}{Sc.\mu}:M'N'$, erit $M'N' = \frac{d\mu}{rSc.\mu}.\overline{rc + zSc.\mu}$; ergo areola $M'm'n'N' = \frac{d\mu}{rSc.\mu}.\overline{rcdz + Sc.\mu.zdz}$. Vocato $= dq$ arcu minimo deſcripto a puncto D, fiat ut $XD:XM'$, ſive $\frac{ySc.\mu - rc}{Sc.\mu}:z:dq:\frac{zSc.\mu.dq}{y.\overline{Sc.\mu - rc}}$, qui erit arcus minimus deſcriptus a puncto $M'$: ergo ſolidum in minima revolutione ortum ab areola $M'm'n'N'$ erit $= \frac{d\mu.dq}{r.\overline{ySc.\mu - rc}}.\overline{rczdz + Sc.\mu z^2dz}$. Integremus poſitis $\mu$, $d\mu$, $dq$ conſtantibus, fiet ſolidum ortum ab area $XM'N'x = \frac{z^2.d\mu.dq}{6r.\overline{ySc.\mu - rc}}.\overline{3rc + 2zSc.\mu}$. Pone $z = XD = \frac{ySc.\mu - rc}{Sc.\mu}$, ut habeas ſolidum in minima rotatione genitum ab area XDdx $= \frac{d\mu dq}{6r.\overline{Sc.\mu}^2}.\overline{2ySc.\mu + rc}.\overline{ySc.\mu - rc}$. Hanc formulam ſi primum integres poſita ſola $q$ variabili, tum elimines $q$, eam formulam invenies, quæ num. ſuperiore inventa eſt.

35. Quoniam ſolidum genitum a trapetio XxdD $=$

$\frac{p\,d\mu}{6\,r^3 . \overline{Sc.\mu}^2} . 2y^3 . \overline{Sc.\mu}^3 - 3rcy^2 . \overline{Sc.\mu}^2 + r^3c^3$, & quod oritur a triangulo $FXx = \frac{pc^3 d\mu}{6\,\overline{Sc.\mu}^2}$; ſi hoc a priori deducatur, remanet $\frac{p\,d\mu}{6r^3} . \overline{2y^3 . Sc.\mu - 3rcy^2}$, quæ formula illa ipſa eſt, quam poſuimus num. 30, facta $c$ negativa; & quæ proinde duorum ſolidorum indicat differentiam. Si recta F 2 D ſecet axem rotationis poſt focum, & curvam; tunc formula exhibens ſolidum ortum ex trapezio 2 X 2 x 2 d 2 D evadit negativa: quare ad hoc ſolidum obtinendum poſitive, in formula ſigna omnia mutanda erunt. Plura alia poſſent obſervari, ſed quum ex his, quæ dicta ſunt, facillime intelligantur, hæc ſufficiant ad inveniendam quantitatem ſolidorum, quæ a rotatione producuntur.

36. Nunc breviter de ſuperficiebus. Curva A D relata ad focum F rotetur primo circa axem A F (*Fig.* 19) tranſeuntem per focum F. Conſtat elementum D d, quod ſumi debet in minima rotatione producere rectangulum, cui baſis D d, altitudo arcus minimus deſcriptus a puncto D; ergo integrando ſuppoſita conſtante D d, zona ab hoc elemento deſcripta æquabit rectangulum ex D d in circumferentiam deſcriptam a puncto D: atqui eſt $r : p :: \frac{y\,Sc.\mu}{r} : \frac{p\,y\,Sc.\mu}{rr}$, quæ æquat hanc circumferentiam puncti D (retineo ſpecies ſupra adhibitas); D d vero $= \sqrt{dx^2 + dy^2}$; ergo zona deſcripta a D d $= \frac{p\,y\,Sc.\mu}{rr}\sqrt{dx^2 + dy^2}$; atqui $dx = \frac{y\,d\mu}{r}$; ergo eadem zona $= \frac{p\,y\,Sc.\mu}{r^3}\sqrt{y^2 d\mu^2 + r^2 dy^2}$, & integrando ſuperficies genita ab A D $= \frac{p}{r^3} S\, y\, Sc.\mu \sqrt{y^2 d\mu^2 + r^2 dy^2}$.

37. Si A D ſit circulus deſcriptus centro F, & radio F D $= r$, erit $y = r$, hoc eſt conſtans; ergo ſuperficies genita a curva A D $= \frac{p}{r} S\, d\mu . Sc.\mu$; atqui $d\mu . Sc.\mu = - r\,dCc.\mu$; ergo eadem $= p S - dCc.\mu = p . \overline{r - Cc.\mu}$. Additur conſtans efficiens, ut facto $Cc.\mu = r$, ſuperficies nulleſcat. Quum autem $r - Cc.\mu$ æquet ſinum verſum A B, ſuperficies genita ab arcu A D æquabit rectangulum ex tota circumferentia in ſinum verſum A B. Sit ſpiralis F D B, (*Fig.* 26.) cujus proprietas ſit, ut $y = Sc.\mu$; hoc eſt deſcripto circulo, cujus radius F I $= r$, ut F D $=$ O L. Quare differentiis acceptis erit $dy = dSc.\mu = \frac{d\mu . Cc.\mu}{r}$; ergo ſuperficies genita ab arcu F D $=$ $\frac{p}{r^3} d\mu\, \overline{Sc.\mu}^2 \sqrt{\overline{Sc.\mu}^2 + \overline{Cc.\mu}^2} = \frac{p}{r} . S\, \frac{d\mu\, \overline{Sc.\mu}^2}{r}$. Hæc poteſt integrari in hunc modum $\frac{p}{r} . \frac{r\mu - Sc.\mu . Cc.\mu}{2}$, quæ a ſola quadratura circuli dependet. Verum rem ita facilius conficiemus. Quoniam $\frac{d\mu\, Sc.\mu}{r} = - dCc.\mu$, erit ſu-

per-

perficies quæſita $\frac{p}{r}.S - Sc.\mu.dCc.\mu$; atqui $S - Sc.\mu.dCc.\mu =$ aream IOL; ergo ſuperficies quæſita genita ab arcu $FD = \frac{p}{r}.IOL$. Quare ea quæ gignitur a curva FDB erit ad quadrantem IFB ut circumferentia ad radium. Si quæras, quænam ſit hæc curva FDB, facile invenies: eam eſſe circulum, cujus diameter $= FB = 2r$.

38. Si axis rotationis EG non tranſeat per focum F, ſed diſtet intervallo $= c$ a parallela per focum F (*Fig.* 22) tranſeunte, eadem omnino methodus locum habet; neque aliud adeſt diſcrimen, niſi quod radius circuli deſcripti a puncto D æquat $\frac{ySc.\mu}{r} + c$, & circumferentia $= \frac{p}{rr}.\overline{ySc.\mu + rc}$; ergo ſuperficies genita a curva $AD = \frac{p}{r^3}S\overline{ySc.\mu + rc}.\sqrt{y^2d\mu^2 + r^2dy^2}$. Quod ſi axis rotationis EG (*Fig.* 24) medius ſit inter focum, & curvam; tunc quantitas $c$ evaderet negativa, & formula reſultaret $\frac{p}{r^3}S\overline{ySc.\mu - rc}.\sqrt{y^2d\mu^2 + r^2dy^2}$.

Ex his formulis omnes ſuperficies genitas in rotatione licet derivare. Hæc quidem ſufficerent quo ad ſolida, & ſuperficies genitas; ſed omittenda non & elegantiſſima regula, quæ Guldini appellari ſolet, per quam maxime ſimplices plerumque oriuntur determinationes. Sed ad eam explicandam aliqua præmittenda ſunt.

39. Sint duæ quælibet quantitates A, B collectæ in punctis A, B, & juncta AB (*Fig.* 27) dividatur in E in ratione quantitatum reciproca; ſit pariter quodlibet planum MN, in quod a punctis A, B, E demittantur perpendiculares AM, BN, EP. Ajo fore $A.AM + B.BN = \overline{A+B}.EP$. Ad demonſtrationem per punctum E agatur parallela, quæ ſecet AM, BN productas, ſi opus eſt, in punctis H, K. Quum ſimilia ſint triangula EAH, EBK erit BE : AE :: BK : AH; ſed ex hypotheſi BE : AE :: A : B; ergo A : B :: BK : AH; ergo $A.AH = B.BK$; ſed $A.AH$ eſt exceſſus rectanguli $A.EP$ ſupra rectangulum $A.AM$, & $B.BK$ eſt exceſſus rectanguli $B.BN$ ſupra $B.EP$; igitur $A.AM : A.EP$ erit arithmetice ut $B.EP : B.BN$; ergo $A.AM + B.BN = \overline{A+B}.EP$. Q. E. D. Hinc colligere licet rectangula, quæ fiunt ex quantitatibus A, B in diſtantias a plano MN eadem eſſe vel quantitates ſint in punctis A, B, vel ſint collectæ in puncto E.

40. Addatur jam tertia quantitas C. Jungatur EC, quæ dividatur in F ita, ut ſit A + B : C :: CF : FE, & in idem planum ducantur normales FR, CQ. Ajo $A.AM + B.BN + C.CQ = \overline{A+B+C}.FR$. Intelligantur quantitates A, B collectæ in E. Ex num. ſuperiore erit $\overline{A+B}.EP + C.CQ = \overline{A+B+C}.FR$; ſed ex eodem num. $\overline{A+B}.EP = A.AM + B.BN$; igitur $A.AM + B.BN + C.CQ = \overline{A+B+C}.FR$. Quapropter vel tria corpora A, B, C ſint poſita in punctis A, B, C, vel ſimul collecta in puncto F, rectangula, quæ fiunt ab iiſdem in ſuas diſtantias a plano, ſunt æqualia.

41. Eodem demonſtrationis progreſſu, ſi addatur quartum corpus D, & inveniatur punctum G ita, ut ſit A + B + C : D :: DG : GF, probabo, quatuor



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rectangula, quæ fiunt ex ſingulis quantitatibus in ſuas diſtantias a plano, æqualia eſſe unico rectangulo ex ſumma quatuor quantitatum in diſtantiam G S. Quoniam vero hic progreſſus ad quinque, ad ſex, imo ad quemcumque numerum quantitatum etiam infinitum extenditur, manifeſtum eſt, poſitis quibuſcumque quantitatibus intra ipſas adeſſe punctum ea præditum proprietate, ut ſumma omnium quantitatum in diſtantiam hujus puncti a plano quodlibet æquet omnia rectangula ex ſingulis quantitatibus in ſuas peculiares diſtantias ab eodem plano. Hoc primum detectum eſt ab illis, qui de re mechanica egerunt, illudque a pleriſque nominatum eſt centrum gravitatis, a nonnullis etiam centrum maſſæ. Nos in geometricis inſiſtentes vocabimus centrum magnitudinis. Si omnes quantitates ſitæ ſunt ex una eademque parte plani, rectangulorum ſumma ſemper accipienda eſt, prout antea fecimus. Verum ſi aliquæ jaceant ad partem oppoſitam, earum diſtantiæ a plano tamquam negativæ accipiendæ ſunt; quare rectangulum ex omnibus quantitatibus in diſtantiam centri magnitudinis a plano æquale eſt rectangulis earum quantitatum, quæ ſunt in eadem parte, in ſuas peculiares diſtantias a plano, demptis ſimilibus rectangulis aliarum quantitatum, quæ ſunt in plaga oppoſita. Quod ſi planum tranſeat per centrum ipſum magnitudinis, quando hujus diſtantia nulla eſt, oportet, ut quæ rectangula exhibent quantitates ad unam partem poſitæ, ſint æqualia illis, quæ præbent quantitates ad alteram partem jacentes.

42. Quæret fortaſſe aliquis, utrum datis quantitatibus, earumque poſitionibus, punctum hoc, quod dicimus centrum magnitudinis, ſit determinatum, & unicum? Unicum ſine dubio eſſe, reſpondeo. Etenim tribus planis non parallelis conceptis, ſi ex rectangulorum proprietate determinemus diſtantiam centri magnitudinis a tribus planis, & ſecundum has diſtantias ducamus tria plana prioribus parallela, patet, in ſingulis hiſce planis centrum magnitudinis exiſtere; ergo in communi trium planorum ſectione, quæ eſt punctum, in quo centrum magnitudinis ſitum eſt. Si quantitates omnes in uno eodemque plano jacerent, in eodem plano ductis duabus lineis non parallelis, determinetur diſtantia centri magnitudinis ab his lineis, & ſecundum has diſtantias agantur duæ prioribus parallelæ, earum interſectio centrum magnitudinis determinabunt. Poſui lineas A M, B N, E P &c. normales plano M N, ut claritati ſervirem; verum progreſſus demonſtrationis non deficit, tametſi normales non ſint, dummodo inter ſe parallelæ. Quare deinceps, ſi commodum fuerit, parallelas quidem, ſed non normales conſiderabo.

43. Si quantitates ſint numero finitæ, nihil facilius eſt, quam earum centrum magnitudinis invenire. Etenim ſatis eſt dividere lineam duas conjungentes in earum ratione reciproca, hoc erit centrum duarum quantitatum. Conjungatur hoc cum tertia, & linea conjungens pariter, ut traditum eſt, in ratione reciproca dividatur, & determinabitur centrum magnitudinum trium quantitatum. Ita progredere, donec omnes acceptæ fuerint, atque ita omnium centrum magnitudinis determinabis. Verum ſi quantitates ſint numero infinitæ, ut accidit in lineis, ſuperficiebus, ac ſolidis, res eſt difficilioris indaginis, & poſcit, ut formulas doceamus, per quas hujuſmodi centrum determinare poſſimus.

44. Sed prius advertendum eſſe cenſeo, exiſtere figuras ejuſmodi proprietatibus præditas, ex quibus centrum magnitudinis aut ſtatim determinatur, aut ſaltem determinatur linea, aut planum, in quo exiſtit. Ita lineæ rectæ centrum magnitudinis eſt punctum illam bifariam dividens, quia conſtat utriuſque par-

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tis elementa analoga ab eo æque diſtare, atque adeo rectangula ex ſingulis elementis in diſtantias ab hoc puncto æqualia erunt. Similiter circuli, ejus circumferentiæ, ſphæræ, ejus ſuperficiei centrum magnitudinis idem eſt ac centrum figuræ, quia ducta per centrum qualibet linea vel plano, rectangula ſingularum partium in diſtantias ab hac linea, vel plano hinc inde æqualia ſunt. Præterea ſi linea dividat omnes ordinatas parallelas æqualiter, perſpicuum eſt in hac linea fore centrum magnitudinis; item ſi planum dividat omnia plana parallela in partes æquales, & ſimiles, centrum magnitudinis erit in hoc plano. Hinc determinatur centrum magnitudinis parallelogrammi A B E D. (*Fig.* 28) Nam ducta F G dividente latera A B, D E biſariam; item H K, quæ alia latera A D, B E biſariam ſecet; harum interſectio C dabit centrum magnitudinis, quod punctum idem eſt, quod præbet diagonalium interſectio. Similiter per tria plana parallela tribus parallelepipedi ſuperficiebus, quorum unumquodque dividat alia duo plana in partes æquales, centrum magnitudinis parallelepipedi determinabitur. Hæ omnes figuræ ſunt, in quibus centrum magnitudinis coincidit cum centro ipſius figuræ.

45. Ad alias tranſeuntes inquiramus centrum magnitudinis cujuſcumque trianguli. Sit triangulum quodlibet A B D. (*Fig.* 29) Dividantur latera duo A B, A D biſariam in F, G, & agantur D F, B G ſe interſecantes in C. Quoniam D F ſecat biſariam omnes parallelas lateri B A, in ea erit centrum magnitudinis trianguli. Hoc autem centrum erit etiam in linea B G ſecante biſariam omnes parallelas A D; Ergo centrum erit punctum interſectionis C. Agatur linea F G. Quum utraque linea A B, A D dividatur biſariam in F, G, erit F G parallela B D; ergo erit A F : A B :: F G : B D; ſed A F eſt dimidium A B; ergo F G eſt dimidium B D; ſed quum propter ſimilitudinem triangulorum F C G, B C D ſit F G : B D :: F C : C D, & F G ſit dimidium B D, erit etiam F C dimidium C D; ergo F C tertia pars totius D F. Simili modo demonſtrabo C G eſſe tertiam partem B G. Jungatur A C; quæ producta ſecet latus B D in E; ajo B D eſſe biſariam diviſam in E. Per punctum C agatur M N parallela B D. Similia triangula F C M, F D B dant F C : F D :: M C : B D; ſed F C eſt tertia pars D F; ergo M C tertia pars B D. Pariter ſimilia triangula G C N, G B D dant G C : G B :: C N : B D; ſed C G eſt tertia pars G B; ergo C N tertia pars B D. Itaque C M, C N æquantes ambæ tertiam partem ejuſdem B D, inter ſe æquales erunt; ergo etiam B E, D E æquales erunt inter ſe. Præterea quum ſit M C : B D :: 1 : 3 erit M C : B E :: 2 : 3; ſed M C : B E :: A C : A E; ergo A C : A E :: 2 : 3; ergo A C eſt æqualis duabus tertiis partibus A E, & C E æqualis tertiæ parti A E. Quapropter ad inveniendum centrum magnitudinis in triangulo ſatis eſt ex quolibet angulo ducere lineam dividentem biſariam latus oppoſitum, & hanc in tres partes æquales dividere; punctum illud, quod propius eſt lateri, erit centrum magnitudinis requiſitum. Quoniam figuræ rectilineæ in triangula dividuntur, quorum centra magnitudinis inveniuntur, nihil facilius eſt, quam cujuslibet figuræ rectilineæ centra magnitudinis reperire.

46. Poſſem ex his principiis aliarum figurarum centrum magnitudinis determinare; ſed ad prolixitatem fugiendam præſtat non pauca lectorum induſtriæ relinquere. Ad formulas itaque generales tradendas accedamus. Curvæ cujuslibet A B (*Fig.* 30) centrum magnitudinis reperire. Dividatur curva in ſua elementa D d; in quamlibet A F ducatur normalis D E. Centrum magnitudinis ſit C, & ejus diſtantia a linea A F ſit C G. Arcus A D vocetur $= s$, elementum $Dd = ds$. Centrum magnitudinis elementi D d illud biſariam dividit, ſeu a puncto dividente biſariam diſtat per minimum intervallum, ac proinde vocata $DE = y$, elemen-

mentum D d diſtabit a recta A F diſtantia $= y$; ergo quantitas D d in ſuam diſtantiam ducta $= y\,ds$, & omnia producta in ſuas diſtantias $= Sy\,ds$, & omnia elementa D d $= S\,ds = s$; ergo vocata C G $= u$, erit $Sy\,ds = su$, ſive $\frac{Sy\,ds}{s} = u$, ex qua formula facta integratione, & poſita $s =$ A D B, & $y =$ B F, determinabitur diſtantia centri magnitudinis lineæ A D B a recta A F. Quod ſi curva referatur ad focum F, vocata D F $= y$, & angulo A F D $= \frac{\mu}{r}$, erit D E $= \frac{y\,Sc.\mu}{r}$; ergo diſtantia centri magnitudinis curvæ a recta A F, nempe C G $= u = \frac{S\frac{y\,Sc.\mu.ds}{r}}{s}$. Si ejuſdem centri diſtantia ab alia recta quacumque priori non parallela determinetur, quiſque videt, locum hujuſce centri magnitudinis fore determinatum.

31. Dati arcus circularis A B (*Fig.* 31) centrum magnitudinis reperire: Dividatur arcus biſariam in H, & ducatur ad centrum circuli K radius H K. Quoniam A H $=$ B H, & uterque arcus reſpectu lineæ H K eſt ſimiliter poſitus, manifeſtum eſt, centrum magnitudinis totius arcus A H B poſitum eſſe in linea K H; ergo unice inquirenda eſt ejus diſtantia a centro circuli K. Præterea manifeſtum eſt item, duorum arcuum A H, B H centra magnitudinis æque diſtare a linea K I perpendiculari K H; ergo eadem quoque erit diſtantia centri magnitudinis totius arcus A H B, ergo ſatis eſt invenire, quantum centrum magnitudinis arcus H B diſtet a linea K I. Sit H D $= s$, D I $= y$; ergo $u = \frac{Sy\,ds}{s}$; ſed $ds = \frac{-r\,dy}{\sqrt{rr - yy}}$; ergo $u = \frac{S\frac{-ry\,dy}{\sqrt{rr-yy}}}{s}$. Integretur ita ut facta $y = r$, quantitas $= o$, & invenies $u = \frac{r\sqrt{rr-yy}}{s}$, in qua ponenda eſt $s =$ H B, $y =$ B E; ergo K E $=$ B F $= \sqrt{rr - yy}$. Quapropter diſtantia quæ quæritur K C erit tertia proportionalis poſt arcum H B, ſinum B F, & radium; ſeu duplicatis antecedentibus ut arcus A H B, ad ejus chordam A B, ita radius ad K C. Idem invenies, ſi utaris alia formula focum reſpiciente. Nam facto K D $= y = r$, & vocato H D $= s$, erit $\mu = \omega - s$, ſpecies $\omega$ indicat quadrantem; ergo $Sc.\mu = Sc.\overline{\omega - s} = Cc.s$; ergo $u = \frac{S\,ds\,Cc.s}{s} = \frac{Sr\,d\,Sc.s}{s} = \frac{r\,Sc.s}{s}$, quæ eadem eſt ac ſuperior. Si A H B foret ſemicircumferentia, fiet ſemicircumferentia ad diametrum, ſeu ut quadrans ad radium, ita radius ad diſtantiam inter centrum magnitudinis, & centrum circuli. Si arcus ſit integra circumferentia, ejus chorda $= o$; ergo etiam diſtantia centri magnitudinis a centro circuli $= o$, adeoque centrum circuli erit centrum magnitudinis, quod aliunde eſt certiſſimum.

32. Venio ad ſuperficies. Sit ſuperficies quælibet A B F, (*Fig.* 32, 33) cujus centrum magnitudinis C oportet determinare. Dividatur ſuperficies in ſua elementa D E e d, quorum centra magnitudinis K determinantur, quum ſint parallelogramma. In lineam quamlibet L G demittantur normales K L, C G. Su-

per-

perficies creſcens vocetur $= s$, ejus elementum $ds$, $KL = y$, $CG$ ut ſupra $= u$. Erit $\frac{S\,y\,ds}{s} = u$. Quare determinata $s$ per $y$, aut viceverſa, & facta integratione, ut par eſt, determinabitur quantitas $u$, quæ eſt diſtantia centri magnitudinis ſuperficiei datæ a linea GL. Duplici modo repræſentamus diviſam ſuperficiem in ſua elementa, primo modo, ut lineæ dividentes ſint perpendiculares, ſecundo modo ut ſint parallelæ lineæ GL. (*Fig.* 32) Si primo modo fiat, vocata $IL = IG = c$, erit $KI, = y - c$, ſed $KI$, eſt dimidium $DE$; ergo $DE = 2y - 2c$. Vocata $AE = x$, $Ee = dx$, fiet $s = S\,\overline{2y - 2c}\,.\,dx$, & $ds = \overline{2y - 2c}\,.\,dx$. Ergo $CG = u = \frac{2\,S\,y^2 dx - c\,y\,dx}{2\,S\,y\,dx - c\,dx} = \frac{S\,y^2 dx - c\,y\,dx}{S\,y\,dx - c\,dx}$.
Si fiat ſecundo modo, erit $KI, = FE = y - c$, & $Ee = dy$, (*Fig.* 33) præterea ſit $DE = x$, erit $ds = x\,dy$; ergo $CG = u = \frac{S\,x\,y\,dy}{S\,x\,dy}$. Poteſt etiam accidere, ut juvet dividere ſuperficiem per lineas obliquas illi, in quam perpendiculares ducantur. Quod ſi fiat non eſt difficile formulas huic diviſioni convenientes ex ſuperiore determinare. Duxi in lineam LG perpendiculares KL, CG; attamen etiamſi eſſent obliquæ dummodo parallelæ, formulæ eædem provenirent. Hæc omnia breviter adnotanda judicavi, quia ex opportuna præparatione facilior evadit centri magnitudinis determinatio.

33. Omittenda non eſt methodus, per quam ſuperficiei centrum magnitudinis determinari poteſt ex æquatione curvæ ad focum relatæ. Sit area AFB, (*Fig.* 34) cujus centrum magnitudinis C requiritur. Per focum F tranſeat AF, & ex foco dividatur area in triangula minima FDd. Sit $FD = y$, $Dd = dx$, arcus radii $= r$, qui metitur angulum $AFD, = \mu$. Trianguli FDd centrum magnitudinis K diſtat a foco F per diſtantiam $= \frac{2}{3}y$, ut ex ſuperioribus patet; ergo $KL = \frac{2}{3}\,y\,\frac{Sc.\mu}{r}$; atqui $FDd = \frac{y\,dx}{2}$; ergo $CG = u =$
$\frac{\frac{1}{3}\,S.\,Sc.\mu.\,y^2 dx}{\frac{1}{2}\,r\,S\,y\,dx}$, a qua poteſt $dx$ expelli ſubſtituendo $\frac{y\,d\mu}{r}$. Quod ſi velis denominare $KL = y$, invenies $\frac{S\frac{y^2 dx}{Sc.\mu}}{S\frac{y\,dx}{Sc.\mu}} = CG = u$. Quod ſi velis referre ſuperficiem ad lineam MP, quæ diſtat ab AF diſtantia $= c$; tunc formulæ inventæ adde $GP = c$, ut habeas, poſita $FD = y$, $CP = c + \frac{\frac{1}{3}\,S\,Sc.\mu.\,y^2 dx}{\frac{1}{2}\,r\,S\,y\,dx} =$
$\frac{S\,\frac{1}{2}\,c\,r\,y\,dx + \frac{1}{3}\,S\,Sc.\mu.\,y^2 dx}{\frac{r}{2}\,S\,y\,dx}$, vel poſita $KL = y$,

$$CP = c + \frac{S\frac{y^2 dx}{Sc.\mu}}{S\frac{y dx}{Sc.\mu}} = \frac{S\frac{cy dx}{Sc.\mu} + \frac{Sy^2 dx}{Sc.\mu}}{S\frac{y dx}{Sc.\mu}}$$. Quod ſi velis KM $= y$ ſatis eſſet in his formulis pro $y$ ſcribere $y - c$.

50. Sectoris circularis KBHA ( *Fig.* 31 ) centrum magnitudinis invenire. Arcus AB biſariam dividatur in H, & ducatur radius KH. Perſpicuum eſt, centrum magnitudinis ſectoris reperiri in radio KH, quia hic radius eumdem biſariam, & ſimiliter dividit. Igitur ſatis erit, determinare ejus diſtantiam a puncto K, quæ eadem erit ac diſtantia ſectoris HKB a linea KE. Agantur duo radij KD, Kd conſtituentes angulum infiniteſimum. Trianguli KDd centrum magnitudinis G diſtat a puncto K per duas tertias partes radij. Agatur in radium KH normalis GL, ut KL repræſentet diſtantiam centri magnitudinis G a linea KE. Vocetur radius KH $= r$, ut KG $= \frac{2}{3}r$, arcus HD $= \mu$, ejus elementum Dd $= d\mu$, & minimus ſector KDd $= \frac{r d\mu}{2}$. Conſtat fore LK $= \frac{2}{3}$ $Cc.\mu$; ergo ſector DKd ductus in LK fiet $= \frac{r d\mu . Cc.\mu}{3}$; ergo OK diſtantia centri magnitudinis totius ſectoris AHBK a centro K fiet $= \frac{2 Sd\mu . Cc.\mu}{3\mu} = \frac{2r . SdSc.\mu}{3\mu} = \frac{2r . Sc.\mu}{3\mu}$, facto arcu HB $= \mu$. Quare recta KO erit quarta proportionalis poſt arcum HB, ejus ſinum BF, & $\frac{2}{3}r$. Q. E. Inv. Si radio $= \frac{2}{3}r$ deſcribatur arcus circularis intereeptus ab iiſdem radiis KA, KB, hujus arcus centrum magnitudinis idem probabitur eſſe ac centrum magnitudinis ſectoris AHBK. Quo invento, quum centrum trianguli KAB datum ſit, diſtat enim a centro K per duas tertias partes lineæ KF, facili negotio centrum magnitudinis ſegmenti AHBK determinabitur.

51. Cujuſcumque ſegmenti parabolici AKB ( *Fig.* 35 ) centrum magnitudinis invenire. Dividatur BA biſariam in E, per punctum E, ducatur diameter parabolæ KE, quæ quando dividit chordas omnes parallelas BA biſariam, centrum magnitudinis continebit. Sit hoc C; ſatis eſt invenire valorem rectæ KC. Ducta DH, quæ biſariam dividitur in I, & vocata KI $= x$, ID $= y$, ut poſita parametro $= a$, ſit æquatio $ax = yy$. Ex formulis habemus KC $= \frac{Syx dx}{Sy dx} = \frac{Sx^{\frac{3}{2}} dx}{Sx^{\frac{1}{2}} dx} = \frac{3}{5}x$, facta $x =$ KE. Itaque ſi KE dividamus in C, ut KC : CE ſit :: 3 : 2, punctum C erit centrum magnitudinis ſegmenti parabolici AKB.

52. Quæ dicta ſunt de ſuperficiebus, & lineis, eadem valent etiam in ſolidis. Nam vocato ſolido creſcente $= s$, ejus elementis, in quæ diviſum concipitur, $= ds$, diſtantia horum elementorum a plano dato $= y$, erit diſtantia centri magnitudinis ab eodem plano $= u = \frac{Syds}{Sds}$. Exemplum demus in cono deſcri-

ſcripto in revolutione trianguli ADB ( *Fig.* 2 ) circa AB. Patens eſt centrum magnitudinis eſſe in axe coni; quare ſatis eſt determinare quantum diſtet a puncto A. Sumptis abſciſſis in axe $= x$ & diviſo cono per plana parallela circulo deſcripto a BD, vocatoque arcu radij $= r$, qui metitur angulum A $= e$, ordinata reſpondens abſciſſæ $x$ erit $= \frac{Sc.e}{Cc.e} . x$; ergo circumferentia circuli ab ordinata deſcripti $= \frac{p.Sc.e}{r.Cc.e} x$, & circulus $= \frac{p.\overline{Sc.e}^2}{2r.\overline{Cc.e}^2} x^2$; ergo elementum ſolidi, nempe $ds = \frac{p.\overline{Sc.e}^2}{2r.\overline{Cc.e}^2} . x^2 dx$. Quare centri magnitudinis diſtantia a puncto A erit $u = \frac{Sx^3 dx}{Sx^2 dx} = \frac{3x^4}{4x^3} = \frac{3}{4} x$; ergo diviſa AB in ratione 3:1, punctum diviſionis erit centrum magnitudinis.

53. Hæc ſatis eſſe arbitror de centro magnitudinis. Nunc tradenda eſt regula Guldini, quæ docet, quantitatem quamlibet poſitam in uno eodemque plano tranſeunte per axem rotationis producere, dum rotatur, quantitatem, quæ æqualis eſt quantitati producenti ductæ in viam centri magnitudinis. Ita linea generat ſuperficiem æqualem eidem lineæ multiplicatæ per arcum deſcriptum a magnitudinis centro & ſuperficies producit ſolidum æquale eidem ſuperficiei ductæ in viam centri magnitudinis. Regulam hanc vel maxime elegantem probabimus facile primum in lineis, deinde in ſuperficiebus ſeſe rotantibus. Sit curva AM, ( *Fig.* 6 ) quæ tota jaceat in plano tranſeunte per axem rotationis HK; ejus centrum magnitudinis ſit C; ajo ſuperficiem genitam in rotatione æqualem eſſe rectangulo, cujus latus unum ſit ipſa linea AM, alterum circumferentia deſcripta a centro C. Intelligatur diviſa linea in ſua elementa, quorum unum ſit Dd, cujus centrum magnitudinis ſit o illud biſariam dividens. Vocetur $AD = s$, $Dd = ds$, diſtantia puncti o, ſeu D ab axe rotationis, hoc eſt $DK = y$, & $CE = u$, quæ lineæ ſunt normales rotationis axi. Ex tradita theoria habemus $\frac{Syds}{s} = u$, dummodo $s$ fiat $= AM$; ergo $Syds = us$. Hæc æquatio multiplicetur per $\frac{p}{r}$, ut fiat $S\frac{p}{r} y ds = s . \frac{p}{r} . u$. Verum $\frac{p}{r} u$ nihil aliud eſt, quam circumferentia deſcripta a puncto C, & $\frac{p}{r} y$ circumferentia deſcripta a puncto o, quæ circumferentia ducta in elementum $ds$ dat elementum ſuperficiei genitæ; ergo ſumma omnium elementorum ſuperficiei genitæ, ſeu ſuperficies genita a tota linea AM, æqualis eſt AM ductæ in circumferentiam deſcriptam ab ejus centro magnitudinis C. Quod. E. D.

54. Nunc ad ſuperficies. Superficies ABF ( *Fig.* 33 ) exiſtens in plano tranſeunte per GL, circa axem GL rotetur. Ejus centrum magnitudinis ſit C. Ajo ſolidum genitum in rotatione æquale eſſe cylindro, cujus baſis ſit eadem ſuperficies ABF, altitudo vero circumferentia circuli deſcripti a puncto C. Dividatur ſuperficies ABF per lineas parallelas axi rotationis GL in ſua elementa, quorum unum ſit DEed, habens centrum magnitudinis K. Ductis axi rotationis normalibus CG, KL, vocetur $CG = u$, $KL = y$. Superficies creſcens $ADEF = s$, ejus elementum $= ds$. Habebimus ex theoria centri magnitudi-

nis $S.\,y\,ds = us$. Hæc multiplicetur per $\frac{p}{r}$, ut ſit $S\,\frac{p}{r}y\,ds = \frac{p}{r}us$. Atqui $\frac{py}{r}$ eſt circumferentia deſcripta a puncto K, & $\frac{p}{r}y\,ds$ eſt ſolidum genitum ab elemento DEed; ergo $S\,\frac{p}{r}y\,ds$ erit ſolidum genitum ab area AFB; igitur hoc ſolidum $= \frac{p}{r}us$; ſed $\frac{p}{r}u$ eſt circumferentia deſcripta a centro C: igitur ſolidum genitum a ſuperficie ABF æquat eamdem ſuperficiem ductam in circumferentiam deſcriptam a centro magnitudinis C. Quod. E. D.

55. In hac regula maxime univerſali illud adverte, quod accidit, ſi pars lineæ rotatæ, aut ſuperficiei cadat ad alteram plagam axis rotationis, ubi ſitum non eſt centrum magnitudinis. Quoniam elementa ejus partis habent ab axe rotationis diſtantias negativas, ſuperficies, aut ſolidum ab ea parte in rotatione genitum negative accipiendum eſt; ergo differentia duarum quantitatum, quæ in rotatione generantur, erit æqualis quantitati rotatæ ductæ in centrum magnitudinis. Quapropter ſi axis rotationis tranſeat per centrum magnitudinis, quantitates genitæ a partibus hinc inde poſitis æquales erunt. Quod ſi areas obtinere ſummam quantitatum, quæ generantur a duabus partibus hinc inde poſitis, oportet, ut ſingularum partium peculiaria centra magnitudinis determinentur, atque ex his definiantur ſingula ſolida, eorumque ſumma accipiatur.

56. Incredibile dictu eſt, quantum Guldini regula in pleriſque difficillimis inquiſitionibus creet facilitatem. Aliquot exempla proponam. Sint plures lineæ rectæ quomodocumque diſpoſitæ A, B, D, E (*Fig.* 36) quæ rotentur circa axem LG, invenire ſuperficiem quæ æqualis ſit omnibus ſuperficiebus a lineis in rotatione productis. Omnium linearum A, B, D, E inveniatur centrum magnitudinis C; per hoc agatur parallela axi LG linea MN, quæ æqualis ſit omnibus lineis ſimul ſumptis; hæc rotata circa axem LG producet ſuperficiem cylindricam æqualem illis omnibus, quæ gignuntur a lineis A, B, D, E. Simili modo ſi (*Fig.* 37) ſint plures figuræ rectilineæ A, B, quæ rotentur circa axem LG, quæritur cilinder æqualis omnibus ſolidis ortis a rotatione figurarum A, B. Omnium harum figurarum determinetur centrum magnitudinis C, cujus diſtantia ab axe ſit CG; inveniatur rectangulum, omnibus figuris æquale cujus altitudo $= 2\,CG$; hoc autem ſit HFK; quod rotatum circa eumdem axem dabit cylindrum æqualem ſolidis ortis a ſuperficiebus A, B. Quamquam aſſumpſi rectas lineas, & figuras rectilineas; tamen quiſque per ſe cognoſcit, methodum non deficere, ſi lineæ eſſent curvæ, & ſuperficies curvilineæ.

57. Dato circulo ADBE, qui rotetur circa axem LG (*Fig.* 38) invenire proportionem inter ſuperficies genitas a duabus circumferentijs DAE, DBE. Vocato quadrante $AE = q$, radio $= r$, conſtat ex ſuperioribus centrum magnitudinis ſemicircumferentiæ DAE diſtare a centro circuli C diſtantia $CK = \frac{rr}{q}$. Eodem modo centrum magnitudinis alterius ſemicircumferentiæ DBE determinabitur accepta $CH = \frac{rr}{q}$: ergo vocata $CG = u$, erit $GK = u + \frac{rr}{q}$, & $GH = u - \frac{rr}{q}$. Atqui ſuperficies genitæ a duabus ſemicircumferentijs, quum ipſæ ſint æquales, ſunt ut circumferentiæ deſcriptæ in rotatione a punctis K, H, ſeu ut radii GK, GH; ergo ſuperficies deſcripta a ſemicircumferentia DAE

ad

ad eam quæ deſcribitur a DBE $:: u+\frac{rr}{q} : u-\frac{rr}{q} :: qu+rr : qu-rr$. Quapropter ſi quis velit per determinationem diſtantiæ CG efficere, ut ſuperficies genita a ſemicircumferentia DA ſit ad eam, quæ gignitur a ſemicircumferentia DBE in ratione data majoris inæqualitatis $m:n$, fiet $u+\frac{rr}{q} : u-\frac{rr}{q} :: m:n$; ergo componendo & dividendo $2u : \frac{2rr}{q}$, ſive $u : \frac{rr}{q} :: m+n : m-n$. Quare ſumenda erit CG, quæ ſit ad CH, vel CK $:: m+n : m-n$. Similis methodus valet, ſi quæras proportionem inter ſolida genita a duobus ſemicirculis DAE, DBE. Nam horum ſemicirculorum centra magnitudinis K, H diſtant a centro circuli diſtantia $=\frac{2rr}{3q}$; ergo proportio ſolidorum erit $:: u+\frac{2rr}{3q} : u-\frac{2rr}{3q}$. Quod ſi velis hanc datam eſſe, nempe $m:n$, invenies $u : \frac{2rr}{3q} :: m+n : m-n$.

58. Quadrans ellypticus ABE (*Fig.* 39) rotetur primum circa axem majorem AB, deinde circa minorem BE, quæritur proportio ſolidorum, quæ in duabus rotationibus generantur. Ut hoc facilius determinetur, inveniamus diſtantiam centri magnitudinis C ab axe AB. Vocata AB $=a$, BE $=b$, BF $=x$, FD $=y$, ut æquatio ſit $aa-xx : yy :: aa : bb$, conſtat CG $=\frac{Sy^2dx}{2.ABE}=$

$$\frac{\frac{b^2}{a^2}S\,a^2dx-x^2dx}{2.ABE}=\frac{b^2x-\frac{b^2}{3a^2}x^3}{2.ABE},$$

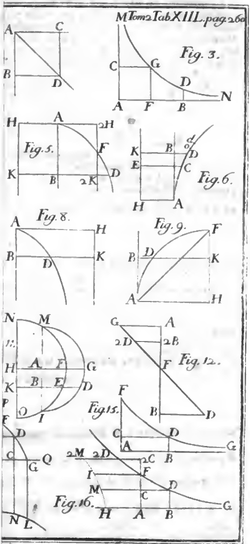
& facta $x=a$, inveniemus CG $=\frac{ab^2}{3.ABE}$. Simili modo determinabitur diſtantia ab axe BE, nempe CH $=\frac{a^2b}{3.ABE}$. Quare ſolidum oriens ex rotatione circa axem AB $=\frac{p}{r}.\frac{ab^2}{3}$, & quod oritur ex rotatione circa axem BE $=\frac{p}{r}.\frac{a^2b}{3}$; ergo primum ſolidum ad alterum erit $:: b:a$, hoc eſt in ratione reciproca axium circa quos fit rotatio. Quod ſi dato ſolido genito per quadrantem ellypticum AEB rotatum circa axem AB velis reperire hemiſphærium æquale, ita analyſim inſtitue. Hoc hemiſphærium ſit illud, quod gignitur a quadrante DCB, (*Fig.* 38) cujus centrum magnitudinis diſtabit a radio DC diſtantia $=\frac{r^3}{3.DCB}$; ergo hemiſphærium $=\frac{pr^2}{3}$; ergo erit $\frac{pr^2}{3}=\frac{pab^2}{3r}$, ſeu $r^3=ab^2$. Igitur radius circuli requiſiti eſt primus ex duabus medijs proportionalibus $b$, $a$. Quod ſi datus fuerit radius circuli, & ſemiaxis $=b$, & quæratur axis $a$, circa quem fit rotatio, hic æquabit quartam proportionalem poſt $a$, $r$. Contra ſi dato $a$ quæratur $b$, hic invenietur æqualis mediæ proportionali inter $a$, & quartam proportionalem poſt $a$, $r$.

59. In univerſo hoc capite ſuppoſuis, rotationem fieri integram & quodlibet punctum deſcribere completam circumferentiam circuli. Verum quiſque facile

cile cognoſcit, ea omnia, quæ dicta ſunt, eodem modo valere, tametſi rotatio non compleatur, dummodo fiat per datum angulum, neque quidquam requiri aliud, niſi ut ſpecies $p$, quæ antea integram circumferentiam radii $r$ deſignabat, deſignet tunc arcum illum datum, per quem fit rotatio. Hæc animadverſio facit, ut regula Guldini aliquando adjuvet calculum integralem; quod ſatis erit illuſtrare exemplo earum curvarum, quæ per modum conchoidis nicomedeæ generantur. Sit curva quælibet A E, ( *Fig.* 40 ) & extra illam poſitus polus C, per quem tranſeat linea C D B, quæ ſecet curvam datam in A, & habeat partes A B, A D æquales. Ita hæc linea moveatur, ut punctum A ſemper permaneat in curva data A E, & ſemper tranſeat per polum C. A punctis B, D deſcribantur curvæ B F f, D G g. Fingamus eam perveniſſe in ſitum C G F; promoveatur in ſitum infinite proximum C g f, & centro C deſcribantur arcus F m, E n, G o. Vocetur $CA = b$, $DA = AB = GE = EF = a$, $CE = y$, $En = dx$. Notum eſt, ſpatium F f g G adæquare zonam circularem F m o G, quæ quum gignatur ex rotatione lineæ F G circa centrum C, cujus centrum magnitudinis eſt E deſcribens arcum E n, erit æqualis $2adx$; ergo ſpatium F f g G $= 2adx$. Dividatur E F bifariam in P, & centro C deſcribatur arcus P p, qui, quum ſit ad $dx :: y + \frac{a}{2} : y$, erit $= dx + \frac{a\,dx}{2y}$; ergo zona F m n E, adeoque ſpatium F f e E $= adx + \frac{a^2 dx}{2y}$. Similiter diviſa G E in Q bifariam deſcriptoque arcu Q q, invenietur ſpatium E e g G $= adx - \frac{a^2 dx}{2y}$. Quapropter differentia ſpatiorum F f e E, E e g G erit $= \frac{a^2 dx}{y}$. Quum autem ex æquatione curvæ E e detur $dx$ per $y, dy$, conſtat omnia quæſita ſpatia inveniri poſſe.

60. Ponamus A E eſſe lineam rectam, & a puncto B gigni conchoidem nicomedeam ſuperiorem, a puncto D inferiorem. Vocetur $AE = s$, erit $CE = y = \sqrt{bb + ss}$, & $Ee = ds$. Ex ſimilitudine triangulorum E e n, C e A, ſive C E A, erit $En = dx : Ee = ds :: CA = b : CE = \sqrt{bb + ss}$; ergo $dx = \frac{b\,ds}{\sqrt{bb + ss}}$. Itaque ſubſtitutis hiſce valoribus habebimus ſpatium F f g G $= 2adx = \frac{2abds}{\sqrt{bb + ss}} = \frac{4a}{b} \cdot \frac{bbds}{2\sqrt{bb + ss}}$; quod pertinet ad quadraturam hyperbolæ. Centro A, ſemiaxe C A deſcribatur hyperbola æquilatera C R, & ex puncto E ordinetur E R, ducaturque A R: notum eſt ſectorem A C R $= S \frac{b^2 ds}{2\sqrt{bb + ss}}$; ergo ſpatium B F G D ad ſectorem A C R erit ut $4a : b$. Differentia inter duo ſpatia F f a E, E e g G inventa eſt $= \frac{a^2 dx}{y}$; ergo ſubſtitutis valoribus $= \frac{a^2 b ds}{bb + ss} = \frac{a^2}{b} \cdot \frac{b^2 ds}{bb + ss}$. Quum autem $S \frac{b^2 ds}{bb + ss}$ ſit arcus circularis radii $= b$, & tangentis $= s$, palam fit differentiam ſpatiorum A B F E, A D G E eſſe ad rectangulum ex prædicto arcu circulari in $a$; ut $a : b$. Quum autem horum

ſpa-




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Tom. 2. Tab. XIV. pag. 260.

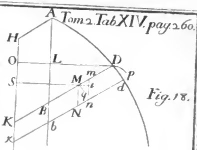



Fig. 18.

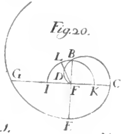



Fig. 20.

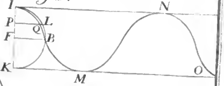



Fig. 21.

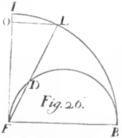
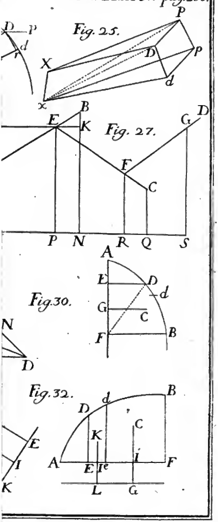


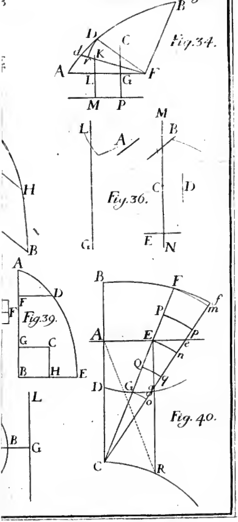

Fig. 26.

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Fig. 25.
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Fig. 27.
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Fig. 30.
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Fig. 32.
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Tom. 2. Tab. XVI pag. 260.



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ſpatiorum ſumma detur ex hyperbolæ quadratura, differentia ex quadratura circuli, apparet, quodlibet ex illis ſpatijs tum a circuli, tum ab hyperbolæ quadratura pendere.

## CAPUT DECIMUM QUINTUM.

### Solvuntur aliquot problemata directa ad quadraturas ſpectantia.

1. PRoblema primum. Quadrare Dioclis Cyſſoidem AMK, ( *Fig.* 1 ) cui eſt æquatio $y^2 = \frac{x^3}{a-x}$, vocata $AB = a$, $AP = x$, $PM = y$. Quare $y\,dx = \frac{x^{\frac{3}{2}}\,dx}{\overline{a-x}^{\frac{1}{2}}} = \frac{x^2\,dx}{\sqrt{ax-xx}}$ exhibet differentiam areæ cyſſoidis APM, quæ ſi integretur, ut evaneſcat, evaneſcente $x$, habebitur area APM quæſita. Inventæ formulæ ſummatoria dependet a quadratura circuli; quod ut pateat ita formula diſponatur $-2.\frac{-x^2\,dx}{2\sqrt{ax-xx}}$. Sepoſito multiplicatore $-2$, cujus opportune rationem habebo, addo & demo $\frac{dx\sqrt{ax-xx}}{2}$, ut evadat $\frac{-dx\sqrt{ax-xx}}{2} + \frac{\overline{ax-2xx}.dx}{2\sqrt{ax-xx}} = \frac{-3}{2}dx\sqrt{ax-xx} + dx\sqrt{ax-xx} + \frac{\overline{ax-2xx}.\,dx}{2\sqrt{ax-xx}}$; ergo integrando $S\frac{-x^2\,dx}{2\sqrt{ax-xx}} = x\sqrt{ax-xx} - \frac{3}{2}S\,dx\sqrt{ax-xx}$. Quantitas algebraica nulleſcente $x$ nulleſcit, ſummatoria autem ſub eadem conditione accipienda eſt. Nunc per coefficiens ſepoſitum $-2$ fiat multiplicatio, ut ſit $S\frac{x^2\,dx}{\sqrt{ax-xx}}$, hoc eſt cyſſoidis area $APM = 3S\,dx\sqrt{ax-xx} - 2x\sqrt{ax-xx}$. Sed producta ordinata PM, ſi opus eſt, uſque ad circulum in N, ſcimus $S\,dx\sqrt{ax-xx}$ æquare aream circularem APN, quæ $= 0$, ſi $x = 0$; & $x\sqrt{ax-xx}$ æquare rectangulum APN; ergo Cyſſoidis area APM æquat differentiam inter tria ſemiſegmenta APN, & duo rectangula APN. Fiat $x = AB = a$, ſemiſegmentum APN evadit ſemicirculus, rectangulum APN evaneſcit. Igitur cyſſoidis ſpatium infinite longum AMKB eſt triplum ſemicirculi genitoris.

2. Alio modo. Ex puncto A agantur AG, Ag in infiniteſimo angulo, ſecantes circulum in N, n, cyſſoidem in F, f, perpendicularem ex B in G, g. Patet

tet, aream minimam FfgG eſſe elementum areæ cyſſoidalis AMFGB. Ut hoc inveniatur, agatur ſu parallela FG. Quoniam angulus ufg = NAn, & ſg = An, erit triangulum ufg = NAn, cujus integrale ſumptum ita, ut evaneſcat facta AN = AB, eſt ſpatium circulare BANSB. Inveniendum nunc quadrilaterum FſuG. Ex f ducatur fi perpendicularis in FG, prædictum quadrilaterum erit = FG.ſi = AN.fi. Ad inveniendam fi, vocetur $AN = z$, $NB = \sqrt{aa-zz}$; ergo No, quæ eſt differentia NB, erit $= \frac{-z\,dz}{\sqrt{aa-zz}}$. Præterea ſcimus AN : AB :: AB : AG, ſeu $z : a :: a : AG = \frac{aa}{z}$; ergo quum $FG = z$, erit $AF = \frac{aa-zz}{z}$. Atqui AN : No :: AF : fi, ſeu $z : \frac{-z\,dz}{\sqrt{aa-zz}} :: \frac{aa-zz}{z} : fi$; ergo $fi = \frac{-dz\sqrt{aa-zz}}{z}$, & $AN.fi = -dz\sqrt{aa-zz}$, hoc eſt minimo ſectori NBn bis ſumpto. Integrale autem NBn ita acceptum, ut evaneſcat facta AN = AB, & $BN = o$, eſt ſegmentum circulare NBS. Igitur area cyſſoidis AMFGB æquat ſpatium ANSB & ſegmentum NSB bis ſumptum, ſeu triplum ſegmentum NSB cum triangulo ANB; ſed facta $AN = o$, triangulum evaneſcit, & ſegmentum NSB evadit ſemicirculus; ergo area cyſſoidis infinite longa AMKB æquat triplicem ſemicirculum genitorem, prorſus ut antea.

3. Problema ſecundum. Si angulus ACB (*Fig. 2*) crurum æqualium AC, CB, quorum CB productum ſit in I, ita moveatur, ut B ſemper remaneat in AD, interea dum CA convertitur circa A, deſcribitur a puncto I curva EID; quæritur area AEIB. Statuatur angulus in ſitu infinite proximo Acb: producantur BC, bc donec concurrant in K, conſtat, angulum K fore æqualem angulo CAc. Nam angulus ABK = AbK + K; ſed ABK = BAC, & AbK = bAc; ergo BAC = bAc + K, & ablato communi bAc, remanet CAc = K. Centro K radio KC deſcribatur arcus Cn, & ducatur AF normalis CB. Angulus ACc = BCn, quia uterque rectus eſt; ergo ablato communi BCc, remanet ACF = cCn; ergo triangula ACF, cCn ſunt ſimilia, adeoque Cc : Cn :: AC : CF; ſed ex ſimilitudine ſectorum Cc : Cn :: AC : CK; ergo AC : CF :: AC : CK; æquantur igitur CF, CK. Dividatur BI biſariam in M, & centro K deſcribatur arcus Mm. Demum centro A radio AC deſcribatur quadrans HCL.

4. His præparatis ſumpto tamquam ſinu toto $AC = r$, vocetur arcus $HC = \mu$, qui metitur angulum HAC, $Cc = d\mu$, quadrans $= \omega$; ergo arcus CL metiens angulum $CAL = \omega - \mu$. Angulus ACG = HAC; ergo ACB = 2. HAC, ac propterea arcus, qui illum metitur, $= 2\mu$; ergo $CF = Cc.2\mu = CK$. Vocetur $BI = b$, erit $KM = r + \frac{1}{2}b + Cc.2\mu$; ſed AC : Cc :: KM : Mm, ſeu $r : d\mu :: r + \frac{1}{2}b + Cc.2\mu : Mm = \frac{r+\frac{1}{2}b.\,d\mu}{r} + \frac{d\mu.\,Cc.2\mu}{r}$, & elementum areæ, hoc eſt BIib = BI.Mm per regulam centri magnitudinis; ergo $BIib = \frac{b}{r}.\,r + \frac{1}{2}b.\,d\mu + \frac{b\,d\mu\,Cc.2\mu}{r} = \frac{b}{r}.\,r + \frac{1}{2}b.\,d\mu + \frac{b}{2}.$

$dSc.2\mu$, & integrando $AEIB = \frac{b}{r}.r + \frac{1}{2}b.\mu + \frac{b.Sc.2\mu}{2}$. Nulla additur conſtans, quia omnia nulleſcunt nulleſcente $\mu$. Propoſita igitur quadratura dependet a quadratura circuli, ut evenire debet, quia curva deſcripta eſt ellypſis, ut probavimus in primo tomo lib. 2, cap. 5., cujus quadraturam conſtat a quadratura circuli dependere.

5. Si quæratur area quadrantis ellyptici ADE, fiet $\mu = \omega$, & $2\mu = 2\omega$; ergo $Sc.2\mu = 0$; igitur area $ADE = \frac{b}{r}.r + \frac{1}{2}b.\omega$. Ex hac formula determinamus quantitatem $b$, ut ellypſis quadrans æquet quadrantem circuli AHL. Quadrans iſte exprimitur per $\frac{r\omega}{2}$; ergo $\frac{b}{r}.r + \frac{1}{2}b = \frac{r}{2}$, ſeu $bb + 2br = rr$; ergo $b + r = \pm r\sqrt{2}$, ſeu $b = \pm r\sqrt{2} - r$. Signum ſuperius valet, quum $b$ ſumitur, ut jacet in figura, ſignum inferius, quum ſumitur ad partes oppoſitas. Recta AD, quæ eſt ſemiaxis primus ellypſis $= 2r + b = \pm r\sqrt{2} + r$; ſed $r$ eſt media proportionalis inter duas $\pm r\sqrt{2} - r$ & $\pm r\sqrt{2} + r$; ergo circulus æqualis ellypſi habet radium medium proportionalem inter duos ſemiaxes ellypſis.

6. Problema tertium. Lineæ datæ BD (*Fig.* 3) motæ intra angulum BAD punctum F deſcribat curvam, quam ſcimus eſſe ellypſim, quæritur area AEFD. Promoveatur BD in ſitum infinite proximum bd, ſeſe decuſſantibus lineis in puncto C. Ut hoc punctum determinetur, agatur AP normalis BD, & centro C deſcribantur minimi arcus bm, Dn; erit CB:CD::bm:Dn, ſeu in ratione compoſita bm:Bm, dn:Dn; ſunt enim æquales Bm, dn; ſed bm:Bm::AP:PB, & dn:Dn::DP:AP; ergo CB:CD in ratione compoſita AP:PB, DP:AP, ſeu ut DP:PB, & componendo CB:BD::DP:BD; igitur CB = DP.

7. His probatis vocetur $FD = a$, $FB = b$, arcus metiens angulum $BAG = \varepsilon$, arcus anguli $BDA = \mu$; ergo arcus anguli $ABD = \varepsilon - \mu$. Conſtat $Sc.\varepsilon : Sc.\overline{\varepsilon - \mu} :: BD = a + b : DA = \frac{\overline{a+b}.Sc.\overline{\varepsilon - \mu}}{Sc.\varepsilon} = \overline{a+b}.\frac{Sc.\varepsilon.Cc.\mu - Cc.\varepsilon.Sc.\mu}{rSc.\varepsilon}$; ſed $AD : DP = CB :: r : Cc.\mu$; ergo

$$CB = \overline{a+b}.\frac{\overline{Cc.\mu}^2}{r^2} \overline{\frac{Cc.\varepsilon.Cc.\mu.Sc.\mu}{r^2Sc.\varepsilon}}.$$

Dividatur FD biſariam in O, & centro C deſcribatur arcus Oo. Angulus DCd eſt differentia anguli ADB; ergo ejus arcus $= -d\mu$. Igitur erit $r : -d\mu :: CO : Oo$; ſed

$$CO = \frac{a}{2} + b - \frac{\overline{Cc.\mu}^2}{r^2}.\overline{a+b} + \frac{Cc.\varepsilon.Sc.\mu.Cc.\mu}{r^2Sc.\varepsilon}\overline{a+b};$$

ergo $Oo =$

$$\frac{-d\mu.\overline{\frac{a}{2} + b}}{r} + \frac{d\mu.\overline{Cc.\mu}^2.\overline{a+b}}{r^3} - \frac{d\mu.Sc.\mu.Cc.\mu.Cc.\varepsilon.\overline{a+b}}{r^3Sc.\varepsilon},$$

quæ du-

ſucta in $\varepsilon$ dabit elementum areæ EFDA. Integratio ita facienda eſt, ut quum $\mu=o$, area evaneſcat. Addita itaque conſtante $=A$ infra determinanda, erit

$$EFDA = A-\frac{a\mu}{r}.\overline{\frac{a}{2}+b}+S\,\frac{ad\mu.\overline{Cc.\mu}^2}{r^3}.\overline{a+b}-$$

$$S\,\frac{ad\mu.Sc.\mu.Cc.\mu.Cc.\varepsilon}{r^3Sc.\varepsilon}.\overline{a+b}$$; atqui ex calculo ſinuum, & coſinuum $\frac{d\mu.Cc.\mu}{r}=dSc\mu$, & $\frac{d\mu.Sc.\mu}{r}=-dCc.\mu$; ergo $EFDA=A-\frac{a\mu}{r}.\overline{\frac{a}{2}+b}+$

$$S\,\frac{adSc.\mu.Cc.\mu}{r^2}.\overline{a+b}+\frac{a\overline{Cc.\mu}^2.Cc.\varepsilon}{2r^2.Sc.\varepsilon}.\overline{a+b}$$; ſed $S\,dSc.\mu.Cc.\mu=$

$\frac{r\mu+Sc.\mu.Cc.\mu}{2}$; ergo $EFDA=A-\frac{ab\mu}{2r}+\frac{a.\overline{a+b}}{2r^2.Sc.\varepsilon}.$

$\overline{Sc.\varepsilon.Sc.\mu.Cc.\mu+Cc.\varepsilon.\overline{Cc.\mu}^2}$; ſed $Sc.\varepsilon.Sc.\mu+Cc.\varepsilon.Cc.\mu=r.Cc.\overline{\varepsilon-\mu}$;

ergo $FDA=A-\frac{ab\mu}{2r}+\frac{a.\overline{a+b}.Cc.\mu.Cc.\overline{\varepsilon-\mu}}{2r.Sc.\varepsilon}$.

8. Si fit $\varepsilon-\mu=o$, unde $Cc.\overline{\varepsilon-\mu}=r$, area evaneſcit; ergo habetur $A-\frac{ab\varepsilon}{2r}+\frac{a.\overline{a+b}Cc.\varepsilon}{2Sc.\varepsilon}=o$, ac proinde $A=\frac{ab\varepsilon}{2r}-\frac{a.\overline{a+b}.Cc.\varepsilon}{2Sc.\varepsilon}$. Quapropter erit area $EFDA=\frac{ab}{2r}.\overline{\varepsilon-\mu}+\frac{a.\overline{a+b}}{2r}.\frac{Cc.\mu.Cc.\overline{\varepsilon-\mu}-rCc.\varepsilon}{Sc.\varepsilon}$.

Supponatur $\varepsilon=$ quadranti, ut $Cc.\varepsilon=o$, erit $EFDA=\frac{ab}{2r}.\overline{\varepsilon-\mu}+\frac{a.\overline{a+b}}{2r}.\frac{Cc.\mu.Cc.\overline{\varepsilon-\mu}}{r}$, & facto $\mu=o$, area erit $=\frac{ab\varepsilon}{2r}$. Si $b=a$ evadit area $=\frac{a^2\varepsilon}{2r}$, ſcilicet quadranti circuli cujus radius $=a$, quemadmodum evenire debet, quum in hac hypotheſi circulus radii $=a$ deſcribatur.

9. Problema quartum. Quadrare, & rectificare curvam problematis quinti cap. 7. Lib. 3. Tom. 1., quæ poſita eſt intra angulum rectum KAH, (*Fig.* 4) & prædita eſt proprietate, ut ducta per X recta MXN æqualem datæ, in eamque demiſſa perpendiculari AO, ſit ſemper MO=XN. Si uterer æquatione ſexti gradus ibi inventa, difficilis, & implicata evaderet problematis ſolutio. Quare per proprietatem expoſitam ita brevius ſolutionem deducam. Per x infinite proximum X duco mxn æqualem datæ, quæ priorem in puncto C ſecabit, atque centro C deſcribo minimos arcus Mp, nq, atque in mn duco normalem Ao, quæ ſecabit MN in t. Quoniam MN, mn æquales ſunt, liquet mp=Nq. Vocata $MN=a$, $MO=z$; ergo $AO=\sqrt{az-zz}$, cujus differentia $to=\frac{adz-2zdz}{2\sqrt{az-zz}}$. Erit $XO=CO=Ct=a-2z$, &

XN

$XN = Cn = z$. Jam vero $Ct : to :: Cn : nq$, ſeu $a - 2z : \frac{adz - 2zdz}{2\sqrt{az - zz}} :: z : nq = \frac{zdz}{2\sqrt{az - zz}}$; ergo elementum ſpatij KXN, ſcilicet $nCq = \frac{z^2 dz}{4\sqrt{az - zz}}$, cujus ſumma, ut ex ſupra inventis num. 1. conſtat, ita ſeſe habet $KXN = \frac{3}{4} S dz\sqrt{az - zz} - \frac{1}{2} z\sqrt{az - zz}$. Nulla additur conſtans, quia facta $z = o$, area KXN evaneſcit. Area itaque noſtræ curvæ a quadratura circuli pendet. Si fiat $z = a$, area KAH invenitur æqualis tribus quartis partibus ſemicirculi radii $= \frac{a}{2}$, & tota area KFGH æqualis tribus ſemicirculis ejuſdem radij. Quare ſpatium infinite longum AMK (*Fig.* 1) cyſſoidis, cujus circulus genitor habet radium $= \frac{a}{2}$, erit æqualis ſpatio integro noſtræ curvæ KFGH (*Fig.* 4).

10. Venio ad rectificationem. $Xx = Cx + CX = xn - XN + qN = mo - MO + qN$; ſed $mo - MO = dz$; ergo $Xx = dz + qN$. Eſt autem $AO : ON :: qn : qN$, ſive $\sqrt{az - zz} : a - z :: \frac{zdz}{2\sqrt{az - zz}} : qN = \frac{dz}{2}$; igitur $Xx = dz + \frac{dz}{2} = \frac{3}{2} dz$, & integrando $KX = \frac{3}{2} z$; non eſt addenda conſtans, quia $KX = o$, quum $z = o$; igitur facta $z = a$, fiet $KXH = \frac{3}{2} a$, & integra curva intra quatuor angulos deſcripta $KHGF = 6a$.

11. Problema quintum. Quadrere, & rectificare curvam problematis 6 Cap. 7. lib. 3. tom. 1., quæ continetur intra angulum rectum MAN, (*Fig.* 5) cujus proprietas eſt, ut per quodlibet punctum O ducta MN æquali datæ, AO eidem ſit perpendicularis. Accipiatur punctum o infinite proximum O, & per illud ducatur mn æqualis datæ, & jungatur Ao ſecans MN in t. Elementum ſpatij quæſiti erit $\frac{AO.Ot}{2}$, quod algebraice eſt determinandum. Quoniam in tertio problemate oſtendimus, punctum interſectionis X linearum MN, mn abſcindere $XN = MO$, fiat $XN = MO = z$; ergo $XO = a - 2z$; voca $MN = a$, & $AO = \sqrt{az - zz}$, & $to = \frac{\left(\frac{a}{2} - z\right).dz}{\sqrt{az - zz}}$.

Præterea ob angulos in O, o rectos & ad verticem in t æquales, erit OAo æqualis OXo, adeoque triangula AOt, tXo ſunt ſimilia; ergo $Xt : to :: AO : Ot$, ſeu $a - 2z : \frac{\left(\frac{a}{2} - z\right).dz}{\sqrt{az - zz}} :: \sqrt{az - zz} : tO = \frac{dz}{2}$. Itaque $AOo = \frac{dz\sqrt{az - zz}}{4}$, cujus ſumma eſt quarta pars ſemiſegmenti circularis, exiſtente diame-

tro $= a$, & ſinu verſo $= z$: Non eſt addenda conſtans, quia ſegmentum A O, & ſegmentum circulare ſimul evaneſcunt. Sequitur, facta $z = a$, ſolium integrum A O F A eſſe ad ſemicirculum radii $= \frac{1}{2}a$, ut 1 : 4, & quatuor folia in quatuor angulis poſita eſſe æqualia eidem ſemicirculo, ſeu circulo radij $= \frac{a}{2\sqrt{2}}$.

12. Quadratam curvam, demus operam, ut rectificemus. Arcus

$$Oo = \sqrt{Ot^2 + ot^2} = \sqrt{\frac{dz^2}{4} + \frac{\frac{aa}{4} - az + zz}{az - zz} dz^2} = \frac{dz\sqrt{aa - 3az + 3zz}}{2\sqrt{az - zz}}.$$

Ut hæc formula ad ſimpliciorem redigatur, pone A O $= u$, ut ſit $\sqrt{az - zz} = u$, erit $z = \frac{a}{2} - \sqrt{\frac{aa}{4} - uu}$, & $dz = \frac{u\,du}{\sqrt{\frac{aa}{4} - uu}}$. Facta ſubſtitutione $Oo = \frac{du\sqrt{aa - 3uu}}{2\sqrt{\frac{aa}{4} - uu}}$. Ex illis, quæ docuimus cap. 12 cognoſces, formulam hanc a rectificatione ellypſis unice dependere. Sed non pigeat, exponere hic conſtructionem. Sumatur A Z $= \frac{a}{2}$ pro ſemiaxe majore, & A H $= \frac{a}{4}$ pro ſemiaxe minore, deſcribaturque ellypſis Z Y H; tum abſcindatur A K $=$ A O $= u$, erigaturque normalis K Y; erit H Y æqualis arcui A O, & F O $=$ Z Y, & A O F æqualis quadranti H Y Z, & totius folii perimeter æqualis perimetro demiellypſis. Facilem demonſtrationem lectoris induſtriæ relinquo.

13. Problema ſextum. Curvæ exponentialis, cujus æquatio, exiſtente protonumero $= f$, ſit $f\left(\frac{x}{f}\right)^{\frac{x}{f}} = y$, quadraturam invenire. Quando nulla alia methodus ſuppetit, ad ſeries neceſſe eſt confugere. Ex num. 15. cap. 6. conſtat, vocato logarithmo $= z$, ejus numerum eſſe æqualem ſeriei

$$f.\,1 + \frac{z}{c} + \frac{z^2}{2c^2} + \frac{z^3}{2.3.c^3} + \frac{z^4}{2.3.4.c^4} \text{ \&c.},$$

exiſtente $c$ ſubtangente ſyſtematis logarithmici. Quoniam quantitatis $f.\left(\frac{x}{f}\right)^{\frac{x}{f}}$ logarithmus æquat $\frac{x}{f}lx$, pro $z$ ſubſtituamus hunc valorem in ſerie ſuperiore, ut habeamus

$$f.\left(\frac{x}{f}\right)^{\frac{x}{f}} = y = f.\,1 + \frac{xlx}{fc} + \frac{x^2\,\overline{lx}^2}{2f^2c^2} + \frac{x^3lx^3}{2.3.f^3c^3} + \frac{x^4lx^4}{2.3.4.f^4c^4} \text{ \&c.}$$

Igitur elementum areæ quæſitæ erit

$$y\,dx = f.\,dx + \frac{x\,dx\,lx}{fc} + \frac{x^2dx\,lx^2}{2f^2c^2} + \frac{x^3dx\,\overline{lx}^3}{2.3.f^3c^3} + \frac{x^4dx\,lx^4}{2.3.4.f^4c^4} \text{ \&c.}$$

Ex num.

num. 2. cap. 9. invenienda ſunt ſingulorum terminorum ſummatoriæ hoc modo

$$S\,dx = x$$

$$S\,x\,dx\,lx = \frac{1}{2}x^2 lx - \frac{1}{2^2}cx^2$$

$$S\,x^2 dx\,\overline{lx}^2 = \frac{1}{3}x^3\overline{lx}^2 - \frac{2}{3^2}cx^3 lx + \frac{2}{3^3}c^2x^3$$

$$S\,x^3 dx\,\overline{lx}^3 = \frac{1}{4}x^4\overline{lx}^3 - \frac{3}{4^2}cx^4\overline{lx}^2 + \frac{2\cdot3\,c^2x^4 lx}{4^3} - \frac{2\cdot3\,c^3x^4}{4^4}$$

$$S\,x^4 dx\,\overline{lx}^4 = \frac{1}{5}x^5\overline{lx}^4 - \frac{4}{5^2}cx^5\overline{lx}^3 + \frac{3\cdot4}{5^3}c^2x^5\overline{lx}^2 - \frac{2\cdot3\cdot4}{5^4}c^3x^5 lx + \frac{2\cdot3\cdot4}{5^5}c^4x^5.$$

Quum progreſſionis lex ſatis pateat, facillimum eſt tabulam ad alios terminos extendere.

14. Hos valores ita integratos ſubſtitusmus in inventa formula areæ, eoſque, prout commodum eſt, in hunc modum diſtribuamus

$$S\,y\,dx = f.\left|\begin{array}{l} x + \frac{1}{2}\cdot\frac{x^2 lx}{fc} + \frac{1}{2\cdot3}\cdot\frac{x^3\overline{lx}^2}{f^2c^2} + \frac{1}{2\cdot3\cdot4}\cdot\frac{x^4\overline{lx}^3}{f^3c^3} + \frac{1}{2\cdot3\cdot4\cdot5}\cdot\frac{x^5\overline{lx}^4}{f^4c^4} \\ -\frac{1}{2^2}\cdot\frac{x^2}{f} - \frac{1}{3^2}\cdot\frac{x^3 lx}{f^2c} - \frac{1}{2\cdot4^2}\cdot\frac{x^4\overline{lx}^2}{f^3c^2} - \frac{1}{2\cdot3\cdot5^2}\cdot\frac{x^5\overline{lx}^3}{f^4c^3} \\ \qquad + \frac{1}{3^3}\cdot\frac{x^3}{f^2} + \frac{1}{4^3}\cdot\frac{x^4 lx}{f^3c} + \frac{1}{2\cdot5^3}\cdot\frac{x^5\overline{lx}^2}{f^4c^2}\ \&c. \\ \qquad\qquad - \frac{1}{4^4}\cdot\frac{x^4}{f^3} - \frac{1}{5^4}\cdot\frac{x^5 lx}{f^4c} \\ \qquad\qquad\qquad + \frac{1}{5^5}\cdot\frac{x^5}{f^4} \end{array}\right.$$

Hæc formula exhibet quadraturam generalem propoſitæ formulæ expreſſam per logarithmos. Verum in caſu peculiari, ubi $x = f$, elegantiſſima evadit. Etenim quum in hoc caſu $lx$, omneſque ejus poſitivæ poteſtates evadant $= o$, omnes termini evaneſcunt præter eos, qui in ſeriebus horizontalibus locum primum tenent; unde proveniet $S\,y\,dx = ff.\ 1 - \frac{1}{2^2} + \frac{1}{3^3} - \frac{1}{4^4} + \frac{1}{5^5} - \frac{1}{6^6}$ &c. Series hæc & elegans eſt, & ita celeriter convergit, ut terminus decimus $= \frac{1}{10^{10}}$ non præbeat niſi unam decimam milleſimam millioneſimam unitatis partem. Quæ de quadratura hujus curvæ diximus, deſumpſimus ex tertio tomo operum Joannis Bernoullij viri celeberrimi.

 15. Pro-

15. Problema ſeptimum. Quadrare curvam quarti gradus, cujus æquatio eſt $y = \frac{r^4 + r^3x + 2r^2x^2 + 2rx^3 + 2x^4}{2.\overline{r+x}.\overline{r^2+x^2}}$. Hoc problema propono, ut diſcant analyſtæ vitare genus paralogiſmi, in quod facillimum eſt labi. Ad ſolutionem hanc primum methodum ſequutus ſum. Quoniam fractio impura eſt, & majus eſt exponens ſpeciei $x$ in numeratore, quam in denominatore, per diviſionem, ſeparatis integris, ad puram deducatur hoc modo $y = x + \frac{r^4 - r^3x}{2.\overline{r+x}.\overline{r^2+x^2}}$. Fractionis tum numerator, tum diviſor multiplicetur per $r-x$, ut fiat

$y = x + \frac{r^3.\overline{r^2 - 2rx + xx}}{2.\overline{r^2-x^2}.\overline{r^2+x^2}}$. Fractio dividatur in duas ita

$y = x + \frac{r^3}{2.\overline{r^2-x^2}} - \frac{r^4x}{r^4-x^4}$. Jam vero multiplicetur per $dx$, ut $ydx$ exhibeat elementum ſpatii quæſiti $ydx = xdx + \frac{r^3dx}{2.\overline{r^2-x^2}} - \frac{r^4xdx}{r^4-x^4}$. In ultima fractione pone $x^2 = rz$, ut oriatur $ydx = xdx + \frac{r^3dx}{2.\overline{r^2-x^2}} - \frac{r^3dz}{2.\overline{r^2-z^2}}$.

16. Duæ formulæ $\frac{r^3dx}{2.\overline{r^2-x^2}}$, $\frac{r^3dz}{2.\overline{r^2-z^2}}$ ſunt elementa ſectoris hyperbolici, cujus ſinus totus $=r$, tangentes autem ſunt $x$, $z$. Sumpta $AC = r$, deſcribatur æquilatera hyperbola, cujus tangens $CB$. (*Fig. 6*) Sume $CD = x$, $CE = z = \frac{x^2}{r}$, & duc $ADF$, $AEG$. Notum eſt $S\frac{r^3dx}{2.\overline{r^2-x^2}}$ = ſectorem $ACF$, & $S\frac{r^3dz}{2.\overline{r^2-z^2}}$ = ſectorem $ACG$. Præterea triangulum $ACE = \frac{rz}{2} = \frac{x^2}{2}$; ergo $Sydx = ACE + ACF - ACG = ACF - ECG$. Nulla in integratione additur conſtans, quia factis $x, z = 0$, omnia evaneſcunt. Igitur curvæ propoſitæ quadratura dependet ab hyperbolæ quadratura. Accepi $x < r$. Verum ſi foret $x > r$, formula ita eſſet diſponenda $Sydx = \frac{x^2}{2} + S - \frac{r^3dx}{2.\overline{x^2-r^2}} - S - \frac{r^3dz}{2.\overline{z^2-r^2}}$, atque formulæ exhiberent ſectores hyperbolicos datos per cotangentes $x$, $z$. Quare ſemiaxe $AV = AC$ deſcribatur hyperbola æquilatera

VLI,

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VLI, cujus cotangens erit CBK. In hac abſcinde $CH = x$, & $CK = z = \frac{x}{r}$; & duc ALK, AHI. Sector $VAI = S - \frac{r^3 dx}{2 . \overline{x^2 - r^2}}$, & ſector $VAL = S \frac{- r^3 dz}{2 . \overline{z^4 - r^2}}$.

17. Hæc omnia apprime cum veritate conſentiunt. Sed ad conſectaria conſtructionis deſcendens animadvertebam, $S y dx$ algebraice obtineri, & eſſe æqualem triangulo ACE, quoties duo ſectores AFC, AGC fuerint æquales, & eorum differentia nulla. Hoc autem continget quotieſcumque $CD = CE$, ſeu $x = z = \frac{rxx}{r}$, aut $x = r$. Igitur ſi ponatur $x = CB = r$, quæ ſuppoſitio trahit $z = r$; $S y dx$ æquabit triangulum $ACB = \frac{rr}{2}$; igitur inferebam, curva propoſita licet generatim habeat quadraturam dependentem a quadratura hyperbolæ, tamen ſi ponatur $x = r$, ſpatium hoc unicum habebit algebraice quadrabile. De hoc conſectario nihil ambigebam.

18. Hoc ut magis magiſque confirmarem, ad aliam methodum me contuli, & reaſſumpta æquatione $y = x + \frac{r^4 - r^3 x}{2 . \overline{r + x} . \overline{r^2 + x^2}}$, fractionem, ut moris eſt, in duas partitus ſum hoc modo $y = x + \frac{r^2}{2 . \overline{r + x}} - \frac{r^2 x}{2 . \overline{r^2 + x^2}}$, quam duxi in $dx$, ut fieret $y dx = x dx + \frac{r}{2} . \frac{r dx}{r + x} - \frac{r x dx}{r^2 + x^2}$. Quoniam utraque formula eſt logarithmica, ad integrationem progrediebar, quæ eſt hujuſmodi: $S y dx = \frac{x^2}{2} + \frac{r}{2} . \overline{l \overline{r + x} - l \sqrt{r^2 + x^2}}$. Non addidi conſtantem, quia ita evaneſcit facta $x = o$, prout evaneſcere debet. Pone jam $x = r$, & invenies $S y dx = \frac{r^2}{2} + \frac{r}{2} . \overline{l 2r - l r \sqrt{2}}$. Quum ad hanc formulam perveniſſem, obſtupui. Nam quum $l 2r - l r \sqrt{2}$ numquam poſſit eſſe $= o$, fieri non poteſt, ut facta $x = r$, $S y dx$ ſit algebraica, & æqualis $\frac{rr}{2}$, quemadmodum ex priori methodo collegeram, ſed ſemper dependet ab hyperbolæ quadratura. Num geometria ſibi ipſi contradicit, & una methodus alteri adverſatur?

19. Obſervabam, conſectarium primæ methodi a veritate non fore alienum, ſi orta fuiſſet æquatio $y dx = x dx + \frac{r}{2} . \frac{r dx}{r + x} - \frac{2 r x dx}{r^2 + x^2}$. Nam ita facta integratione, ut omnia evaneſcant, ſi ſiat $x = o$, invenies

$S y dx$

$S y d x = \frac{x^2}{2} + \frac{r}{2} . \overline{l \overline{r+x} - l \frac{rr+xx}{f} - lr + l \frac{rr}{f}}$. Quiſque videt $r$ eſſe ſubtangentem, $f$ protonumerum ſyſtematis logarithmici. Jam vero fac $x = r$, & habebis $S y d x = \frac{r^2}{2} + \frac{r}{2} . \overline{l 2r - l \frac{2rr}{f} - lr + l \frac{rr}{f}}$. Quoniam $2r : r :: \frac{2rr}{f} : \frac{rr}{f}$, erit arithmetice $l 2r : lr :: l \frac{2rr}{f} : l \frac{rr}{f}$; ergo $l 2r + l \frac{rr}{f} = l \frac{2rr}{f} + lr$, ſive $l 2r - l \frac{2rr}{f} - lr + l \frac{rr}{f} = 0$; ergo $S y d x = \frac{rr}{2}$. Curva igitur licet univerſaliter habeat quadraturam dependentem a quadratura hyperbolæ, tamen unicum ſpatium reſpondens abſciſſæ $x = r$ habet algebraice quadrabile, & æquale dimidio quadranti $rr$. Poſt hanc animadverſionem dubitavi primum, me in calculo faciendo errorem admiſiſſe, ſed iterum iterumque inſtituens diligentiſſime calculum, errorem deprehendi nullum. Quare quomodo ſe res haberet, anceps, & incertus hærebam.

20. Inter has dubitationes omnia diligentiſſime inſpiciens, obſervavi, utrumque ſectorem ACF, ACG evadere infinitum, ſi tam CD, quam CE evadit $= r$. Poſt hanc obſervationem ſuſpicatus ſtatim ſum, differentiam duorum ſectorum infinitorum, licet reſpectu ipſorum ſit infiniteſima, tamen poſſe eſſe quantitatem finitam. Quod ſi accidat, ſine dubio paralogiſmum committit, qui eam facit $= 0$. Quid eveniat in re noſtra videndum eſt. Sume $CB = r$, & duc, ac produc AB, quæ erit aſymptotum hyperbolæ CGF. Accipe infiniteſimam $Bn = s$, ut $Cn = r - s$. Fiat ut $CB : Cn :: Cn : Cm$ ſive $r : r - s :: r - s : Cm = \frac{rr - 2rs + ss}{r} = r - 2s$, omiſſis, quæ reſpectu $r$ infiniteſima ſunt; igitur $Bm = 2s$, & $nm = s$. Agantur jam Anq, Amp. Quæritur quinam futurus ſit ſector pAq, qui eſt differentia duorum ſectorum ACq, ACp. Facile eſt probare, poſita $CD = x$, fore rectam $AF = \frac{r\sqrt{rr+xx}}{\sqrt{rr-xx}}$. Fiat jam $CD = Cn = r - s$, & proveniet $Aq = \frac{r\sqrt{2rr - 2rs + ss}}{\sqrt{2rs - ss}} = \frac{r\sqrt{r}}{\sqrt{s}}$, quæ infinita eſt. Similiter facta $x = Cm = r - 2s$, invenietur $Ap = \frac{r\sqrt{2rr - 4rs + 4ss}}{\sqrt{4rs - 4ss}} = \frac{r\sqrt{r}}{\sqrt{2s}}$. Quare $Aq : Ap$ erit ratio finita, & eadem ac ratio $\sqrt{2} : 1$, & ipſarum differentia $= \frac{r\sqrt{r}}{\sqrt{s}} . \frac{\sqrt{2} - 1}{\sqrt{2}}$. Centro A intervallis Am, Ap deſcribantur minimi arcus mo, pr. Triangulum mno eſt ſimile AnC, quia præter angulos rectos, habent communem angulum in n; ergo erit $An : AC :: nm : mo$, ſive $r\sqrt{2} : r :: s : mo = \frac{s}{\sqrt{2}}$. Præterea $Am : mo :: Ap : pr$, ſive $r\sqrt{2} : \frac{s}{\sqrt{2}} :: \frac{r\sqrt{r}}{\sqrt{2s}} : pr = \frac{\sqrt{rs}}{2\sqrt{2}}$. Igitur triangulum $Apr = \frac{r\sqrt{r}}{2\sqrt{2s}} . \frac{\sqrt{rs}}{2\sqrt{2}} = \frac{rr}{8}$,

quod

quod finitum eſt. Inſuper ducta recta pq invenietur triangulum $Apq = \frac{r\sqrt{r}}{2\sqrt{r}} \cdot \frac{\sqrt{rr}}{2\sqrt{2}} = \frac{rr}{4\sqrt{2}}$, quod pariter finitum eſt; ergo ſector Apq, qui medius eſt inter duas quantitates finitas, finitus erit; atque adeo omitti non poteſt differentia duorum ſectorum, qui habentur facta $x = o$. Itaque ſi unquam accidit, ut in formula ſeſe offerant duæ quantitates infinitæ, quarum differentia capienda ſit, diſcat analyſta ex hoc exemplo, eam differentiam non negligere, niſi prius probet, eam eſſe minorem quacumque data relate ad quantitates finitas.

21. Sed quæret fortaſſe aliquis, ut determinem quantitatem finitam æqualem differentiæ duorum ſectorum ACq, ACp, (*Fig.* 7) ſeu ſectorem Apq. Centro A radijs Ap, Aq deſcribe arcus circulares ps, qu deſinentes in aſymptotum. Conſtat $As = Ap$, $Au = Aq$; ſupra probatum eſt $Ap : Aq :: 1 : \sqrt{2}$; ergo $As : Au :: 1 : \sqrt{2}$. Præterea perſpicuum eſt, quadrilaterum spqu æquare ſectorem Aps. Ex vertice C in aſymptotum duc CD normalem, ut $AD = \frac{r}{\sqrt{2}}$. Abſcinde $AE = r$, & duc EF parallelam CD, demum rectam AF. Quoniam $AD : AE :: \frac{1}{\sqrt{2}} : 1 :: 1 : \sqrt{2}$, erit $AD : AE :: As : Au$; ergo ex natura hyperbolæ ſpatium $DCFE = spqu$; ſed primum æquat ſectorem ACF, ſecundum æquat ſectorem Apq: ergo $ACF = Apq$ ſcilicet differentiam duorum ſectorum infinitorum. His poſitis, facile eſt determinare tangentem CG. Nam ex natura hyperbolæ $EF = \frac{r}{2}$; ergo $AF = \frac{r}{2}\sqrt{5}$; ſed vocata $CG = x$, eadem $AF = \frac{r\sqrt{rr+xx}}{\sqrt{rr-xx}}$; ergo $\frac{r\sqrt{5}}{2} = \frac{r\sqrt{rr+xx}}{\sqrt{rr-xx}}$, aut $\sqrt{5} \cdot \sqrt{rr-xx} = 2\sqrt{rr+xx}$, ſive $5rr - 5xx = 4rr + 4xx$, ex qua $9xx = rr$, & $x = \frac{r}{3}$, ergo differentia ſectorum infinitorum æquat ſectorem, cujus tangens eſt pars tertia ſemidiametri hyperbolici.

22. Quod ſi in formula $Sydx = \frac{x^2}{2} + S\frac{r^3 dx}{2 \cdot r^2 - x^2} - S\frac{r^3 dz}{2 \cdot r^2 - z^2}$, ita daretur $z$ per $x$, ut fieret $z = x$; poſita $x < r$, tunc quidem licet $Sydx$ generatim dependeret ab hyperbolæ quadratura, tamen unicum valorem haberet algebraice integrabilem. Hoc autem continget, ſi fuerit $z = \frac{mx^2}{r}$, exiſtente $m > 1$. Nam tunc facta $x = \frac{r}{m}$, erit quoque $z = \frac{r}{m}$. Quum autem in hoc caſu ſectores proveniant finiti, eorum differentia non poteſt non eſſe minor quacumque data: ergo facta $x = \frac{r}{m}$, fiet $Sydx = \frac{rr}{2mm}$.

23. Problema octavum. Quadrare, & rectificare curvam, quæ generatur, ſi ex quodlibet puncto A circumferentiæ circuli AOC (*Fig.* 8) ducantur infinitæ AOP, & abſcindantur ubique OP æquales datæ. De hac egimus in Tom. 1. lib.

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2. lib. 3. num. 9. Cap. 7. Per centrum circuli B agatur AD, & vocetur $AB = a$, $CD = 2b$. In angulo infinite exiguo aganiur duæ AP, Ap, quæ ſecent circumferentiam in O, o. Ducantur CO, Co, quæ ſecabit AO in m. Demum OP dividatur bifariam in V, & centro A deſcribantur minimi arcus Vu, pn. Sit $CO = z$; ergo $mo = dz$, & $AO = \sqrt{4aa - zz}$, & $AV = b + \sqrt{4aa - zz}$. Quoniam AO : mo :: AV : Vn habebimus $\sqrt{4aa - zz} : dz :: b + \sqrt{4aa - zz} : Vu = \frac{b\,dz}{\sqrt{4aa - zz}} + dz$. Elementum POop per regulam centri magnitudinis fiet $= \frac{2b^2 dz}{\sqrt{4aa - zz}} + 2b\,dz$, ſeu $CDPO = \frac{b^2}{a^2} S. \frac{2a^2 dz}{\sqrt{4aa - zz}} + 2bz$; ſummatoria ita capienda, ut, $z$ evaneſcente, evaneſcat. Ad hanc ſummatoriam inveniendam adverte, ſimilia eſſe triangula mOo, OCA; ergo OA : AC :: om : Oo, ſeu $\sqrt{4aa - zz} : 2a :: dz : Oo = \frac{2a\,dz}{\sqrt{4aa - zz}}$; ergo ſector $OBo = \frac{aa\,dz}{\sqrt{4aa - zz}}$; & integrando, ut par eſt, $S \frac{2a^2 dz}{\sqrt{4a^2 - zz}} =$ duplici ſectori CBO; ergo ſpatium quæſitum CDPO æquat duplicem ſectorem CBO ductum in $\frac{bb}{aa}$, & duplex triangulum COP.

24. Applicemus formulam hanc caſui peculiari, & oſtendamus cautionem, quæ uſurpanda eſt. Sit $b = a$. Perſpicuum eſt in hoc caſu ſpatium COPD eſſe æquale duplici ſectori COB, & duplici triangulo COP. Per punctum A agatur AG tangens circulum. Spatium DPGAOC clauſum inter ſemicircumferentiam AOC, inter curvam DPG, & duas rectas æquales CD, AG æquat duplicem ſemicirculum AOC, & duplex triangulum GAC, ſeu quadratum AC. Hactenus nihil difficile fuit applicare formulam. Verum quum punctum O procedit ad alteram partem circumferentiæ in 2O, & P in 2P cavendum eſt erroris periculum. Promoveatur 2O2P in 2o2p. Si quæratur ſpatium AOCD2P, huic nihil additur aliud, quam elementum A2P2p: & formulæ additur ſpatium 2PA2p, & demitiur 2OA2o: ergo duplex ſector major ſemicirculo BCO2O ſimul cum duplici triangulo C2O2P æquare debet ſpatium CD2PAOC, dempto circulari ſegmento A2O. Itaque ſi punctum 2O progrediatur uſque ad C, quum ſegmentum fiat æquale ſemicirculo, & $C2O = 0$, fiet ſpatium clauſum a ſemicircumferentia, a tota curvà APD, & a recta CD, dempto ſemicirculo A2OC, æquale duplici circulo; ergo idem ſpatium æquale quinque ſemicirculis. Hæc cautio ſufficiet etiam in omnibus illis caſibus, in quibus $CD > CA$.

25. Ut modum diſcamus, quo tractari poſſunt caſus, ubi $CD < CA$, (*Fig.* 9.) ponamus $CD = a$. Curva ingredietur circulum, & intra ipſum faciet folium, cujus dimidium eſt A3PB. Accommodetur $AI = AB$, & jungatur CI. Quæ dicta ſunt numero ſuperiore oſtendunt ſpatium clauſum inter duas curvas COA, DPGA æquale duplo ſectori majori ſemicirculo BCOAI, ſimul cum ſegmento IA, & duplici triangulo AIC. Quare reſtat, ut quadremus folium.

Eam-

Eamdem methodum uſurpantes agamus A 3 P 3 O, A 3 p 3 o in angulo infinite acuto. Inveniemus ſpatium 3 O 3 P 3 p 3 o $= adz - \frac{1}{4}\frac{2a^2 dz}{\sqrt{4aa - zz}}$ vocata C 3 O $= z$; ſeu integrando, & ponendo $z = a\sqrt{3}$ habebimus ſpatium C B 3 P A I C $= aa\sqrt{3}$ dempto dimidio ſectoris C B I; ſeu duplici triangulo C I A dempto dimidio ſectoris C B I. Quare folium A 3 P B æquabit ſpatium C A I addito dimidio C B I, dempto duplici triangulo C I A, ſeu $\frac{3}{2}$ C B I, dempto triplici triangulo B I A, ſeu tribus ſectoribus B I A demptis tribus triangulis B I A, ſeu tribus ſegmentis A I. Hiſce determinatis omnia remanent determinata. Eamdem methodum in reliquis caſibus ſequere.

26. Ad rectificationem obtinendam opportunum eſt denominare A O $= u$, (*Fig.* 8.) unde C O $= \sqrt{4aa - uu}$, & ejus differentia m o $= \frac{-udu}{\sqrt{4aa - uu}}$; atqui A O : m o :: A P : n p ſeu $u : \frac{-udu}{\sqrt{4aa - uu}} :: 2b + u : $ n p $= \frac{-du . \overline{2b + u}}{\sqrt{4aa - uu}}$.

Patet n P $=$ m O $= du$; ergo P p $= -\sqrt{du^2 + \frac{du^2 . \overline{2b + u}^2}{4aa - uu}} =$

$$- \frac{2du\sqrt{aa + bb + bu}}{\sqrt{4aa - uu}} = 2\sqrt{b} . \frac{-du\sqrt{\frac{a^2 + bb}{b} + u}}{\sqrt{2a + u} . \sqrt{2a - u}}.$$

Appoſui ſignum $-$, quia arcus D P, quem inquiro, creſcit decreſcente recta A O, & viceverſa. Formula, ad quam pervenimus generatim dependet a rectificatione ellypſis, ſed in hypotheſi, ubi $b = a$, eſt algebraice integrabilis; provenit enim P p $= 2\sqrt{a} . \frac{-du}{\sqrt{2a - u}}$, & integrando D P $= 4\sqrt{a} . \sqrt{2a - u}$; nulla additur conſtans, quia facta $u = 2a$, evaneſcit. Si fiat $u = o$, proveniet arcus D P G $= 4a\sqrt{2}$. Deinceps A 2 O $= u$ evadit negativa, & formula exhibet arcum D P 2 P, ita ut dum fit $u = -2a$, totus arcus D G A proveniat $= 8a$, quare arcus A 2 P G $= 4a . \overline{2 - \sqrt{2}}$.

27. Ad evolvendos caſus alios fiat $2a - u = \frac{tt}{2a}$, $\frac{4aa - tt}{2a} = u$, per quam ſubſtitutionem in hanc mutabitur æquatio P p $= \frac{2\sqrt{2} . \sqrt{b}}{\sqrt{a}}$.

$\frac{dt\sqrt{\frac{2a . \overline{a + b}^2}{b} - tt}}{\sqrt{8aa - tt}}$, quam in hunc modum placet exponere,



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$$Pp = \frac{dt\sqrt{16.\,\overline{a+b}^2 - 8\,\frac{b}{a}\,tt}}{\sqrt{8aa-tt}}.$$ Hujuſce formulæ integrationem obtineri rectificata ellypſi ex cap. 12. cognoſcimus. Conſtructionem exhibeamus paucis. Accipio $FE = 4.\overline{a+b}$, $EH = 4.\overline{a-b}$, ſeu $4.\overline{b-a}$, prout $a$ erit aut major, aut minor quam $b$, & duobus ſemiaxibus $EF$, $EH$ (*Fig.* 8, 10) deſcribo ellypſim FLH; tum abſcindo. $EI = t.\frac{4.\overline{a+b}}{2\sqrt{2}.a} = \frac{t.\sqrt{2}.\overline{a+b}}{a} = \frac{\sqrt{2}.\overline{a+b}}{a}\sqrt{2a.\overline{2a-u}} = \frac{2.\overline{a+b}}{\sqrt{a}}\sqrt{\overline{2a-u}}$, & ordino IL, erit arcus DP æqualis arcui HL. Facta $u = o$, abſciſſa EI tranſibit in $EM = 2\sqrt{2}.\overline{a+b}$; ergo arcus DPG æquabit arcum HN. Denique poſito $u = -2a$, fiet abſciſſa $EI = 4.\overline{a+b}$, hoc eſt æqualis ſemiaxi FE; ergo curvæ integer arcus æquabit quadrantem ellypticum FNH. Hæc conſtructio maxime ſimplex duo docet. Primum ſi fuerit $a = b$, evaneſcente ſecundo axe EH, ellypſis coincidit cum primo axe FE; ergo curva evadet algebraice rectificabilis, quod ſupra per integrationem probavimus: Deinde vel ſit $a > b$, vel $a < b$, dummodo differentia ſit eadem, eadem ellypſis arcuum quantitatem deſignat, qui proinde æquales ſint neceſſe eſt.

28. Si ſuper circulo AGO (*Fig.* 11) cujus centrum C conſtituatur æqualis circulus OM, cujus centrum B, & in ejus diametro vel producta, vel non, ſumatur punctum D; tum circulus MO rotetur ſupra circulum OG; ajo a puncto D deſcribi quamdam epicycloidem protractam, aut contractam, quæ eadem eſt cum curva, quam in hoc problemate conſideravimus. Hoc autem demonſtratu non eſt difficile. Supponamus circulum rotantem perveniſſe in locum G2M, exiſtente ejus centro in 2B, & puncto O in 2O, erit arcus G2O = GO. Ducatur per centrum 2B diameter 2O2M, ſectaque 2M2D = MD, erit punctum 2D in epicycloide; centro C intervallo BD deſcribatur circulus NSV, conjungatur 2DV; jungatur 2BC, quæ tranſibit per punctum G; ajo 2DV, 2BC eſſe parallelas: nam producta 2D2O in F; liquet 2OF = OF; ergo 2BF = CF, ſed 2D2B = CV; ergo 2DV, 2BC ſunt parallelæ. Agatur radius CS; hic eſt parallelus 2D2B, quia anguli 2D, CSV, utpote ambo æquales CVS, ſunt æquales inter ſe; ergo 2DS = 2BC, adeoque conſtans. Igitur punctum 2D eſt in curva, quæ deſcribitur, ſi ex puncto V ducantur infinitæ VS2D, & abſcindantur S2D æqualis datæ 2BC, ſeu BC. Quapropter hujuſmodi epicyclois coincidit cum curva noſtri problematis. Per regulas vulgaris algebræ facile eſt, curvam uno modo deſcriptam, altero quoque modo deſcribere, ſeu, quod idem eſt, ex datis, quæ habentur in una deſcriptione, data determinare, quæ altera deſcriptio expoſcit.

29. Problema nonum. Quadrare, & rectificare curvam, cujus proprietas eſt, ut ex puncto dato F (*Fig.* 12) ducta qualibet chorda FE, eique perpendiculari EH definente in rectam poſitione datam FH, ſit ubique $EH = a$ datæ. Ut formulas eleganter determinemus, centro F radio $FB = a = EH$ deſcribatur circulus BAC, quem FE ſecet in Z. Per Z agatur tangens circuli KZI. His poſitis eſt FE : EH :: FZ : ZI; ſed FZ : ZI :: KZ : FZ; ergo EF : EH :: KZ : FZ; conſequentes æquales ſunt; ergo etiam antecedentes, ſcilicet FE, KZ, quæ vocen-

centur $= x$. In angulo infiniteſimo cum F E ducatur F z e, quæ ſecet E H in b. Quoniam K Z $= x$ eſt tangens arcus B Z, erit hic arcus $= S \frac{a^2 dx}{a^2 + x^2}$; ergo ejus elementum Z z $= \frac{a^2 dx}{a^2 + x^2}$. Atqui F Z : Z z :: F E : E b, ſeu $a : \frac{a^2 dx}{a^2 + x^2} :: x :$ E b $= \frac{ax dx}{a^2 + x^2}$; ergo areæ F A E Z F elementum, hoc eſt F E e, ſive

$$FEb = \frac{a}{2} \cdot \frac{x^2 dx}{a^2 + x^2} = \frac{a}{2} \cdot dx - \frac{a^2 dx}{a^2 + x^2},$$

& integrando ſine additione conſtantis, quæ non requiritur, $\frac{ax}{2} - \frac{a}{2} S \frac{a^2 dx}{a^2 + x^2}$: atqui $\frac{ax}{2}$ præbet triangulum F Z K, $\frac{a}{2} S \frac{a^2 dx}{a^2 + x^2}$ dat ſectorem F B Z; ergo area F A E Z F æquat differentiam trianguli F Z K, & ſectoris F Z B, ſeu æquat trilineum B A Z K.

30. Eleganti calculo inventa eſt quadratura, longiorem poſcit rectificatio. Quoniam inventa eſt E b $= \frac{ax dx}{a^2 + x^2}$, & b e $= dx$; erit E e $= dx \sqrt{1 + \frac{a^2 x^2}{(a^2 + x^2)^2}}$. Ut formulam hanc reducam, utor ſubſtitutione $\frac{aa + xx}{x} = 2z$, quæ formulam in hanc convertit $dx \sqrt{1 + \frac{aa}{4zz}} = \frac{dx}{z} \sqrt{zz + \frac{aa}{4}}$. Ut arceatur $dx$, invenienda eſt $x$ per $z$, quod præſtat reſolutio æquationis quadraticæ $xx - 2xz = -aa$, quæ dabit $x = z \pm \sqrt{zz - aa}$. Ex hac docemur, duplicem valorem $x$ reſpondere cuicumque $z$. Quare opportunum erit in figura valorem $z$ determinare. Quoniam eſt ZK : FK :: FK : KI ſeu $x : \sqrt{aa + xx} : \sqrt{aa + xx}$ : KI $= \frac{aa + xx}{x} = 2z$. Agatur F A dividens bifariam angulum rectum B F C, quæ tranſibit per interſectionem circuli B A C, & curvæ F A E, quia ob angulum ſemirectum A F B tangens arcus B A æqualis eſt radio. Fac angulum A F D $=$ A F Z, & linea F D producta ſecet circulum in V. Manifeſtum eſt tangentem circuli ductam per V terminatam ad rectas F B, F C æquare K Z I. Quare duplex $x$, quæ reſpondet eidem $z$, erit F D, F E, quarum prima $= z - \sqrt{zz - aa}$, altera $= z + \sqrt{zz - aa}$.

31. His determinatis, quæ elegantiam non mediocrem conciliant, neceſſarium eſt advertere, arcum minimum E e, prout eſt elementum arcus A E exprimi quidem per formulam $\frac{dx}{z} \sqrt{zz + \frac{aa}{4}}$; at arcus minimus D d, prout eſt elementum arcus A D, debet exprimi per formulam ſigno $-$ affectam, neme



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pe per $-\frac{dx}{z}\sqrt{zz+\frac{aa}{4}}$. Nunc vero proſequens calculum, differentio æquationem $x=z\pm\sqrt{zz-aa}$, & invenio $dx=dz\pm\frac{z\,dz}{\sqrt{zz-aa}}$. Qui valor ſubſtitutus in formula rectificationis dabit $\frac{dz}{z}\sqrt{zz+\frac{aa}{4}}\pm\frac{dz\sqrt{zz+\frac{aa}{4}}}{\sqrt{zz-aa}}$. Quæ dicta ſunt ſatis demonſtrant, quid ſignificet ſignorum ambiguitas. Namque elementum $Ee=\frac{dz}{z}\sqrt{zz+\frac{aa}{4}}+\frac{dz\sqrt{zz+\frac{aa}{4}}}{\sqrt{zz-aa}}$; contra elementum

$Dd=\frac{-dz}{z}\sqrt{zz+\frac{aa}{4}}+\frac{dz\sqrt{zz+\frac{aa}{4}}}{\sqrt{zz-aa}}$. Anguli infiniteſimi E F e, D F d æquales ſumendi ſunt. Ex his colligimus, differentiam arcuum, hoc eſt $Ee-Dd=\frac{2dz}{z}\sqrt{zz+\frac{aa}{4}}$; ſummam vero nempe $Ee+Dd=\frac{2dz\sqrt{zz+\frac{aa}{4}}}{\sqrt{zz-aa}}$. Quare facta integratione erit $AE-AD=S\frac{2dz}{z}\sqrt{zz+\frac{aa}{4}}$, quæ a ſola quadratura hyperbolæ dependet. Præterea $AE+AD=S\frac{2dz\sqrt{zz+\frac{aa}{4}}}{\sqrt{zz-aa}}$. Iſtæ ſummatoriæ ita accipiendæ ſunt, ut facta $z=x=a$, evaneſcant, & nihilo æquales fiant. Hoc adverte, non poſſe eſſe $z<a$, ſed ab $a$ uſque in infinitum augeri.

32 Jam vero per logarithmos integremus formulam $\frac{dz}{z}\sqrt{zz+\frac{aa}{4}}$, quæ ita erit diſponenda $\frac{z\,dz}{\sqrt{zz+\frac{aa}{4}}}+\frac{\frac{a^2}{4}dz}{z\sqrt{zz+\frac{aa}{4}}}$. Primæ formulæ ſumma $=\sqrt{zz+\frac{aa}{4}}-\frac{a}{2}\sqrt{5}$; ita enim evaneſcit, poſita $z=a$. Ut ſecunda ad logarithmos reducatur, adhibenda eſt ſubſtitutio $\sqrt{zz+\frac{aa}{4}}=y$, & formula in hanc mutatur $\frac{\frac{a^2}{4}dy}{yy-\frac{a^2}{4}}$, quæ reſolvitur in haſce duas $\frac{\frac{a}{4}dy}{y-\frac{a}{2}}-\frac{\frac{a}{4}dy}{y+\frac{a}{2}}$.

Iſtæ

Iſtæ, ſi integrentur in ſyſtemate, cujus tam ſubtangens, quam protonumerus $= \frac{a}{4}$, fient $l\,\frac{a}{4}.\frac{\sqrt{5}+1}{\sqrt{5}-1}.\frac{y-\frac{a}{2}}{y+\frac{a}{2}} = l\,\frac{a}{4}.\frac{\sqrt{5}+1}{\sqrt{5}-1}.\frac{\sqrt{zz+\frac{aa}{4}}-\frac{a}{2}}{\sqrt{zz+\frac{aa}{4}}+\frac{a}{2}}$; quia poſita $z=a$, oritur $l\,\frac{a}{4}=0$. Igitur $AE - AD = S\,\frac{2dz}{z}\sqrt{zz+\frac{aa}{4}} =$

$$2\sqrt{zz+\frac{aa}{4}} - a\sqrt{5} + 2l\,\frac{a}{4}.\frac{\sqrt{5}+1}{\sqrt{5}-1}.\frac{\sqrt{zz+\frac{aa}{4}}-\frac{a}{2}}{\sqrt{zz-\frac{aa}{4}}+\frac{a}{2}}.$$

33. Nunc me converto ad formulam $dz\frac{\sqrt{zz+\frac{aa}{4}}}{\sqrt{zz-aa}}$, quæ per arcus ellypticos, & hyperbolicos integratur. Poſito ſemiaxe majore $GM = \frac{a\sqrt{5}}{2}$, (*Fig.* 12, 13) & minore $GN = a$, deſcribe ellypſim MN, abſcinde $GR = a\frac{\sqrt{zz-aa}}{\sqrt{zz+\frac{aa}{4}}}$, & determina arcus NQ, MQ. Deinde ſecta $GL = \frac{a}{2}$ cum ſemiaxibus GL, GN deſcribe hyperbolam LO, accipe abſciſſam $GS = \frac{az}{2\sqrt{zz-aa}}$, & determina arcum LO. Non curata conſtante, quam paullo infra determinabimus, inveniemus ex cap. 12. n. 40. $S\frac{dz\sqrt{zz+\frac{aa}{4}}}{\sqrt{zz-aa}} = \frac{z\sqrt{zz-aa}}{\sqrt{zz+\frac{aa}{4}}} + \frac{25a^2z}{16\sqrt{zz-aa}.\sqrt{zz+\frac{aa}{4}}} = -\frac{5}{4}$ $NQ - MQ - \frac{5}{4}LO$, ſive neglecto quadrante ellyptico, qui conſtans eſt

$$S\,\frac{dz\sqrt{zz+\frac{aa}{4}}}{\sqrt{zz-aa}} = \frac{16z^3+9a^2z}{16\sqrt{zz-aa}.\sqrt{zz+\frac{aa}{4}}} - \frac{1}{4}NQ - \frac{5}{4}LO.$$

Addenda eſt ejuſmodi conſtans, ut facta $z=a$, omnia evaneſcant. Verum poſita $z=a$, cum quantitas algebraica, tum arcus LO evadit infinitus; quare quantitas addenda exprimeretur per differentiam duarum quantitatum infinitarum. Hoc incommodum vitabis utens artificio, ut determinatis arcubus VO, VY, quorum differentia ſit rectificabilis algebraice, pro arcu LO opportune ſubſtituas arcum LY. Hanc ob rem ſeca $GT = a\sqrt{\frac{1}{4}+\frac{2}{2\sqrt{5}}}$, tum $GX = \frac{\sqrt{aa+4zz}}{2\sqrt{5}}$, habebis $\frac{z\sqrt{aa+4zz}}{2\sqrt{zz-aa}} - a.\overline{2+\sqrt{5}} = VO - VY$; igitur

tur $\frac{z\sqrt{aa+4zz}}{2\sqrt{zz-aa}} - a.\overline{2+\sqrt{5}} + 2LV - LY = LO$. Peracta ſubſtitutione, factoque calculo proveniet

$$S\,\frac{dz\sqrt{zz+\frac{aa}{4}}}{\sqrt{zz-aa}} = \frac{-z\sqrt{zz-aa}}{2\sqrt{aa+4zz}} + \frac{5}{4}a.\overline{2+\sqrt{5}} - \frac{1}{4}NQ - \frac{5}{2}LV + \frac{5}{4}$$

LY. Ejuſmodi quantitas addenda eſt, ut, facta $z = a$, omnia evaneſcant; atqui, facta $z = a$, evaneſcit quantitas algebraica, NQ fit quadrans ellypticus, & evaneſcit LY; igitur formula ita erit enuncianda

$$S\,\frac{2dz\sqrt{zz+\frac{aa}{4}}}{\sqrt{zz-aa}} = AE + AD\,(Fig.\,12) = -\frac{z\sqrt{zz-aa}}{\sqrt{aa+4zz}} + \frac{MN-NQ}{2} + \frac{5}{2}LY = z$$

$= \frac{z\sqrt{zz-aa}}{\sqrt{aa+4zz}} + \frac{1}{2}MQ + \frac{5}{2}LY$. Determinatis ita conſtantibus adverto, hoc æquationem habere incommodi, quod facta $z$ infinita duplex infinitum habeatur, nempe quantitas algebraica, & arcus LI. Quare iterum pro LY ſubſtituamus LO, qui facta $z$ infinita erit nullus. Habebimus itaque

$$AE + AD = z.\frac{9a^2+16z^2}{4\sqrt{zz-aa}.\sqrt{aa+4zz}} - \frac{5a}{2}.\overline{2+\sqrt{5}} + 5LV -$$

$\frac{5}{2}LO + \frac{1}{2}MQ$, in qua ſi $z$ ſit infinita, ſola quantitas algebraica infinita eſt, evaneſcente LO, & degenerante MQ in quadrantem ellypticum.

34. Quoniam tam ſumma quam differentia arcuum AE, AD inventa eſt, liquet dari arcus AE, AD. Verum non debeo omittere determinationem arcus AF. Hanc ob rem primam æquationem deme ex ſecunda, hoc eſt differentiam ex ſumma arcuum, & invenies $2\,AD = -\sqrt{aa+4zz}$

$$+ z.\frac{9aa+16zz}{4\sqrt{zz-aa}.\sqrt{aa+4zz}} + a\sqrt{5} - \frac{5}{2}a.\overline{2+\sqrt{5}} - 2l\frac{a}{4}.\frac{\sqrt{5}+1}{\sqrt{5}-1}.$$

$$\frac{\sqrt{zz+\frac{aa}{4}} - \frac{a}{2}}{\sqrt{zz+\frac{aa}{4}} + \frac{a}{2}} + 5LV - \frac{5}{2}LO + \frac{1}{2}MQ.$$ In hac formula licet duæ quantitates algebraicæ, facta $z$ infinita, ambæ ſint infinitæ; tamen ſeſe deſtruunt, quum earum differentia non finita ſit, ſed infiniteſima, ut mox oſtendam. Reliquæ omnes finitæ ſunt. Quare contractis terminis inveniemus

$2AF = 5a - \frac{3}{2}a\sqrt{5} - 2l\frac{a}{4}.\frac{\sqrt{5}+1}{\sqrt{5}-1} + 5LV + \frac{1}{2}MQN$. Nihil jam reliquum eſt, niſi ut oſtendam, differentiam duarum quantitatum $\sqrt{aa+4zz}$, $z.\frac{9aa+16zz}{4\sqrt{zz-aa}.\sqrt{aa+4zz}}$, facta $z$ infinita non finitam, ſed infiniteſimam eſſe.

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eſſe. Quantitates duas ita diſpono $\dfrac{\overline{aa+4zz}.\sqrt{zz-aa}}{\sqrt{aa+4zz}.\sqrt{zz-aa}}$, $z.\dfrac{9aa+16zz}{4\sqrt{aa+4zz}.\sqrt{zz-aa}}$. Pono, extracta radice, $\sqrt{zz-aa}=z-\dfrac{a^2}{2z}$; terminos reliquos negligo, quia in eo ſunt infiniteſimorum ordine, quem in calculo negligere oportet. Prima quantitas mutatur in hanc $\dfrac{\begin{matrix}aaz\\-2aaz\end{matrix}+4z^3-\dfrac{a^4}{2z}}{\sqrt{aa+4zz}.\sqrt{zz-aa}}=$ $\dfrac{4z^3-aaz+\dfrac{a^4}{2z}}{\sqrt{aa+4zz}.\sqrt{zz-aa}}$; atqui altera quantitas $=\dfrac{4z^3+\dfrac{9}{4}aaz}{\sqrt{aa+4zz}.\sqrt{zz-aa}}$; ergo, hac a prima dempta, differentia fit $\dfrac{-\dfrac{13}{4}aaz+\dfrac{a^4}{2z}}{\sqrt{aa+4zz}.\sqrt{zz-aa}}$, in qua omittenda $\dfrac{a^4}{2z}$ infiniteſima reſpectu $\dfrac{13}{4}a^2z$; atqui $\dfrac{-12a^2z}{4\sqrt{aa+4zz}.\sqrt{zz-aa}}$ eſt infiniteſima, quia in diviſore $z$ elevatur ad ſecundam poteſtatem, in numeratore tantum ad primam; differentia itaque inter duas expoſitas quantitates algebraicas infiniteſima eſt.

35. Problema decimum. Quadrare, & rectificare lemniſcatam. Ex lib. 3. Tom. 1. cap. 12. n. 10. conſtat, vocatis $CM=x$, $ML=y$, $CB=a$, (*Fig.* 14) lemniſcatæ CLB æquationem eſſe $xx+yy=a\sqrt{xx-yy}$. Quoniam hæc æquatio tum quadraturæ, tum rectificationis formulas implicitas exhibet, propterea ex ea lemniſcatæ æquationem inquiro relatam ad focum C, hac methodo utens. Excito CX normalem, & æqualem CB, & centro C deſcribo quadrantem XYB; ago CLY, cum qua minimum angulum faciat Cly, & centro C radio CL deſcribo minimum arcum Lm. Voco arcum $XY=\mu$, ejus elementum $Yy=d\mu$, ſumo pro ſinu toto $CB=a$, & denomino $CL=z$, ejuſque elementum $ml=dz$. His poſitis æquatio provenit $zz=a\sqrt{xx-yy}$; atqui $z:x::a:Sc.\mu$, $z:y::a:Cc.\mu$; ergo $x=\dfrac{z.Sc.\mu}{a}$, $y=\dfrac{z.Cc.\mu}{a}$; igitur $zz=z\sqrt{\overline{Sc.\mu}^2-\overline{Cc.\mu}^2}$, ſive $zz=\overline{Sc.\mu}^2-\overline{Cc.\mu}^2$; quæ duas præbet æquationes, nimirum $zz=2.\overline{Sc.\mu}^2-aa$, $zz=aa-2.\overline{Cc.\mu}^2$, ergo $Sc.\mu=\dfrac{\sqrt{aa+zz}}{\sqrt{2}}$, $Cc.\mu=\dfrac{\sqrt{aa-zz}}{\sqrt{2}}$. Prima ex his differentietur $dSc.\mu=$ $\dfrac{zdz}{\sqrt{2}.\sqrt{aa+zz}}$; atqui $dSc.\mu=\dfrac{d\mu Cc.\mu}{a}=\dfrac{d\mu\sqrt{aa-zz}}{a\sqrt{2}}$; ergo $\dfrac{d\mu\sqrt{aa-zz}}{a\sqrt{2}}=\dfrac{zdz}{\sqrt{2}.\sqrt{aa+zz}}$, ſeu $d\mu=\dfrac{azdz}{\sqrt{aa-zz}.\sqrt{aa+zz}}$, ſed

$a:d\mu$

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$a:d\mu = \frac{azdz}{\sqrt{aa-zz}.\sqrt{aa+zz}} :: z : \mathrm{Lm} = \frac{z^2dz}{\sqrt{aa-zz}.\sqrt{aa+zz}}$. His poſitis elementum areæ $\mathrm{CLl} = \frac{z^3dz}{2\sqrt{aa-zz}.\sqrt{aa+zz}} = -\frac{z^3dz}{2\sqrt{a^4-z^4}}$; ergo integrando area $\mathrm{CL} = A - \frac{\sqrt{a^4-z^4}}{4}$. Quoniam poſita $z=o$, debet area evaneſcere, fiet $A = \frac{aa}{4}$; ergo ſegmentum $\mathrm{CL} = \frac{aa}{4} - \frac{\sqrt{a^4-z^4}}{4}$, in qua, ſi fiat $\mathrm{CL} = \mathrm{CB} = a$, totum ſpatium lemniſcaiæ $= \frac{aa}{4}$, hoc eſt quadrato dimidiæ CB.

36. Alio modo quadratura determinatur. Quoniam inventa eſt $\mathrm{CL} = \sqrt{\overline{Sc.\mu}^2 - \overline{Cc.\mu}^2}$, & eſt CY : Yy, ſeu $a:d\mu :: \mathrm{CL} = \sqrt{\overline{Sc.\mu}^2 - \overline{Cc.\mu}^2} : \mathrm{Lm} = \frac{d\mu.\sqrt{\overline{Sc.\mu}^2 - \overline{Cc.\mu}^2}}{a}$, erit elementum areæ $\frac{\mathrm{CL.Lm}}{2} = \frac{d\mu\,\overline{Sc.\mu}^2 - d\mu\,\overline{Cc.\mu}^2}{2a}$; atqui $\frac{d\mu\,Sc.\mu}{a} = -dCc.\mu$, & $\frac{d\mu\,Cc.\mu}{a} = dSc.\mu$; ergo $\frac{\mathrm{CL.Lm}}{2} = \frac{-Sc.\mu.dCc.\mu - Cc.\mu.dSc.\mu}{2}$, & integrando, ſegmentum $\mathrm{CL} = A - \frac{Sc.\mu.Cc.\mu}{2}$. Adverte, lemniſcatam in puncto C facere angulum ſemirectum; ergo in hoc puncto $Sc.\mu = Cc.\mu = \frac{a}{\sqrt{2}}$; ſed in hoc puncto area debet evaneſcere; ergo $A = \frac{aa}{4}$; igitur vera quadratura ſegmenti $\mathrm{CL} = \frac{aa}{4} - \frac{Sc.\mu.Cc.\mu}{2}$, quæ, ſi $\mu$ evadat quadrans, dat ſpatium $\mathrm{CLB} = \frac{aa}{4}$, ut antea.

37. Venio ad rectificationem. Quando $\mathrm{ml} = dz$, & inventa eſt $\mathrm{Lm} = \frac{z^2dz}{\sqrt{aa-zz}.\sqrt{aa+zz}}$, facile eſt invenire lemniſcatæ arcum directum

$$\mathrm{CL} = S\frac{a^2dz}{\sqrt{a^2+z^2}.\sqrt{a^2-z^2}}; \text{inverſum vero } \mathrm{BL} = S\frac{-a^2dz}{\sqrt{a^2+z^2}.\sqrt{a^2-z^2}}.$$

Formulæ iſtæ integrantur per rectificationem ellypſis, & hyperbolæ, atque ex methodis traditis hæc oritur conſtructio. Perpendicularis CB excitetur $\mathrm{CA} = a\sqrt{2}$, & poſitis ſemiaxibus CA, CB deſcribatur ellypſis ADB; tum poſito ſemiaxe CB delineetur hyperbola æquilatera BVO, abſcinde $\mathrm{CF} = \sqrt{aa-zz}.\sqrt{2}$, cui reſpondet ordinata $\mathrm{FD} = \mathrm{CG} = z$, & determina ar-

arcum BD. Deinde abſcinde $CS = \frac{a\sqrt{aa+zz}}{z\sqrt{2}}$, & determina arcum BO; Habebimus $BL = S\frac{-a^2dz}{\sqrt{aa+zz}.\sqrt{aa-zz}} = \frac{-\sqrt{aa+zz}.\sqrt{aa-zz}}{z} +$ $BD + BO$. Quoniam omnia evaneſcunt facta $z = a$, nulla eſt addenda conſtans. Rectificatio hæc lemniſcatæ hoc habet incommodi, quod ſi punctum L valde accedat ad punctum C vel maxime augetur, tum arcus BO, tum quantitas algebraica, immo coincidente puncto L cum C utraque evadit infinita. Remedium in ſuperiore problemate uſurpatum huc quoque eſt transferendum: nimirum determinandi ſunt arcus VO, VN, quorum differentia ſit rectificabilis, & pro arcu BO ſubſtituendus eſt in formula arcus BN, hanc ob rem abſcinde $CT = a\sqrt{1+\frac{1}{\sqrt{2}}}$, & determina arcum BV; tum ſeca $CP = \frac{aa}{\sqrt{aa-zz}}$, & determina arcum BN. His poſitis erit $\frac{aa\sqrt{aa+zz}}{z\sqrt{aa-zz}} - a.\sqrt{2+1} = VO -$ $VN = BO + BN - 2BV$. Itaque opportune ſubſtituto in ſuperiore æquatione valore arcus BO, prodibit $BL = S\frac{-a^2dz}{\sqrt{aa+zz}.\sqrt{aa-zz}} =$

$$\frac{-\sqrt{aa+zz}.\sqrt{aa-zz}}{z} + \frac{aa\sqrt{aa+zz}}{z\sqrt{aa-zz}} - a.\sqrt{2+1} + BD - BN + 2BV$$

$$= \frac{z\sqrt{aa+zz}}{\sqrt{aa-zz}} - a.\sqrt{2+1} + BD - BN + 2BV.$$

38. Fiat nunc $z = 0$, & obtinebimus quadrantem lemniſcatæ $BLC = -a.\sqrt{2+1} + BDA + 2BV$. Dematur ex hac æquatio ſuperior, & orietur $CL = S\frac{a^2dz}{\sqrt{aa+zz}.\sqrt{aa-zz}} = \frac{-z\sqrt{aa+zz}}{\sqrt{aa-zz}} + AD + BN$. In hac ſi ſubſtituas valorem arcus BN datum per BO obtinebis $CL =$

$$S\frac{a^2dz}{\sqrt{aa+zz}.\sqrt{aa-zz}} = \frac{-z\sqrt{aa+zz}}{\sqrt{aa-zz}} + \frac{aa\sqrt{aa+zz}}{z\sqrt{aa-zz}} - a.\sqrt{2+1} +$$

$AD - BO + 2BV = \frac{\sqrt{aa+zz}.\sqrt{aa-zz}}{z} - a.\sqrt{2+1} + AD - BO +$ $2BV$. Si fiat $z = a$ orietur quadrans lemniſcatæ $CLB = -a.\sqrt{2+1} +$ $ADB + 2BV$. Hujus, & ſuperioris curvæ rectificationem deſumpſimus ex Vincentij Riccati litteris ſecunda ad Mariſcottum, & ad Fantonum, quæ editæ ſunt in ſecundo Opuſculorum tomo, opuſculo ſecundo. Si autem integrum hoc opuſculum perleges, quam utilitatem habeant arcus elliptici, & hyperbolici, plenius cognoſces.

39. Si quæreres per functionem ſolius $z$ lemniſcatæ rectificationem, ita com-

poſ fieres. Quoniam $CL = \sqrt{\overline{Sc.\mu}^2 - \overline{Cc.\mu}^2}$ erit differentiando

$$lm = \frac{Sc.\mu . dSc.\mu - Cc.\mu . dCc.\mu}{\sqrt{\overline{Sc.\mu}^2 - \overline{Cc.\mu}^2}} = \frac{2Sc.\mu . Cc.\mu . d\mu}{a\sqrt{\overline{Sc.\mu}^2 - \overline{Cc.\mu}^2}}$$; ſed

$$Lm = \frac{d\mu\sqrt{\overline{Sc.\mu}^2 - \overline{Cc.\mu}^2}}{a}$$; ergo minimus arcus

$$Ll = d\mu\sqrt{\frac{\overline{Sc.\mu}^2 - \overline{Cc.\mu}^2}{a^2} + \frac{4\,\overline{Sc.\mu}^2 . \overline{Cc.\mu}^2}{aa.\overline{Sc.\mu}^2 - \overline{Cc.\mu}^2}} = \frac{d\mu}{a}.$$

$$\frac{\overline{Sc.\mu}^2 + \overline{Cc.\mu}^2}{\sqrt{\overline{Sc.\mu}^2 - \overline{Cc.\mu}^2}} = \frac{ad\mu}{\sqrt{\overline{Sc.\mu}^2 - \overline{Cc.\mu}^2}}.$$ Quaproter ex illis, quæ in hoc problemate demonſtrata ſunt, conſtat, formulam inventam dependere a rectificatione ellypſis, & hyperbolæ.

40. Problema undecimum. Rectificare, & quadrare epicycloides omnes, quæ generantur a puncto accepto in circumferentia circuli rotantis ſupra alium circulum. Circulus rotans ſit TNQ, (*Fig.* 15) cujus diameter QT; circulus, ſupra quem ſit rotatio, ſit VL, cujus diameter RG; punctum curvam deſcribens ſit N; & arcus curvæ deſcriptæ FNL. Circulus rotans promoveatur in poſitionem infinite proximam RMP, exiſtente infiniteſimo angulo PAQ effecto a diametris tranſeuntibus per contactus. A diametrorum QT, PR extremis punctis ad puncta N, M, in quibus epicyclois a circulis ſecatur, agantur rectæ QN, TN, PM, RM, & centro A intervallo AP deſcribatur arcus PQ. Efformatum eſt triangulum PXQ, in quo angulus QPX eſt infiniteſimus; anguli vero ad baſim QX finiti ſunt; ergo PQ, PX æquantur. Simili modo oſtendam arcum RT æquare minimum arcum circuli QNT interceptum inter punctum T, & rectam NR, ſcilicet Th. Præterea arcus curvæ MN confunditur cum chorda QN producta. Nam quum Th non differat ab RT niſi quantitate infiniteſima ſecundi ordinis, erit in primis Rh quantitas infiniteſima ordinis ſecundi, & Nh non differet ab RN niſi quantitate infiniteſima ſecundi ordinis, adeoque chorda Nh, & per conſequens NR non differt ab MR niſi quantitate infiniteſima ordinis ſecundi; igitur angulus MNR, adeoque etiam angulus MNT non differt a recto; directio itaque rectæ MN non differt a directione NQ. Tandem ducta RO parallela NT, erit arcus ZN æqualis Th ob arculum ſecundi ordinis interceptum inter rectas OR, NR, adeoque erit etiam æqualis TR; igitur NZ eſt elementum arcus QN; & NO erit elementum chordæ NQ, ut clarum fiet, ſi ex puncto Q ad punctum Z ducatur QZ.

41. His prænotatis ad exactam demonſtrationem; producatur QN donec concurrat cum PM in p; & MR, NT donec concurrant in X. Quoniam anguli pNX, pMX ſunt recti, erit angulus X æqualis PpQ, ſeu PMQ ob angulum pPN ſecundi ordinis infiniteſimorum, ſed angulus x = NRO ob parallelas NX, pR. Ergo anguli QNP, MRO æquantur; atqui angulus PNQ eſt ad RNT :: Px : Th; quia anguli ſunt ad circumferentiam ejuſdem cir-

culi

culi ; & Pa:Th::PQ : RT::PA : RA ; ergo angulus MRO : RNT ſeu ORN::PA:RA, & componendo MRN:ORN, ſeu MN:NO::PG: GA, in ratione nimirum conſtanti: igitur integrando curva FN ad chordam QN, ut utriuſque circuli diametri ſimul ſumptæ ad radium circuli, ſuper quem ſit rotatio.

42. Si radius RA evadat infinitus, arcus circuli VL convertitur in lineam rectam, & curva in cycloidem vulgarem, in qua erit arcus FN duplus chordæ QN; & ſemicyclois FL dupla diametri FV, & tota cyclois quadrupla. Si radius RA ſit æqualis radio PB, curva FNL erit epicyclois quarti gradus, in qua arcus FN erit ad chordam QN::4:1, & ſemiepicyclois quadrupla diametri FV, & integra octupla. Si circulus QT (*Fig.* 16) rotetur intra circulum VGD, deſcribatque epicycloidem FNL, facta eadem conſtructione, eademque uſurpata methodo, quam ſupra uſurpavimus, quando rotatio externe ſiebat, inveniemus arcum FM ad chordam PM ut differentia diametrorum, nempe PG ad radium GA circuli ſuper quem ſit rotatio. Si GA ſit infinita, circulus VL convertitur in lineam rectam, & curva FML in cycloidem vulgarem, unde eadem eruuntur, quæ ſupra. Si PR diameter circuli rotantis ſit æqualis GA alterius circuli radio, arcus FM invenitur æqualis PM, & FL æqualis AL. Hoc autem fieri non poteſt, niſi curva FL convertatur in lineam rectam AL, & niſi rotato circulo QT per integram periferiam, a puncto M deſcribatur bis diameter circuli VL. Hoc autem contingat neceſſe eſt. Etenim quum circulus AMR (*Fig.* 17) tranſire debeat per centrum A, ducta OM tranſibit per centrum B; quare angulus MBA æquabit duplicem BAO: ergo arcus VR = AM = RO; igitur punctum O eſt ſemper in radio VA, & punctum M in radio AL. Si ſit GA < PR, tunc arcus FM (*Fig.* 16) minor eſt chorda PM ita, ut exiſtente $GA = \frac{1}{2} PR$, arcus FM, immo tota curva FL omnino evaneſcat, atque ad punctum redigatur. Tandem ſi diameter circuli rotantis evadat major diametro alterius, ſuper quem ſit rotatio, eadem omnino valebit proportio, & eadem methodo demonſtrabitur.

43. Ut expedita rectificatione ad quadraturam veniam, iiſdem poſitis, quæ ſupra, obſervo, trapetium MNTR (*Fig.* 15) eſſe elementum ſpatii comprehenſi ab arcu epicycloidis MF, arcu circuli RV, & rectis MR, VF, ſcilicet FMRV. Similiter obſervo triangulum NRO eſſe elementum ſegmenti circularis comprehenſi ab arcu circulari QN, chorda NT, & diametro TQ, quia quum arcus MR = LR, & NT = LT, erit ZN elementum arcus QN = RT; ergo ſi intelligatur ducta ZT, erit triangulum ZTN elementum prædicti ſegmenti circularis; ſed ZTN non differt a ZRN, & hoc non differt ab ORN; ergo ORN eſt elementum prædicti circularis ſegmenti. Jam vero triangulum MRN eſt ad triangulum ORN, ut MN:NO, ſeu ex ſupra demonſtratis ut GP:GA; ergo componendo trapetium MRTN ad RNO, ut PG + GA:GA, in ratione ſcilicet conſtanti; igitur facta integratione, ſpatium FNTV ad ſegmentum circuli QYNT, quæ duæ quantitates ſimul evaneſcunt in F, ſeſe habet ut PG + GA:GA. Q. E. Inv.

44. Quando radius GA eſt infinitus, circulus VL convertitur in lineam rectam, & curva FL in cycloidem vulgarem, in qua proinde erit ſpatium FMRV triplum ſegmenti circularis PMR, & area ſemicycloidis FLV tripla ſemicirculi PMR. Si duorum circulorum radii æquales ſint, ſpatium FMRV



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erit ad ſegmentum circulare PMR, ut 5 : 1. Igitur epicycloïs, quæ in hac hypotheſi eſt curva quarti gradus, habet ſpatium integrum ſitum inter ejus curvam, & circumferentiam circuli, quod ad circulum ipſum eſt ut 5 : 1. Quod ſi addatur circulus idem, ſpatium ab integra epicycloide clauſum erit ſextuplum circuli genitoris, ut alias vidimus. Si rotatio fiat, intra circulum VLG, ſervata eadem demonſtrationis methodo invenietur ſpatium VFMR ( *Fig.* 16 ) ad ſegmentum PYMR ut differentia diametrorum addito GA radio circuli VLG ad eumdem radium GA. Si GA infinita ſit, oritur cycloïs, in qua eadem deducuntur, quæ antea. Si PR æqualis ſit radio VA, epicycloïs, ut vidimus, eſt diameter AL circuli VLG. In hoc caſu ſpatium VFMR eſt duplum ſegmenti PYRM, adeoque area VFL, ſeu VAL æqualis circulo genitori, quod patet ex elementis. Si fiat GA = BR, curva colligitur in punctum, & ejus area accepta ut antea evadit æqualis circulo genitori, quod per ſe clarum eſt. Eadem methodo determinabis, quid accidat, ſi diameter PR fiat major diametro RG. Atque hæc ſatis ſint de epicycloïdibus.

45. Problema duodecimum. Super linea poſitione data AB ( *Fig.* 18) ita moveatur data BE, ut angulus variabilis ABE detur per lineam AB: quæritur quadratura, & rectificatio curvæ deſcriptæ a puncto E. Feratur BE per ſpatium infiniteſimum Bb, & tranſeat in eb, deſcribente puncto E elementum Ee. Intelligantur EB, eb productæ, ut coeant in C. Centro C deſcribatur arcus bm, & ex E demittatur EF normalis rectæ AB. Conſtans BE fiat $= r$, quam ut ſinum totum accipiemus. Arcus metiens EBA $= \mu$, AB $= x$. Dabitur ex conditione problematis æquatio inter $x$, $\mu$. Erit FE $= Sc.\mu$, BF $= Cc.\mu$. Differentia duorum angulorum ebA, EBA æqualis eſt angulo C; ergo arcus dimetiens angulum C $= d\mu$. Demum Bb $= dx$. Similia ſunt triangula EBF, bmB, quia præter angulum ad verticem habent angulos rectos in F, m: ergo erit EB : EF :: bB : bm, aut $r : Sc\mu :: dx :$ bm $= \frac{dx.Sc.\mu}{r}$. Item EB : BF :: bB : Bm, aut $r : Cc.\mu :: dx :$ Bm $= \frac{dx.Cc.\mu}{r}$. Eſt autem ut $d\mu : r ::$ bm $= \frac{dx.Sc.\mu}{r}$ : BC $= \frac{dx.Sc.\mu}{d\mu}$.

46. His ſtatutis ad inveniendam quadraturam, deſcripto arcu en, adverte ſpatium EBbe adæquare enmb, quod generatur a linea mn conſtante, quæ convertitur circa punctum C. Ejus itaque valor nullo negotio per regulam Guldini determinatur. Diviſa itaque mn bifariam in D, centro C deſcribatur arcus Dd; erit EBbe = BE . Dd. Ad inveniendum Dd; fiat $r : d\mu ::$ CD $=$

$\frac{dx.Sc.\mu}{d\mu} + \frac{r}{2}$ : Dd $= \frac{dx.Sc.\mu}{r} + \frac{d\mu}{2}$; ergo ſpatium quæſitum

EBbe $= dx\, Sc.\mu + \frac{r\,d\mu}{2}$, & integrando

EBE $= S dx\, Sc.\mu + \frac{r\mu}{2}$, cui, prout opus fuerit, addenda eſt opportuna quantitas.

47. Quod ſpectat ad rectificationem, deſcripto centro C arcu en, advertendum En = Bm $= \frac{dx.Cc.\mu}{r}$. Ut inveniatur arcus ne, fiat $r : d\mu ::$ Cn

=

$= \frac{dx.Sc.\mu}{d\mu} + r : en = \frac{dx.Sc.\mu}{r} + d\mu$; igitur

$$Ee = \sqrt{\frac{dx^2.\overline{Cc.\mu}^2}{r^2} + \frac{dx^2.\overline{Sc.\mu}^2}{rr} + \frac{2dxd\mu.Sc.\mu}{r} + d\mu^2} =$$

$\sqrt{dx^2 + \frac{2dxd\mu.Sc.\mu}{r} + d\mu^2}$. Quum autem detur $x$ per $\mu$ proveniet formula ex una duntaxat determinata coaleſcens.

48. Ut problema hoc alicui caſui peculiari applicarem, ſuppoſui $x = \mu$, hoc eſt A B ( *Fig.* 19 ) æqualem arcui metienti angulum E B A. Hinc æquationem ſtatim cognovi pertinere ad cycloidem vulgarem. Sit cyclois H G D, cujus circulus genitor F K D habeat radium $= r$. Ex centro C parallelam baſi H F duc C G, in qua abſcinde G A $= r$. Age quemlibet radium C I, & I E, parallelam H F; ex E duc E B parallelam radio I C. Ajo A B fore æqualem arcui K I, qui metitur angulum I C K, ſeu E B A. Etenim arcus D I K = G K, arcus D I = E I = B C; ergo arcus K I = G K − B C = G B − K C; ſed G A = K C; ergo K I = A B. Itaque ſi A ita moveatur, ut A B confecta a puncto A ſit æqualis arcui anguli E B A, deſcribetur cyclois. Si A movetur ab A verſus C deſcribitur pars G D, ſi in oppoſitam partem, fiunt $x$, $\mu$ negativæ, & deſcribitur pars G H. Adverte punctum B eſſe locum, ubi invenitur centrum circuli deſcribentis, dum punctum deſcribens eſt in E.

49. Incipiamus a rectificatione, quam ſcimus algebraice obtineri. Poſito $d\mu$ pro $dx$ habebimus

$$GE = S\sqrt{2d\mu^2 + \frac{2\overline{d\mu}^2.Sc.\mu}{r}} = S\frac{\sqrt{2}}{\sqrt{r}}.d\mu\sqrt{r+Sc.\mu};$$

atqui $d\mu = \frac{rdSc.\mu}{Cc.\mu}$; ergo $GE = S\sqrt{2r}.\frac{dSc.\mu\sqrt{r+Sc.\mu}}{Cc.\mu} =$

$S\sqrt{2r}.\frac{dSc.\mu\sqrt{r+Sc.\mu}}{\sqrt{rr-\overline{Sc.\mu}^2}} = S\frac{\sqrt{2r}.dSc.\mu}{\sqrt{r-Sc.\mu}} = A - 2\sqrt{2r}.\sqrt{r-Sc.\mu}$. Ut determinetur conſtans addita $A$, advertendum eſt G E nulleſcere, ſi $\mu$, & $Sc.\mu = 0$; ergo $A = 2r.\sqrt{2}$; ergo $GE = 2r\sqrt{2} - 2\sqrt{2}\sqrt{rr - r.Sc.\mu}$. Si B E veniat in C D, ut fiat $Sc.\mu = r$, fiet $GD = 2r\sqrt{2}$; ergo $DE = 2\sqrt{2r.}\sqrt{r - Sc.\mu}$. Ducatur I L normalis C D, ut fiat $CL = Sc.\mu$, & $DL = r - Sc.\mu$. Quare arcus D E æquabit duplam mediam proportionalem inter F D, D L, ſeu duplam chordam D I, quod per alias methodos item eſt demonſtratum.

50. Veniamus ad quadraturam. Subſtituta pro $dx$ in formula inventa $d\mu$, fiet $GBE = Sd\mu.Sc.\mu + \frac{r\mu}{2}$; ſed $d\mu.Sc.\mu = -rdCc.\mu$; ergo $GBE = S - rdCc.\mu + \frac{r\mu}{2}$; & integrando addita conſtante, $GBE = rr - rCc.\mu + \frac{r\mu}{2}$; ita enim facta $\mu = 0$, & $Cc\mu = r$ evaneſcit. Formula hæc docet, quadraturam ſpatij GBE dependere a circuli quadratura. Centro B intervallo B E deſcribe arcum E O; manifeſtum eſt ſectorem $BEO = \frac{r\mu}{2}$; ergo ſpatium $GEO = rr - rCc.\mu$

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$rCc.\mu$; quare erit algebraice integrabile. Transferatur B in C, & BE in CD, ut arcus EO evadat quadrans DK, quem voco $= \omega$. Habebimus ſpatium $GDC = rr + \frac{r\omega}{2}$, quia $Cc.\omega = o$, & ſpatium $GDIK = rr$.

51. Si punctum A ad partes oppoſitas moveatur, & perveniat in 2B, non minus $x$, quam $\mu$ evadit negativa; ergo mutato ſigno ſpeciei $\mu$, fiet ſpatium $G2E2E = rr - rCc.\mu - \frac{r\mu}{2}$. Hoc ſpatium prodit negativum, quod ſemper verum eſt, donec punctum 2B perveniat in G, in quo caſu ſpatium evadit ſegmentum cycloidale, cujus chorda $= r$. Tum puncto B ulterius procedente in 3B, linea 3B3E antequam ad 3E perveniat ſecabit cycloidem in V exſtente trilineo G3BV poſitivo, ſegmento autem V3E negativo: quare formula præbebit horum ſpatiorum differentiam, quæ nulla erit, quum $r - Cc.\mu = \frac{\mu}{2}$, ſeu quum dimidium arcus erit æquale ſuo ſinui verſo. Progrediente motu ſpatium negativum decreſcet, & nullum fiet, quum linea AG venerit in ſitum HN normalem GC; poſitivum autem augebitur, & tandem fiet NHG. Si centro 2B deſcribatur arcus 2E2O, ſector $2B2E2O = \frac{r\mu}{2}$; igitur $G2B2O = rr - rCc.\mu$. Quod ubique verum eſt, etiam quum $Cc.\mu = o$; ergo centro N deſcripto quadrante HQ, $GQH = rr$. Quum etiam $GDK = rr$; erit $GQH = GDK$; ergo addito ſpatio HGKF, fiet $DGHFK = QHFK$; ſed vocato quadrante $= \omega$, $QHFK = 2r\omega$: ergo $DGHFK = 2r\omega$. ſed $2r\omega$ æquat duplicem ſemicirculum DKF: ergo prædictum ſpatium æquale eſt duplici ſemicirculo genitori, quod per alias methodos demonſtratur.

52. Quod ſi AB non ſit æqualis arcui metienti angulum ABE, (*Fig.* 18) ſed ad illum habeat rationem datam, tum gignetur cyclois, quæ dicitur protracta, & contracta, prorſus ea, quæ produceretur, ſi punctum deſcribens ſitum eſſet extra circumferentiam circuli ſuper rectam rotantis; quod ita demonſtro. Curva NOQ (*Fig.* 20) deſcribatur a puncto O, dum linea BO ita movetur, ut AB ad arcum deſcriptum radio $BO = r$ eſt in ratione data $a:r$. Abſcinde $BE = a$, conſtat AB fore æqualem arcui deſcripto centro B, radio BE dimetienti angulum ABE, & GED fore cycloidem, cujus circulus genitor ſit DKF radii $= a$. Sume $CQ = r$; & deſcribe circulum QRS; Quoniam BO, CQ æquales ſunt, conſtat, punctum Q reperiri in O, dum punctum D deſcribens cycloidem eſſet in E; ergo curva NQ eſſet cyclois protracta deſcripta a puncto Q, dum circulus DKF rotatur in linea HF.

53. Hujus curvæ æquatio eſt $x:\mu::a:r$, ſeu $x = \frac{a\mu}{r}$. Quod pertinet ad ejus quadraturam; eadem ferme eruuntur & eadem methodo, quæ in caſu ſuperiore. Quare ea utpote facilia analyſtarum induſtriæ relinquo. Aliqua dicam de rectificatione. Quoniam eſt $x:\mu::a:r$ poſito pro $dx$ in formula rectificationis ejus valore $\frac{ad\mu}{r}$ proveniet $\frac{d\mu}{r}\sqrt{rr + aa + 2a.Sc.\mu}$, quæ eſt elementum arcus NO. (*Fig.* 21) Agatur OIG; patet arcum $MI = \mu$, $Sc.\mu = CG$, & $r - Sc.\mu = GQ$. Agatur circuli chorda QI, quæ appelletur $= z$; ergo erit

$\sqrt{ar}$.

$\sqrt{2r}.\sqrt{r - Sc.\mu} = z$. His poſitis quoniam $\frac{d\mu}{r} = \frac{dSc.\mu}{Cc.\mu} = \frac{dSc.\mu}{\sqrt{rr - \overline{Sc.\mu}^2}} = \frac{dSc.\mu}{\sqrt{r - Sc.\mu}.\sqrt{r + Sc.\mu}}$; ejicienda eſt ex hac formula $Sc.\mu$, & introducta $z$, tranſformanda eſt. Habebimus $\frac{dSc.\mu}{\sqrt{r - Sc.\mu}} = \frac{-dz\sqrt{2}}{\sqrt{r}}$, & $Sc.\mu = r - \frac{zz}{2r}$; ergo elementum arcus NO $=$

$$\frac{-dz\sqrt{2}.\sqrt{rr + aa + 2ar - \frac{azz}{r}}}{\sqrt{r}.\sqrt{2r - \frac{zz}{2r}}} = \frac{-2dz\sqrt{\overline{r+a}^2 - \frac{a}{r}zz}}{\sqrt{4rr - zz}}.$$

Formula $\frac{-dz\sqrt{\overline{r+a}^2 + \frac{a}{r}zz}}{\sqrt{4rr - zz}}$ dependet a rectificatione ſolius ellypſis, atque ita conſtruitur. Sumpto primo ſemiaxe QF $= r + a$, & ſecundo QE $= r - a$ deſcribe quadrantem ellypticum FE. Accipe abſciſſam QP, quæ ſit ad QI $= z :: r + a : 2r$, & ordina PR. Accipe pariter QT $= \overline{r + a}.\sqrt{2}$, & duc ordinatam TV. His effectis arcus NO $= 2.$VR, NQ $= 2.$VE, QO $= 2$ER. Et quoniam facta $z = 2r$, abſciſſa ellypſis ſit $= r + a$, $=$ QF, totus arcus QNL $= 2.$EVF. Quare ſi deſcribas ellypſim ſimilem, cujus axes eſſent dupli, arcus hujus ellypſis æquales eſſent arcubus noſtræ cycloidis.

54. Si recta AB $= x$ (*Fig.* 18) non detur niſi per ſinum, aut coſinum anguli ABE $= \mu$, tunc curva erit ſemper algebraica. Ut unum proponamus exemplum, ſit curva, in qua AB ſit ad ſinum verſum anguli ABE $:: m : 1$ in ratione data. Quæritur ejus quadratura, & rectificatio. Hanc curvam eſſe ellypſim apollonianam, facile conſtat. Ducto EF, quæ vocetur $= y$, erit $y = Sc.\mu$. Sit AF $= z$; ergo $z = x - Cc.\mu$; ſed ex proprietate $x = m.\overline{r - Cc.\mu}$; ergo $z + Cc.\mu = mr - mCc.\mu$; ſive $\overline{m+1}.Cc.\mu = mr - z$; ſed $Cc.\mu = \sqrt{rr - yy}$; ergo $\overline{m+1}.\sqrt{rr - yy} = mr - z$, & quadrando $\overline{m+1}^2.r^2 - (\overline{m+1}^2.y^2 = m^2r^2 - 2mrz + zz$, ergo $\overline{m+1}^2.r^2 - (\overline{mr - z}^2 : yy :: \overline{m+1}^2.r^2 : r^2$. Quare ſumpto primo ſemiaxe CG $= \overline{m+1}.r$, ſecundo CD $= r$ deſcribatur ellypſis GED: (*Fig.* 22) hæc erit curva quæſita, in qua ſumpto quocumque puncto E, & applicata EB $= r$, ſectaque GA $= r$, erit ſemper AB $: r - Sc.\mu :: m : 1$. Ex his proprietas hæc ellypſis dimanat. Ex quocumque puncto ellypſis E accomodetur EB $=$ ſemiaxi CD, quem voco $= r$, cui pariter ſit æqualis GA, erit AB $: r - Sc.$ABE $::$ CG $-$ CD $:$ CD. Quod ſi ſit CG $<$ CD, AB proveniet negativa.

55. Sed ad quadraturam curvæ progrediamur. Spatium GBE ex formula $= Sdx.Sc.\mu + \frac{r\mu}{2}$; ſed $dx = -mdCc.\mu$; ergo GBE $= mS - Sc.\mu.d.Cc.\mu + \frac{r\mu}{2}$; ſed $-dCc.\mu = \frac{d\mu\, Sc.\mu}{r}$; ergo GBE $= mS\frac{d\mu.\overline{Sc.\mu}^2}{2} + \frac{r\mu}{2}$; atqui

Qui $S d\mu . \overline{Sc.\mu}^2 = \frac{r^2\mu}{2} - \frac{r\,Sc.\mu.Cc.\mu}{2}$; ergo $GBE = \frac{\overline{m+1}.r\mu}{2} - m$ $\frac{Sc.\mu.Cc.\mu}{2}$, quæ dependet a quadratura circuli. Centro B deſcribatur arcus OE, & ducatur ſinus EF. Ex formula habebimus $GEO + OEB = m\,OEF + OEB$; ergo $GEO = m.OEF$.

56. Nunc ad rectificationem, in cujus formula pro $dx$ ſcribe $-m\,dCc.\mu$, ut habeas $\sqrt{m^2 d.\overline{Cc.\mu}^2 - \frac{2m\,d\mu\,dCc.\mu.Sc.\mu}{r} + d\mu^2}$; atqui $-dCc.\mu$ $= \frac{d\mu.Sc.\mu}{r}$; ergo elementum arcus $= \int\sqrt{\frac{m^2 d\mu^2.\overline{Sc.\mu}^2}{r^2} + \frac{2m\,d\mu^2.\overline{Sc.\mu}^2}{r^2} + d\mu^2}$ $= \frac{d\mu}{r}\sqrt{m^2 + 2m.\overline{Sc.\mu}^2 + r^2}$, quæ certe exhibet rectificationem ellypſis. Reducitur autem ad notas formulas, ſi pro $d\mu$ ſubſtituas $\frac{r\,dSc.\mu}{Cc.\mu} = \frac{r\,dSc.\mu}{\sqrt{rr - \overline{Sc.\mu}^2}}$.

Hæc ſatis ſint. Quiſque videt, me ſuppoſuiſſe lineam deſcribentem AG primum ſitam eſſe in recta, in qua movetur punctum B, quod recedit a puncto G. Cæterum methodus uſurpata declarat, quo pacto analyſis inſtituenda ſit in aliis ſuppoſitionibus.

57. Problema decimum tertium. Complanare ſuperficiem cylindri ſcaleni ADEC. (*Fig.* 23) Secetur cylinder plano HSK normali ad axem BF. Ducantur in ſuperficie cylindrica rectæ MN, mn infinite proximæ, quæ erunt parallelæ axi BF, & efficient parallelogrammum MNnm, quod eſt elementum ſuperficiei cylindricæ. Hoc autem parallelogrammum æquale eſt rectangulo MN. Ss, quia quum planum HSK ſit normale BS, erit etiam normale rectis MN, mn, ergo etiam normale plano MNnm, & communis interſectio Ss normalis MN; hinc parallelogrammum MNnm æquale erit rectangulo MN in Ss. Igitur omnium parallelogrammorum ſumma, hoc eſt ſuperficies cylindrica æqualis erit rectangulo rectæ MN ductæ in perimetrum curvæ HSK; ſed hæc curva eſt ellypſis, ergo ſuperficies cylindrica æquat rectangulum rectæ MN in perimetrum ellypſis, adeoque ejus complanatio dependet ab ellypſis rectificatione. Si velis ellypſim magis determinare, per centrum F duc diametrum DFE, in quam lineam productam, ſi opus eſt caderet normalis a puncto B ducta in planum circuli DE. Vocetur $= \mu$ arcus anguli BFE, & ſumatur $FE = r$ pro ſinu toto. Ductis in ſuperficie DA, EC incidentes in ellypſim in H, K, erit unus axis HK, alter æqualis diametro circuli $= 2r$. Facile eſt determinare $HK = 2\,Sc.\mu$.

58. Problema decimum quartum. Complanare ſuperficiem coni ſcaleni AFBP, (*Fig.* 24) cujus baſis ſit circulus AFBM, vertex P. Secetur conus plano ABP tranſeunte per diametrum ACB, in quam productam, ſi opus eſt, cadat PD baſis plano normalis. Sumatur minimus arcus Ff, cujus tangens ſit TFQ, & ex F ducatur ordinata FE, tum radius FC. Rectæ TQ ſit normalis QD, & jungantur FP, fP, QP. Perſpicuum eſt, ſuperficiei conicæ elementum $FPf = \frac{Ff.PQ}{2}$. Etenim quum planum PDQ tranſeat per PD normalem

malem plano baſis, eidem plano normale erit planum PQD; ſed tangens TFQ normalis eſt QD communi ſectioni planorum TQD, QDP; ergo TQ normalis erit QP; adeoque triangulum $FPf = \frac{QP.Ff}{2}$. Sit modo $CA = r$, $CD = b$, $CE = x$, erit $FE = \sqrt{rr - xx}$. Quoniam angulus CFT rectus eſt, erit $CE : CF :: CF : CT$, ſive $x : r :: r : CT = \frac{rr}{x}$; ſed $CT : CF :: TD : DQ$, ſive $\frac{rr}{x} : r :: \frac{rr}{x} + b : DQ = \frac{rr + bx}{r}$. Vocetur $PD = a$, fiet $PQ = \sqrt{aa + \frac{\overline{rr + bx}^2}{rr}}$. Minimus arcus circularis $Ff = \frac{-rdx}{\sqrt{rr - xx}}$. Igitur

$$FPf = \frac{PQ.Ff}{2} = \frac{-rdx . \sqrt{aa + \frac{\overline{rr+bx}^2}{rr}}}{2\sqrt{rr - xx}} =$$

$$\frac{-bdx . \sqrt{\frac{a^2rr + r^4}{b^2} + \frac{2rr}{b}x + xx}}{2\sqrt{rr - xx}}.$$ Hæc autem formula ſui generis judicanda eſt, quia neque per quadraturas, neque per rectificationes ſectionum conicarum conſtrui poteſt. Quapropter tum ipſa, tum omnes, quæ ad eam reduci poſſunt, complanato cono ſcaleno, conſtructionem recipient.

59. Problema decimum quintum. Curva AMB, (*Fig.* 25) de qua loquuti ſumus tomo primo lib. 3. cap. 7. num. 22, revolvatur circa AB, quæritur ſolidum generatum. Curvæ hæc eſt æquatio $y^2 = \frac{a^2x - 2ax^2 + x^3}{2a - x} = -x^2 - a^2 + \frac{2a^3}{2a - x}$, in qua $AB = a$, $AP = x$. Igitur $y^2dx = -x^2dx - a^2dx + \frac{2a^3dx}{2a - x}$, quæ integretur adjuncta neceſſaria conſtante deinceps determinanda; erit $Sy^2dx = A - \frac{x^3}{3} - a^2x - 2a^3 l\overline{2a - x}$; logarithmus ſumptus eſt in ſyſtemate ſubtangentis $= a$. Ponamus primo ſummatoriam nullam fieri quum $x = 0$. In hac hypotheſi invenies $A = 2a^3 l2a$; ergo $Sy^2dx = 2a^2 l2a - \frac{x^3}{3} - a^2x - 2a^3 l\overline{2a - x}$; igitur $\frac{p}{2r} Sy^2dx$, ſive ſolidum generatum ab area AMP $= \frac{p}{2r} . \overline{2a^2 l2a - \frac{x^3}{3} - a^2x - 2a^3 l\overline{2a - x}}$; & facta $x = a$, fiet ſolidum genitum

ab area AMB $= \frac{p}{r} . a^2 . \overline{l2a - la} - \frac{2a^3}{3}$, quæ formula brevior fiet, ſi ſumatur protonumerus $= a$, ut $la = o$. Ponamus deinde $Sy^2dx$ nullam eſſe, quum $x = a$. In hac hypotheſi inveniemus $A = \frac{4a^3}{3} + 2s^2 la$; ergo $Sy^2dx = \frac{4a^3}{3} + 2a^2 la - \frac{x^3}{3} - a^2x - 2a^2 l\overline{2a - x}$, quæ multiplicata per $\frac{p}{2r}$ dabit ſolidum genitum ab area B2M2P. Si fiat $x = 2a$, determinabitur ſolidum deſcriptum ab area infinite longa BDQ $= \frac{p}{r} . a^2 la - a^2 lo - \frac{5a^3}{3}$, quæ brevior fiet, ſi $la = o$. Formula autem indicat, ſolidum eſſe infinitum, quia $lo$ eſt quantitas negativa, & infinita.

60. Problema decimum ſextum. Si curva KXH, (*Fig.* 4) de qua egimus Tom. 1. Lib. 3. cap. 7. prob. 5, rotetur circa AK, quæritur tum ſuperficies genita a curva KX, tum ſolidum genitum ab area KXI. Hujuſce curvæ quadraturam, & rectificationem invenimus prob. 4, ibique MN $= a$, MO $=$ NX $= z$, invenimus curvam KX $= \frac{3}{2} z$; ergo X$x = \frac{3}{2} dz$. Facile eſt probare XI $= \frac{z^{\frac{3}{2}}}{a^{\frac{1}{2}}}$; atqui elementum ſuperficiei genitæ a KX $= \frac{p}{r}$. XI. X$x$; ergo idem elementum $= \frac{3p}{2r} . \frac{z^{\frac{3}{2}} dz}{a^{\frac{1}{2}}}$, & integrando ſine additione conſtantis, quæ neceſſaria non eſt, ſuperficies deſcripta a KX $= \frac{3p}{5r} . \frac{z^{\frac{5}{2}}}{a^{\frac{1}{2}}}$, & poſita $z = a$ tota ſuperficies genita a curva KXH $= \frac{3p}{5r} . aa$.

61. Ut ſolidum determinetur, oportet invenire elementum I$i$. Quiſque videt AN $= \sqrt{aa - az}$; ſed eſt NM : AN :: X$x$ : I$i$.; ergo $a : \sqrt{aa - az}$, ſeu $\sqrt{a} : \sqrt{a - z} :: \frac{3}{2} dz :$ I$i = \frac{3}{2} \frac{dz . \sqrt{a - z}}{\sqrt{a}}$; atqui elementum ſolidi quæſiti $= \frac{p}{2r}$. IX$^2$. I$i$; ergo idem elementum $= \frac{3p}{4r} . \frac{z^3 dz . \sqrt{a - z}}{a\sqrt{a}}$. Hæc integrationem algebraicam recipit, quod tibi palam fiet, ſi ponas NO $= t$, ut ſit $a - z = t$, & $dz = - dt$. Facta enim ſubſtitutione evadet

$$z^3 dz \sqrt{a - z} = \overline{a - t}^3 . - dt . t^{\frac{1}{2}} = - a^3 t^{\frac{1}{2}} dt + 3a^2 t^{\frac{3}{2}} dt - 3a t^{\frac{5}{2}} dt + t^{\frac{7}{2}} dt,$$

& integrando non neglecta conſtante

$$Sz^3dz\sqrt{a-z}=A-\frac{2}{3}a^3t^{\frac{3}{2}}+\frac{2.3}{5}a^2t^{\frac{5}{2}}-\frac{2.3}{7}at^{\frac{7}{2}}+\frac{2}{9}t^{\frac{9}{2}}.$$

Hæc debet nulleſcere ſi $z=o$, ſeu $t=a$; ergo fiet

$$A=\frac{2}{3}a^{\frac{9}{2}}-\frac{2.3}{5}a^{\frac{9}{2}}+\frac{2.3}{7}a^{\frac{9}{2}}-\frac{2}{9}a^{\frac{9}{2}}=\frac{96}{945}a^{\frac{9}{2}}=\frac{32}{315}a^{\frac{9}{2}};$$

ergo integratio completa erit

$$Sz^3dz\sqrt{a-z}=\frac{32}{315}a^{\frac{9}{2}}-\frac{2}{3}a^3t^{\frac{3}{2}}+\frac{2.3}{5}a^2t^{\frac{5}{2}}-\frac{2.3}{7}at^{\frac{7}{2}}+\frac{2}{9}t^{\frac{9}{2}}.$$

Quæ, ſi multiplicetur per $\frac{3p}{4ra^{\frac{3}{2}}}$, dabit ſolidum quæſitum =

$$\frac{3p}{r}\cdot\frac{8a^3}{315}-\frac{1}{6}a^{\frac{3}{2}}t^{\frac{3}{2}}+\frac{3}{10}a^{\frac{1}{2}}t^{\frac{5}{2}}-\frac{3t^{\frac{7}{2}}}{14a^{\frac{1}{2}}}+\frac{1t^{\frac{9}{2}}}{18a^{\frac{3}{2}}}:$$

Fiat jam $t=o$, & totum ſolidum genitum ab area KHA erit $=\frac{p}{r}\cdot\frac{8a^3}{105}$.

62 Problema decimum ſeptimum. Revolvatur cyclois AFM (*Fig.* 26) circa tangentem AK. Quæritur ſuperficies deſcripta ab arcu AF. Ducta ordinata FSB, & chorda AS, vocetur circuli genitoris diameter $AH=2a$, $AB=x$, erit chorda $AS=\sqrt{2ax}$; ergo arcus AF, qui duplus eſt chordæ $=2\sqrt{2a}.\sqrt{x}$, & ejus elementum $Ff=\frac{dx\sqrt{2a}}{\sqrt{x}}$. Atqui elementum quæſitæ ſuperficiei eſt æquale $\frac{p}{r}$. AB. Ff; ergo $=\frac{p}{r}\sqrt{2a}.dx.\sqrt{x}$, & integrando nulla addita conſtante, quæ neceſſaria non eſt, ſuperficies genita ab arcu $AF=\frac{p}{r}\sqrt{2a}.\frac{2}{3}x^{\frac{3}{2}}=\frac{2p}{3r}.x\sqrt{2ax}$. Si $x=a=AC$, fiet ſuperficies genita ab $AE=\frac{2\sqrt{2}.pa^2}{3r}$; ſi $x=2a=AH$, tota ſuperficies deſcripta a demicycloide $AFM=\frac{8p}{3r}.aa$. Solutiones problematum, quas in hoc capite exhibuimus, ſatis oſtendere judicamus, qua methodo alia hujus generis problemata ſint reſolvenda.

## CAPUT DECIMUM SEXTUM.

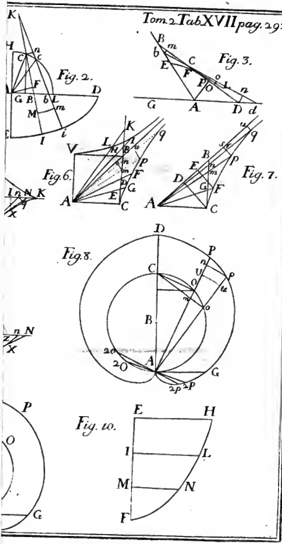
### Aliquot problematum inverſorum ſolutio exhibetur.

1. Problema primum. Invenire curvam BC ( *Fig.* 1 ) relatam ad ordinatas orthogonales AE, EC, in qua area ABCE æquet rectangulum ex curva BC in datam $= a$. Vocatis $AE = x$, $EC = y$ ſtatim prodit æquatio $S y dx = a S \sqrt{dx^2 + dy^2}$, ſive $y dx = a \sqrt{dx^2 + dy^2}$. Antequam progrediar, conſiderans æquationem, video huic ſatisfacere lineam rectam parallelam AE; nam poſita $AB = a$, ductaque BF parallela AE, fit $y = a$, $dy = o$, quibus valoribus ſubſtitutis naſcitur æquatio identica. Hoc autem ex ipſis elementis cognoſcimus. Verum elata æquatione ad quadratum fit $\overline{y^2 - a^2} . dx^2 = a^2 dy^2$; ergo $dx = \frac{a dy}{\sqrt{y^2 - a^2}}$, quæ eſt curva coſinuum hyperbolicorum. Si itaque deſcribatur curva BC, cujus ordinatæ CE æquent coſinus hyperbolicos, abſciſſæ AE logarithmos analogos, exiſtente $AB = a$ ſinu toto, erit area $ABCE = AB . BC$. Adverte, quantitates iſtas algebraicas eſſe; invenies enim $BC = \sqrt{y^2 - a^2}$, ſcilicet æqualem ſinui hyperbolico.

2. Problema ſecundum. Invenire curvam AB ( *Fig.* 2 ) relatam ad focum C, in qua area CAB ſit æqualis curvæ AB ductæ in conſtantem $= \frac{1}{2} a$. Facto angulo minimo BCb, & centro C deſcripto minimo arcu Bm, ſit $CB = y$, $Bm = dx$: igitur ex conditione problematis erit $y dx = a . \sqrt{dx^2 + dy^2}$. Huic æquationi ſatisfacit $y = a$. Quare circulus AD deſcriptus radio $CA = a$ præditus eſt requiſita conditione. Ut videamus, utrum præter circulum alia reperiatur curva eadem conditione donata, ad hanc redigamus æquationem $dx = \frac{a dy}{\sqrt{y^2 - a^2}}$. Hæc eſt æquatio curvæ quæſitæ, ſed nondum conſtat, utrum algebraica ſit vel traſcendens. Deſcripto circulo radio $CA = a$ vocetur arcus $AD = \mu$, & $Dd = d\mu$. Quum ſit $y : dx :: a : d\mu$, æquatio fiet $d\mu = \frac{a^2 dy}{y \sqrt{y^2 - a^2}}$. Hæc ultima formula eſt differentia arcus circularis, cujus radius $= a$, ſecans $= y$; ergo arcus AD, cujus radius pariter $= a$, habet ſecantem $CB = y$; ſed CB ſecans eſſe non poteſt, niſi AB fuerit linea recta. Igitur præter circulum ſola linea recta prædita eſt proprietate, ut ſpatium CAB æquet rectangulum ex AB in $\frac{1}{2}$ CA.

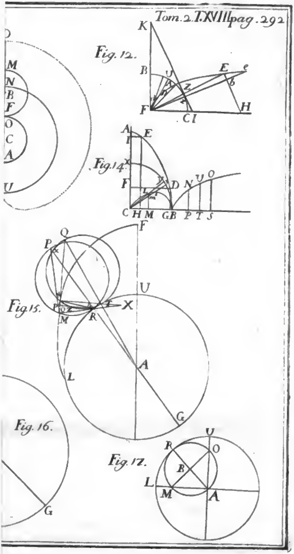
3. Problema tertium. Quæritur curva relata ad coordinatas orthogonales,

cujus

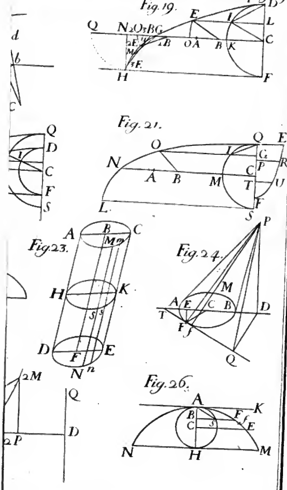



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cujus area datur per ſolam abſciſſam. Curva eſt ſemper algebraica, ſi functio abſciſſæ, per quam datur, aut algebraica eſt, aut involvat unicum ſignum ſummatorium omnem quantitatem afficiens; ut ejus differentia exprimi poſſit per $P\,dx$. Etenim quum debeat eſſe $y\,dx = P\,dx$, facta diviſione per $dx$, fiet $y = P$, quæ eſt æquatio curvæ algebraicæ. Si $P$ ſit poteſtas $x$, aut poſitiva, aut negativa, curvæ erunt aut parabolæ, aut hyperbolæ. Ut aliquod exemplum proferam poſito ſemicirculo AEB, quæratur curva AIM, (*Fig.* 3) cujus area AID æquet ſegmentum circuli clauſum inter arcum, & ſuam chordam AE. Vocata $AD = x$, $DI = y$; conſtat ſegmentum exprimi per $S\frac{r\,dx\sqrt{x}}{2\sqrt{2r-x}}$; ergo erit $y\,dx = \frac{r\,dx\sqrt{x}}{2\sqrt{2r-x}}$, ſeu $y = \frac{r\sqrt{x}}{2\sqrt{2r-x}}$, quæ eſt curva tertij gradus, quæ incipit in A, ſecat ſemicirculum in F, ubi abſciſſa $AG = \frac{3}{2}r$, ſeu dividit radium CB bifariam in G, tum accedit ad aſymptotum BM. Hujus curvæ ſpatium AID æquat ſegmentum circulare AE; addatur utrique ſpatium AIE interceptum inter chordam AE, & curvam, & fiet AIE interceptum inter curvam & circulum æquale triangulo rectilineo AED. Itaque ſpatium infinite lungum AFMB clauſum inter curvam & aſymptotum æquale erit ſemicirculo, & communi ablato fiet ſpatium infinite lungum $FBM = AEFIA$; atqui hoc æquat triangulum rectilineum AFG; ergo idem FBM eſt æquale triangulo AFG.

4. Quæramus ad alterum exemplum curvam AI, (*Fig.* 4) cujus area AID æquet ſegmentum hyperbolicum AE clauſum inter curvam, & chordam. Hoc ſegmentum, vocata $AD = x$, erit $S\frac{r\,dx\sqrt{x}}{2\sqrt{2r+x}}$; ergo æquatio proveniet $\frac{r\,dx\sqrt{x}}{2\sqrt{2r+x}} = y\,dx$, ſeu $\frac{r\sqrt{x}}{2\sqrt{2r+x}} = y$, quæ eſt pariter curva tertij gradus, quæ, abſciſſa $CH = CK = \frac{r}{2}$ & ductis parallelis primo axi HM, K2M, diſcedit ab A, & accedit ad aſymptotum HM, tum prædita eſt alio ramo, cujus aſymptota ſunt 2AN, K2M. In hac ſpatium AID æquat ſegmentum AE: ergo addito trilineo AIE clauſo inter chordam AE, rectam IE, & curvam AI, fiet ſpatium clauſum inter arcum AE, arcum AI, & rectam IE æquale triangulo rectilineo AED. Quoad alterum ramum ducta qualibet A2E ad hyperbolam, actaque ordinata hyperbolæ 2E2D, quæ curvam ſecet in 2I, ſector hyperbolicus A2E2A æquabit ſpatium infinite lungum N2A2D2I.

5. Problema quartum. Quæratur curva, quæ, ſi convertatur circa axem abſciſſarum, exhibeat ſolidum datum per functionem abſciſſæ ductam in $\frac{p}{r}$, quæ eſt ratio circumferentiæ ad radium. Curva eſt ſemper algebraica, ſi aut algebraica ſit functio, aut unicum ſignum ſummatorium habeat totam quantitatem afficiens. Ad exemplum quæratur curva, quæ converſa circa axem exhibeat ſolidum $= \frac{p}{r} S\frac{a^3 dx}{x}$. Habebimus æquationem $\frac{p}{2r} Sy^2 dx = \frac{p}{r} S\frac{a^3 dx}{x}$; er-

go

go $y^2 = \frac{a^3}{x}$, ſeu $y^2x = a^3$, quæ eſt hyperbolica.

6. **Problema quintum**. Poſito $FA = b$, $FB = y$, deſcripto radio FA (*Fig.* 5) arcu circulari $AD = \mu$, quæritur curva, in qua $r^3y = Cc.\mu . S\,dTc.\mu . S\overline{Cc.m\mu.dSc.\mu}$, in qua ita accipi debet ſummatoria, ut poſito $\mu = 0$, ſit $y = r$. Facta integratione invenies $S\,dTc.\mu.S\overline{Cc.m\mu.dSc.\mu} = Tc.\mu.S\overline{Cc.m\mu.dSc.\mu} - S\overline{Tc.\mu.Cc.m\mu.dSc.\mu}$; igitur $r^3y = Cc.\mu.Tc.\mu.S\overline{Cc.m\mu.dSc.\mu} - Cc.\mu.S\overline{Tc.\mu.Cc.m\mu.dSc.\mu}$; ſed $Cc.\mu.Tc.\mu = rSc.\mu$, & $dSc.\mu = \frac{d\mu\, Cc.\mu}{r}$; igitur $r^3y = Sc.\mu\, S\overline{Cc.m\mu.Cc.\mu.d\mu} - Cc.\mu\, S\overline{Cc.m\mu.Sc.\mu.d\mu}$

Poſtquam ad hanc formam redacta eſt æquatio, inveniendæ ſunt ſeparatim ſummatoriæ illæ duæ, quas continet. Quam ob rem notandum eſt, haberi ex cap. 12. lib 2. Tom. 1. numero ſexto quatuor ſequentes formulas, in quibus ſupponimus $\varphi > \pi$; $Sc.\varphi.Cc.\pi = \frac{r}{2} Sc.\overline{\varphi + \pi} + \frac{r}{2} Sc.\overline{\varphi - \pi}$;

$Cc.\varphi.Sc.\pi = \frac{r}{2} Sc.\overline{\varphi + \pi} - \frac{r}{2} Sc.\overline{\varphi - \pi}$; $Sc.\varphi.Sc.\pi = \frac{r}{2} Cc.\overline{\varphi - \pi} - \frac{r}{2} Cc.\overline{\varphi + \pi}$; $Cc.\varphi.Cc.\pi = \frac{r}{2} Cc.\overline{\varphi + \pi} + \frac{r}{2} Cc.\overline{\varphi - \pi}$.

7. His notatis poſito $\varphi = m\mu$, $\pi = \mu$, ex quarta formula habemus $Cc.m\mu.Cc.\mu = \frac{r}{2} Cc.\overline{m+1}.\mu + \frac{r}{2} Cc.\overline{m-1}.\mu$; ergo facta multiplicatione per $d\mu$, erit $d\mu\, Cc.m\mu.Cc.\mu = \frac{r}{2} d\mu.Cc.\overline{m+1}.\mu + \frac{r}{2} d\mu.Cc.\overline{m-1}.\mu = \frac{r}{2.\overline{m+1}}.\overline{m+1}\, d\mu.Cc.\overline{m+1}.\mu + \frac{r}{2.\overline{m-1}}.\overline{m-1}.d\mu.Cc.\overline{m-1}.\mu$; ſed $\overline{m+1}.d\mu.Cc.\overline{m+1}.\mu = r\,dSc.\overline{m+1}.\mu$, & $\overline{m-1}.d\mu.Cc.\overline{m-1}.\mu = rSc.\overline{m-1}.\mu$; ergo $d\mu Cc.m\mu.Cc.\mu = \frac{r^2}{2}\frac{dSc.\overline{m+1}.\mu}{m+1} + \frac{r^2}{2}\frac{dSc.\overline{m-1}.\mu}{m-1}$, & integrando addita conſtante $S\,d\mu.Cc.m\mu.Cc.\mu = \frac{r^2}{2}.A + \frac{Sc.\overline{m+1}.\mu}{m+1} + \frac{Sc.\overline{m-1}.\mu}{m-1}$. Multiplicemus per $Sc.\mu$, ut prodeat $Sc.\mu.S\overline{d\mu.Cc.m\mu.Cc.\mu} = \frac{r^2}{2}.ASc.\mu + \frac{Sc.\mu\, Sc.\overline{m+1}.\mu}{m+1} + \frac{Sc.\mu.Sc.\overline{m-1}.\mu}{m-1}$.

Advocemus hic tertiam formulam, per quam habebimus $Sc.\mu.Sc.\overline{m+1}.\mu = \frac{r}{2} Cc.m\mu - \frac{r}{2} Cc.\overline{m+2}.\mu$, &

Sc.

$Sc.\mu.Sc.\overline{m-1}.\mu = \frac{r}{2}.Cc.\overline{m-2}.\mu - \frac{r}{2}.Cc.m\mu$; ergo

$\frac{Sc.\mu.Sc.\overline{m+1}.\mu}{m+1} + \frac{Sc.\mu.Sc.\overline{m-1}.\mu}{m-1} =$

$\frac{r}{2}.\overline{\frac{-2Cc.m\mu}{mm-1} - \frac{Cc.\overline{m+2}.\mu}{m+1} + \frac{Cc.\overline{m-2}.\mu}{m-1}}$; igitur

$$Sc.\mu S\overline{d\mu Cc.m\mu.Cc.\mu} = \frac{r^3}{4}.\overline{\frac{2ASc.\mu}{r} - \frac{2Cc.m\mu}{mm-1} - \frac{Cc.\overline{m+2}.\mu}{m+1} + \frac{Cc.\overline{m-2}.\mu}{m-1}}.$$

8. Eadem methodo tractanda eſt alia formula $Cc.\mu S\overline{Cc.m\mu.Sc.\mu.d\mu}$. Nam ex formula ſecunda inveniemus

$Cc.m\mu.Sc.\mu = \frac{r}{2}Sc.\overline{m+1}.\mu - \frac{r}{2}Sc.\overline{m-1}.\mu$; ergo

$d\mu Cc.m\mu.Sc.\mu = \frac{r}{2}d\mu Sc.\overline{m+1}.\mu - \frac{r}{2}d\mu Sc.\overline{m-1}.\mu =$

$\frac{r}{2.\overline{m+1}}.\overline{m+1}.d\mu\, Sc.\overline{m+1}.\mu - \frac{r}{2.\overline{m-1}}.\overline{m-1}.d\mu.Sc.\overline{m-1}.\mu$; atqui

$\overline{m+1}.d\mu\, Sc.\overline{m+1}.\mu = -r\,dCc.\overline{m+1}.\mu$, &

$\overline{m-1}.d\mu\, Sc.\overline{m-1}.\mu = -r\,dCc.\overline{m-1}.\mu$; ergo

$d\mu Cc.m\mu.Sc.\mu = \frac{rr}{2}.\overline{\frac{-dCc.\overline{m+1}.\mu}{m+1} + \frac{dCc.\overline{m-1}.\mu}{m-1}}$, & integrando adjecta conſtante

$S\overline{d\mu Cc.m\mu.Sc.\mu} = \frac{r^2}{2}.\overline{B\frac{-Cc.\overline{m+1}.\mu}{m+1} + \frac{Cc.\overline{m-1}.\mu}{m-1}}$, & per $Cc.\mu$ multiplicando $Cc.\mu S\overline{d\mu.Cc.m\mu.Sc.\mu} =$

$\frac{r^2}{2}.\overline{BCc.\mu - \frac{Cc.\mu.Cc.\overline{m+1}.\mu}{m+1} + \frac{Cc.\mu.Cc.\overline{m-1}.\mu}{m-1}}$. Adhibenda jam eſt quarta formula, & inveniemus

$Cc.\mu.Cc.\overline{m+1}.\mu = \frac{r}{2}.Cc.\overline{m+2}.\mu + \frac{r}{2}.Cc.m\mu$, &

$Cc.\mu.Cc.\overline{m-1}.\mu = \frac{r}{2}.Cc.\mu + \frac{r}{2}.Cc.\overline{m-2}.\mu$; ergo

$Cc.\mu.S\overline{d\mu Cc.m\mu.Sc.\mu} =$

$\frac{r^3}{4}.\overline{\frac{2BCc.\mu}{r} + \frac{2Cc.m\mu}{mm-1} - \frac{Cc.\overline{m+2}.\mu}{m+1} + \frac{Cc.\overline{m-2}.\mu}{m-1}}.$

9. Demum ſi hanc formulam deducamus ab ea, quæ inventa eſt numero ſeptimo, reperiemus $Sc.\mu S\overline{d\mu Cc.m\mu.Cc.\mu} - Cc.\mu.S\overline{d\mu Cc.m\mu.Sc.\mu}$, ſeu $r^3y = \frac{r^3}{4}.\overline{\frac{ASc.\mu - BCc.\mu}{r} - \frac{4Cc.m\mu}{mm-1}}$, ſive

$y =$

$y = \frac{A Sc.\mu - B Cc.\mu}{4r} - \frac{Cc.m\mu}{mm-1}$. Ut conditio impleatur, ut poſito $\mu$, & $Sc.\mu = o$, adeoque $Cc.\mu$, & $Cc.m\mu = r$, ſit $y = r$, debet valere hæc æquatio $r = \frac{-B}{4} - \frac{r}{mm-1}$; ergo $\frac{B}{4} = \frac{-mmr}{mm-1}$; igitur æquatio

$y = \frac{A Sc.\mu}{4r} + \frac{mm Cc.\mu}{mm-1} - \frac{Cc.m\mu}{mm-1}$, in qua $A$ quemcumque valorem habere poteſt. Hæc æquatio clariſſime indicat, curvam ſemper eſſe algebraicam, ſed eo altioris gradus, quo altiores ſunt numeri ſimpliciſſimi exhibentes rationem $m:1$.

10. Hæc quidem ſolutio locum habet, ſi non ſit $m = 1$, ſed prorſus deficit exiſtente $m = 1$. Ut hunc quoque caſum tractemus, accipiamus æquationem jam præparatam, quæ in hypotheſi eſt hujuſmodi

$r^3 y = Sc.\mu\, S\overline{Cc.\mu}^2.d\mu - Cc.\mu.S\overline{Cc.\mu.Sc.\mu.d\mu}$. Ex num. 18. Cap. 10. Si addamus neceſſariam conſtantem, erit

$S\overline{Cc.\mu}^2.d\mu = \frac{r}{2}.\overline{Ar + r\mu + Cc.\mu.Sc.\mu}$, & multiplicando per $Sc.\mu$, eſt

$Sc.\mu\, S\overline{Cc.\mu}^2.d\mu = \frac{r}{2}.\, Ar Sc.\mu + r\mu Sc.\mu + Cc.\mu.\overline{Sc.\mu}^2$. Inſuper $Cc.\mu.Sc.\mu.d\mu = r Sc.\mu.d Sc.\mu$, & integrando

$S\, Cc.\mu.Sc.\mu.d\mu = \frac{r}{2}.\, Br + \overline{Sc.\mu}^2$, & multiplicando per $Cc.\mu$ fiet

$Cc.\mu.S Cc.\mu.Sc.\mu.d\mu = \frac{r}{2}.\overline{Br Cc.\mu + Cc.\mu.Sc.\mu}$. Quare deducta hac a ſuperiore formula erit $2ry = A Sc.\mu - B.Cc.\mu + \mu\overline{Sc.\mu}^2$. Facto $\mu$, & $Sc.\mu = o$; eſt $Cc.\mu = r$, ergo $2rr = -rB$, ſeu $B = -2r$; ergo æquatio $2ry = A Sc.\mu + 2r Cc.\mu + \mu Sc.\mu$, ſeu $2r.\overline{y - Cc.\mu} = \overline{A + \mu}.Sc.\mu$, quæ ſemper dependet a circuli quadratura.

11. Problema ſextum. Invenire curvam relatam ad coordinatas orthogonales, cujus abſciſſæ $x = Cb.\mu$, ordinatæ

$y = \frac{S Cb.m\mu.Sb.\mu.d\mu \pm S Sb.m\mu.Cb.\mu.d\mu}{r^2}$, exiſtente $y = o$, ſi $\mu = o$. Inveniamus ſingulas ſummatorias hoc modo. Ab elementis ſinuum & coſinuum hyperbolicorum hæc theoremata deſumamus $Sb.\varphi.Cb.\pi = \frac{r}{2}.Sb.\overline{\varphi + \pi} + \frac{r}{2} Sb.\overline{\varphi - \pi}$,

$Cb.\varphi.Sb.\pi = \frac{r}{2} Sb.\overline{\varphi + \pi} - \frac{r}{2} Sb.\overline{\varphi - \pi}$,

$Cb.\varphi.Cb.\pi = \frac{r}{2} Cb.\overline{\varphi + \pi} + \frac{r}{2} Cb.\overline{\varphi - \pi}$,

$Sb.\varphi.Sb.\pi = \frac{r}{2} Cb.\overline{\varphi + \pi} - \frac{r}{2} Cb.\overline{\varphi - \pi}$. His poſitis veniamus ad primam.

mam. Ex ſecunda formula $Cb.m\mu.Sb.\mu = \frac{r}{2}Sb.\overline{m+1}.\mu - \frac{r}{2}Sb.\overline{m-1}.\mu$, multiplicando per $d\mu$, fit $d\mu\, Cb.m\mu.Sb.\mu = \frac{r}{2.\overline{m+1}}.\overline{m+1}.d\mu.Sb.\overline{m+1}.\mu - \frac{r}{2.\overline{m-1}}.\overline{m-1}.d\mu.Sb.\overline{m-1}.d\mu$, ſive $d\mu\, Cb.m\mu.Sb.\mu =$

$\frac{r^2}{2}.\frac{+Cb.\overline{m+1}.\mu}{\overline{m+1}} - \frac{r^2 dCb.\overline{m-1}.\mu}{2.\overline{m-1}}$, & integrando addita conſtante habemus

$S d\mu\, Cb.m\mu.Sb.\mu = \frac{r^2}{2}.\overline{A + \frac{Cb.\overline{m+1}.\mu}{m+1} - \frac{Cb.\overline{m-1}.\mu}{m-1}}$. Ad ſecundam erit

$Sb.m\mu.Cb.\mu = \frac{r}{2}Sb.\overline{m+1}.\mu + \frac{r}{2}Sb.\overline{m-1}.\mu$, ergo

$d\mu\, Sb.m\mu.Cb.\mu = \frac{r}{2.\overline{m+1}}.\overline{m+1}.d\mu\, Sb.\overline{m+1}.\mu +$

$\frac{r}{2.\overline{m-1}}.\overline{m-1}.d\mu\, Sb.\overline{m-1}.\mu$, ergo $d\mu\, Sb.m\mu.Cb.\mu =$

$\frac{r^2 dCb.\overline{m+1}.\mu}{2.\overline{m+1}} + \frac{r^2 dCb.\overline{m-1}.\mu}{2.\overline{m-1}}$, & facta integratione

$S d\mu.Sb.m\mu.Cb.\mu = \frac{r^2}{2}.\overline{B + \frac{Cb.\overline{m+1}.\mu}{m+1} + \frac{Cb.\overline{m-1}.\mu}{m-1}}$; ergo ſi in

propoſita formula valeat ſignum + habebimus $2y = A + B + \frac{2Cb.\overline{m+1}.\mu}{m+1}$, Ad implendam conditionem oportet ponere $A + B = \frac{-2r}{m+1}$. Quare æquatio erit $y = \frac{-r + Cb.\overline{m+1}.\mu}{m+1}$. Si vero valeat ſignum − proveniet

$2y = A - B - \frac{2Cb.\overline{m-1}.\mu}{m-1}$, in qua conditioni ſatisfiet, ſi ſit $A - B = \frac{2r}{m-1}$; unde æquatio $y = \frac{r - Cb.\overline{m-1}.\mu}{m-1}$.

12. Problema ſeptimum. Invenire curvam æquationis $\frac{y}{r} =$

$$\&c.\ S\frac{\left(d\mu\, Sc.\frac{1}{8}\mu\right.}{r^2} S\frac{\left(d\mu\, Sc.\frac{1}{4}\mu\right.}{r^4} S\frac{\left(d\mu.Sc.\frac{1}{2}\mu\right.}{r^2} S\frac{d\mu\, Cc.\mu}{r^2}$$

, cujus abſciſſa eſt aliqua functio $\mu$, ordinata vero eſt $y$, ea conditione, ut nulleſcente $\mu$ nulleſcat etiam $y$. Ut æquatio evolvatur, paullatim procedendum eſt, & initium ſumendum ab ultimis ſummatorijs hac methodo. Quoniam $\frac{d\mu\, Cc.\mu}{r^2} = \frac{dSc.\mu}{r}$ erit integrando $S\frac{d\mu\, Cc.\mu}{r^2} = \frac{Sc.\mu}{r}$; nulla additur conſtans,

ſtans, quia ita evaneſcente $\mu$ evaneſcit ſummatoria. Fiat multiplicatio per $Sc\frac{1}{2}\mu$, ut ſit $Sc\frac{1}{2}\mu.\,S\overline{\frac{d\mu\,Cc.\mu}{r^2}} = \frac{Sc\frac{1}{2}\mu.Sc.\mu}{r} = \frac{1}{2}Cc.\frac{1}{2}\mu - \frac{1}{2}Cc.\frac{3}{2}\mu$, quæ ducatur in $\frac{d\mu}{r^2}$, & erit $\frac{d\mu Sc.\frac{1}{2}\mu}{r^2}\,S\overline{\frac{d\mu.Cc.\mu}{r^2}} = \frac{\frac{1}{2}d\mu.Cc.\frac{1}{2}\mu}{r^2} - \frac{\frac{3}{2}d\mu Cc.\frac{3}{2}\mu}{3r^2} = \frac{dSc.\frac{1}{2}\mu}{r} - \frac{dSc.\frac{3}{2}\mu}{3r}$, & integrando $S\overline{\left(\frac{d\mu Sc.\frac{1}{2}\mu}{r^2}\,S\overline{\frac{d\mu Cc.\mu}{r^2}}\right.} = \frac{Sc.\frac{1}{2}\mu}{r} - \frac{Sc.\frac{3}{2}\mu}{3r}$. Multiplica per $Sc.\frac{1}{4}\mu$, ut ſit $Sc.\frac{1}{4}\mu\;S\overline{\left(\frac{d\mu Sc.\frac{1}{2}\mu}{r^2}\,S\overline{\frac{d\mu Cc.\mu}{r^2}}\right.} = \frac{1}{r}Sc.\frac{1}{4}\mu.Sc.\frac{1}{2}\mu - \frac{1}{3r}Sc.\frac{1}{4}\mu.Sc.\frac{3}{2}\mu = \frac{1}{2}Cc.\frac{1}{4}\mu - \frac{1}{2}Cc.\frac{3}{4}\mu - \frac{1}{2.3}Cc.\frac{5}{4}\mu + \frac{1}{2.3}Cc.\frac{7}{4}\mu$. Fiat multiplicatio per $\frac{d\mu}{r^2}$, ut oriatur

$$\frac{d\mu Sc.\frac{1}{4}\mu}{r^2}\,S\overline{\left(\frac{d\mu Sc.\frac{1}{2}\mu}{r^2}\,S\overline{\frac{d\mu Cc.\mu}{rr}}\right.} = \frac{2.\frac{1}{4}d\mu Cc.\frac{1}{4}\mu}{r^2} - \frac{2}{3}.\frac{\frac{3}{4}d\mu Cc.\frac{3}{4}\mu}{r^2} - \frac{2}{3.5}.\frac{\frac{5}{4}d\mu Cc.\frac{5}{4}\mu}{r^2} + \frac{2}{3.7}.\frac{\frac{7}{4}d\mu Cc.\frac{7}{4}\mu}{r^2} = \frac{2.dSc.\frac{1}{4}\mu}{r} - \frac{2}{3}.\frac{dSc.\frac{3}{4}\mu}{r} - \frac{2}{3.5}.\frac{dSc.\frac{5}{4}\mu}{r} + \frac{2}{3.7}.\frac{dSc.\frac{7}{4}\mu}{r}$$

Quare integrando erit

$$S\overline{\left(\frac{d\mu Sc.\frac{1}{4}\mu}{r^2}\,S\overline{\left(\frac{d\mu Sc.\frac{1}{2}\mu}{r^2}\,S\overline{\frac{d\mu Cc.\mu}{r^2}}\right.}\right.} = \frac{2Sc.\frac{1}{4}\mu}{r} - \frac{2}{3}.\frac{Sc.\frac{3}{4}\mu}{r} - \frac{2}{3.5}.\frac{Sc.\frac{5}{4}\mu}{r} + \frac{2}{3.7}.\frac{Sc.\frac{7}{5}\mu}{r}$$

Multiplicetur de more per

Sc.

$Sc.\frac{1}{8}\mu$, ut proveniat $Sc.\frac{1}{8}\mu\ \mathrm{S}\frac{d\mu Sc.\frac{1}{4}\mu}{r^2}\left(\mathrm{S}\frac{d\mu Sc.\frac{1}{2}\mu}{r^2}\left(\mathrm{S}\frac{d\mu Cc.\mu}{r^2}\right.\right. =$

$\frac{2}{r}.Sc.\frac{1}{8}\mu.Sc.\frac{1}{4}\mu - \frac{2}{3r}.Sc.\frac{1}{8}\mu Sc.\frac{3}{4}\mu - \frac{2}{3.5r}Sc.\frac{1}{8}\mu Sc.\frac{5}{4}\mu + \frac{2}{3.7r}Sc.\frac{1}{6}\mu Sc.\frac{7}{4}\mu = Cc.\frac{1}{8}\mu - Cc.\frac{3}{8}\mu - \frac{1}{3}Cc.\frac{5}{8}\mu + \frac{1}{3}Cc.\frac{7}{8}\mu - \frac{1}{3.5}Cc.\frac{9}{8}\mu + \frac{1}{3.5}Cc.\frac{11}{8}\mu + \frac{1}{3.7}Cc.\frac{13}{8}\mu - \frac{1}{3.5}Cc.\frac{15}{8}\mu$. Demum ſi multiplicatio fiat per $\frac{d\mu}{r^2}$, reperietur $\frac{d\mu Sc.\frac{1}{8}\mu}{r^2}\left(\mathrm{S}\frac{d\mu Sc.\frac{1}{4}\mu}{r^2}\left(\mathrm{S}\frac{d\mu Sc.\frac{1}{2}\mu}{r^2}\mathrm{S}\frac{d\mu Cc.\mu}{r^2}\right.\right. =$

$$\frac{8.\frac{1}{8}d\mu Cc.\frac{1}{8}}{rr} - \frac{8}{3}.\frac{\frac{3}{8}d\mu Cc.\frac{3}{8}\mu}{rr} - \frac{8}{3.5}.\frac{\frac{5}{8}d\mu Cc.\frac{5}{8}d\mu}{r^2} + \frac{8}{3.7}.\frac{\frac{7}{8}d\mu Cc.\frac{7}{8}\mu}{r^2} - \frac{8}{3.5.9}.\frac{\frac{9}{8}d\mu Cc.\frac{9}{8}\mu}{r^2} + \frac{8}{3.5.11}.\frac{\frac{11}{8}d\mu Cc.\frac{11}{8}\mu}{r^2} + \frac{8}{3.7.13}.\frac{\frac{13}{8}d\mu Cc.\frac{13}{8}\mu}{r^2} - \frac{8}{3.7.15}.\frac{\frac{15}{8}d\mu Cc.\frac{15}{8}\mu}{r^2} = 8.\frac{dSc.\frac{1}{8}\mu}{r} - \frac{8}{3}.\frac{dSc.\frac{3}{8}\mu}{r} - \frac{8}{3.5}.\frac{dSc.\frac{5}{8}\mu}{r} + \frac{8}{3.7}.\frac{dSc.\frac{7}{8}\mu}{r} - \frac{8}{3.5.9}.\frac{dSc.\frac{9}{8}\mu}{r} + \frac{8}{3.5.11}.\frac{dSc.\frac{11}{8}\mu}{r} + \frac{8}{3.7.13}.\frac{dSc.\frac{13}{8}\mu}{r} - \frac{8}{3.7.15}.\frac{dSc.\frac{15}{8}\mu}{r}.$$

Quapropter facta integratione habebimus ſummatoriam quæſitam, atque adeo æquationem

$$y = 8Sc.\frac{1}{8}\mu - \frac{8}{3}Sc.\frac{3}{8}\mu - \frac{8}{3.5}Sc.\frac{5}{8}\mu + \frac{8}{3.7}Sc.\frac{7}{8}\mu - \frac{8}{3.5.9}Sc.\frac{9}{8}\mu + \frac{8}{3.5.11}Sc.\frac{11}{8}\mu + \frac{8}{3.7.13}Sc.\frac{13}{8}\mu - \frac{8}{3.7.15}Sc.\frac{15}{8}\mu.$$

Hæc tria problemata propoſui, ut methodus aperiretur, qua hujuſmodi quæſtiones tractandæ ſunt.

13. Problema octavum. Invenire curvam, cujus quadratura dependeat abſolute a quadratura circuli, habeat tamen unum, aut plura ſpatia algebraice quadrabilia. Pluribus modis ſolutio problematis obtineri poteſt, duos maxime ſimplices uſurpabo, quorum unus per differentiam, alter per ſummam peragitur. Accipe

cipe primum. Sit ſemicirculus AFB, ( *Fig. 6* ) in cujus diametro accipe duas abſciſſas AG, AD, & ordinatis GF, DE duc radium CF. Sit $AG = x$, $AD = z$, & ſume quamcumque $M$ algebraice datam per $x$. Sit radius circuli $= r$, & conſtans quæcumque $= a$. Si concipias curvam, cujus area æqualis ſit rectangulo $aM$, & ſectori ACF dempto ſemiſegmento ADE, ejus quadratura generaliter dependebit a quadratura circuli, & erit æqualis rectangulo $aM$ addito mixtilineo CDEFC. Verum ſi accidat, ut linea AD fiat æqualis AG ideſt $z = x$, tunc evaneſcente ſpatio EDGF area curvæ fiet æqualis rectangulo $aM$, & triangulo FGC, adeoque erit algebraica. Quare ad hoc reducitur problema, ut talis $z$ determinetur per $x$, quæ generaliter non æquet eamdem $x$, ſed hæc æqualitas detur in uno, duobus aut, pluribus valoribus $x$; quod non eſt difficile, ut mox apparebit. Sector CAF $= \frac{r^2 dx}{2\sqrt{2rx - xx}}$, ſemiſegmentum AED $= dz\sqrt{2rz - zz}$; ergo elementum areæ curvæ quæſitæ erit $adM + \frac{r^2 dx}{2\sqrt{2rx - xx}} - dz\sqrt{2rz - zz}$. Supponamus curvæ quæſitæ eſſe ordinatam $= y$, abſciſſam $= P$, exiſtente $P$ data per $x$; ponamus item $dP = Q\,dx$, ergo areæ curvæ elementum erit $yQ\,dx$: igitur $adM + \frac{r^2 dx}{2\sqrt{2rx - xx}} - dz\sqrt{2rz - zz} = yQ\,dx$. Ad determinationem $z$ quod attinet, ut ea ſemel fiat æqualis $x$, ponatur $z = \frac{x . \overline{x + A}}{r}$; ergo tunc fiet $z = x$, quum $r = x + A$, ſive $x = r - A$. Subſtituto valore $z$ in ultima æquatione inventa proveniet æquatio

$$adM + \frac{r^2 dx}{2\sqrt{2rx - xx}} - \left(\frac{2x\,dx + A\,dx}{rr}\right) . \sqrt{-x^4 - 2Ax^3 - A^2x^2 + 2r^2Ax + 2rrx^2} = yQ\,dx,$$

ſive poſito $dM = N\,dx$

$$\frac{aN}{Q} + \frac{r^2}{2Q\sqrt{2rx - xx}} - \frac{1}{Q} . \frac{2x + A}{rr} \sqrt{-x^4 - 2Ax^3 - A^2x^2 + 2r^2Ax + 2r^2x^2} = y.$$

Quare ſi conſtruas curvam, cujus abſciſſæ $= P$ quocumque modo datæ per $x$ ordinatæ autem $y$, quæ determinatæ ſunt per $x$ in noviſſima æquatione, curva habebit ſpatium algebraice quadrabile pro caſu $x = r - A$, licet generatim ejus quadratura dependeat a quadratura circuli. In determinatione quantitatis $A$, hoc obſervandum eſt, ut tam $x$, quam $z$ proveniat poſitiva, & minor quam $2r$; quia extra hos limites areæ circulares evaderent imaginariæ.

14. Ut caſum aliquem breviter conſideremus, fiat $A = \frac{1}{2}r$, ut tunc evadat $x = z$, quum, & $x$, & $z = \frac{1}{2}r$, præterea $P = M = x$, quæ erit curvæ abſciſſa, unde $Q = M = 1$, & proveniet æquatio curvæ

$a+$

$$a + \frac{r^2}{2\sqrt{2rx - xx}} - \left(\frac{2x + \frac{1}{2}r}{rr}\right)\sqrt{-x^2 - rx^3 + \frac{7}{4}rrxx + r^3x} = y.$$

Figura exhibet progreſſum curvæ PLN, cujus abſciſſæ AG $= x$, ordinatæ GM $= y$. Si radius CA dividatur bifariam in H, & per H ducatur normalis KHL, area PAHL æquabit $a$. AH + triang. CHK, ſeu $\frac{ar}{2} + \frac{rr\sqrt{3}}{4}$. Hoc eſt in curva ſpatium unicum quadraturam recipiens. Verum in hac curva infinita aſſignari poſſunt ſpatia, quorum ſumma ſit algebraice quadrabilis. Hoc ut demonſtrem, adverto, quod ſi AG $= x$ ſit minor AH $= \frac{r}{2}$, erit AD $= z < x$; ſed ſi Ag $= x$ ſit $> \frac{r}{2}$, Ad $= z$ erit $> x$. Jam vero data AG geometrice determinari poteſt punctum g, ut ſect's AD, Ad prout oportet, ordinatæ intercipiant arcus æquales EF, ef; quare æqualia erunt ſegmenta EF, ef; Hiſce ſuppoſitis area PAGM æquabit rect. $a$AG, quadril. DEFG, ſegment. EF, & triangulum CFG; item area PAgm æquabit rect. $a$Ag, trian. Cod — triang. ofe — ſegm. fe; ergo quum ſegmenta EF, ef æqualia ſint, ſumma arearum PAGM, PAgm æquabit rectang. $a$. $\overline{\text{AG} + \text{Ag}}$, quadrilaterum DEFG, triangulum CFG, triangulum Cod dempto triangulo ſoe; ergo eſt quadrabilis algebraice.

15. Si velis ſpatia duo integrabilia algebraice, neceſſe erit ponere $z = x \cdot \frac{xx + Ax + rB}{rr}$; ergo in caſu $z = x$, proveniet $\frac{xx + Ax + rB}{-rr} = o$. In hac æquatione ita oportet determinare valores $A$, $B$, ut duæ radices $x$ ſint reales, & inæquales, ſingulæ mindres quam $2r$. Hoc methodo directa nullo negotio determinabis. Nam finge areas algebraicas eſſe oportere tum quum $x = r$, quum $x = \frac{1}{2}r$; ergo valebunt factores $x - r = o$, $x - \frac{r}{2} = o$, qui ſimul multiplicati dant $xx - \frac{3}{2}x + \frac{rr}{2} = o$. Conferatur hæc cum ſuperiore æquatione, & invenies $A = -\frac{3}{2}r$, $B = \frac{3}{2}r$; ergo, poſita $z = x \cdot \frac{xx - \frac{3}{2}rx + \frac{3}{2}rr}{rr}$, habebis in curva duas areas algebraicas exiſtente $x = r$, & $x = \frac{1}{2}r$. Simili modo ſi velis tria ſpatia algebraice integrabilia, ponenda erit $z = x \cdot \frac{x^3 + Ax^2 + rBx + r^2C}{r^3}$, ut in caſu $x = z$, habeatur æquatio $x^3 + Ax^2 + rBx + r^2C = o$, in qua ita determinandi ſunt
$$-r^3$$
coefficientes, ut omnes valores $x$ ſint $< 2r$, & omnes ſint reales, & inæquales: nam ſi duo eſſent imaginarii, unum tantum prodiret ſpatium algebraice

ſe quadrabile. Si ponas $z = x . \frac{x^4 + Ax^3 + rBx^2 + r^2Cx + r^3D}{r^4}$ & eamdem methodum ſequaris, invenies curvam, cujus quatuor ſpatia aream algebraicam habebunt; atque ita deinceps.

16. Modum, qui fit addendo, breviter expedio. Poſita tangente $AG = x$, ( *Fig.* 7 ) ſector AFC erit $= S \frac{r^3dx}{2.rr+xx}$; ſimiliter poſita tangente $AD = z$, trilineum $ADE = S \frac{rz^2dz}{2.r^2+z^2}$. Jam vero concipe curvam, cujus area ſit æqualis $aM + S \frac{r^3dx}{2.rr+xx} + S \frac{rz^2dz}{2.rr+zz}$, hæc generatim dependebit a quadratura circuli, ſed ſi evadat $AD = AG$ area ejuſdem curvæ erit algebraica, utpote æqualis triangulo rectilineo $ACG = \frac{rx}{2}$. Curvæ hujus abſciſſa ſit $= P$, ordinata $= y$; pone $dP = Qdx$, $dM = Ndx$, & erit $aNdx + \frac{r^3dx}{2.rr+xx} + \frac{rz^2dz}{2.rr+zz} = yQdx$. Jam vero $z$ ita determinanda eſt, ut generatim differat ab $x$, ſed in uno, duobus, aut pluribus caſibus fiat æqualis $x$. Ut hoc obtineas methodum ſuperiorem uſurpare potes.

17. Verum ut cognoſcas, hoc pluribus modis obtineri poſſe, due rectam AM facientem cum AG angulum ſemirectum; tum deſcribe curvam, quæ ſecet rectam AM in uno, in duobus, tribus, aut pluribus punctis, & accipe ſemper $AD = z$ æqualem hujus curvæ ordinatæ HG. Manifeſtum eſt, $z$, $x$ fore ubique inæquales exceptis punctis illis, ubi curva ſecat rectam AM: clarum eſt, curvam ad minimum tot graduum eſſe oportere, quot ſunt puncta ſectionum. Quare hæc determinatio $z$ præbebit curvam, quæ tot habebit ſpatia algebraica, quot exiſtunt ſectionum puncta.

18. Evolvamus caſum maxime ſimplicem. Ponamus $a = o$, $P = x$, unde $Q = 1$, ut æquatio fiat $\frac{r^3dx}{2.rr+xx} + \frac{rz^2dz}{2.r^2+z^2} = ydx$. Completo quadrato ACBL, ( *Fig.* 8 ) ex puncto L agatur LN, quæ faciat angulum ſemirectum cum LA. Poſita $AG = x$, ſit $GH = z = r - x$. In ſuperiore æquatione pro $z$ ſcribe $r - x$, ut fiat $\frac{r^3dx}{2.rr+xx} + \frac{r.\overline{r-x}^2.-dx}{2.2rr-2rx+xx} = ydx$. Facta diviſione per $dx$, & redactis terminis ad eamdem denominationem fiet

$$\frac{r^5 - r^3x^2 + 2r^2x^3 - rx^4}{2.2r^4 - 2r^3x + 3r^2x^2 - 2rx^3 + x^4} = y,$$

quæ eſt curva quinti gradus. Hæc curva generaliter habebit aream dependentem a quadratura circuli. Poſita enim abſciſ-

abſciſſa curvæ $= AG$, ejus area erit æqualis ſectori ACF, addito trilineo AED, in quo $AD = GH$. Verum ſi diviſa AL biſariam in O, ubi ordinata OI incidit in punctum ſectionis linearum AM, LN, & eſt $= AO$, poſita curvæ abſciſſa $= AO$, ejus area æquabit ſectorem ACV, addito trilineo AOV, hoc eſt triangulum CAO, quod æquat quadratum dimidij radij CA. Itaque curva generatim dependet a quadratura circuli; habet tamen unicum ſpatium algebraice quadrabile.

19. Problema nonum. Invenire curvam, cujus quadratura generaliter dependet a quadratura hyperbolæ; habeat tamen unum aut plura ſpatia algebraice quadrabilia. Inter infinitos modos, quibus problema ſolvi poteſt, unum eligo. Inter aſſymptota RC, CS ( *Fig.* 9 ) deſcripta ſit hyperbola RGS, in qua $CA = AG = b$. Sumatur $AB = x$, ut $BD = \frac{bb}{b+x}$, & ſpatium $GABD = S\frac{b^2dx}{b+x}$. Sume item $AE = z$, ordinataque EF, junctaque chorda GF erit ſegmentum

$$GOF = S\frac{bz^2dz}{2\cdot\overline{b+z}^2}.$$

Data $M$ per $x$, ſtatuo aream curvæ quæſitæ eſſe

$$aM + S\frac{b^2dx}{b+x} + S\frac{bz^2dz}{2\cdot\overline{b+z}^2}.$$

Sit curvæ quæſitæ abſciſſa $= P$, data per $x$, ordinata $= y$; ergo æquatio curvæ erit $adM + \frac{b^2dx}{b+x} + \frac{bz^2dz}{2\cdot\overline{b+z}^2} = ydP$, & facta $dM = Ndx$, & $dP = Qdx$ fiet $aNdx + \frac{b^2dx}{b+x} + \frac{bz^2dz}{2\cdot\overline{b+z}^2} = yQdx$.

Si $z$ ſit major, aut minor $x$, quadratura hujus curvæ ſine dubio dependet ab hyperbolæ quadratura, ejuſque area eſt æqualis figuræ rectilineæ AGFE addito ſpatio curvilineo FEDB, ſi $AE < AB$, vel dempto curvilineo, ſi ſit $AE > AB$. Verum ſi $AE = AB$, ſpatium curvilineum nulleſcit, & area invenitur æqualis figuræ rectilineæ, adeoque algebraice quadrabilis. Itaque nihil reliquum eſt ad ſolutionem problematis, niſi ut ejuſmodi $z$ determinetur, quæ generatim non ſit æqualis $x$, ſed in uno, pluribuſve punctis hanc æqualitatem obtineat. Huc transfer methodos, quas adhibuimus in problemate ſuperiore.

20. Placet addere modum, per quem obtinetur curva generatim dependens a quadratura hyperbolæ, habens tamen infinita ſpatia algebraice integrabilia. Ducta AN, quæ faciat angulum ſemirectum cum AB, eique ducta normali $AH = \frac{1}{2}r$, & parallela HM, poſitoque ſinu toto $= r$, deſcribe ſuper baſi HLM curvam ſinuum circularium AKILMN, quæ ſecabit lineam AN in punctis infinitis K, I, N &c. Ex quolibet ejus puncto V ducatur VB normalis AB. Si, ſumptis $AB = x$, fiant $VB = z$, generatim $z$ erit aut minor, aut major $x$; ſed in infinitis punctis, ubi curva ſecat AN, fit $z = x$, adeoque infinitas areas algebraice integrabiles curva continebit. Quæ curva, ut poſſit aliqua ratione deſcribi, invenienda eſt tam $x$, quam $z$ per HT, ejuſque functio-

ces; quod ubi inventum erit, facile erit determinare coordinatas $p, y$ curvæ deſcribendæ per eamdem H T.

21. Hanc ob rem voca $HT = AZ = \mu$, erit $TV = TY = Sc.\mu$, $HY = \mu + Sc.\mu$, $ZV = ZX = Sc.\mu - \frac{r}{2}$, $AX = \mu + Sc.\mu - \frac{1}{2}r = x\sqrt{2}$; ergo datur $x$ per $\mu$ ita, $x = \frac{\mu + Sc.\mu}{\sqrt{2}} - \frac{1}{2\sqrt{2}}r$, & differentiando $dx = \frac{d\mu + dSc.\mu}{\sqrt{2}} = \frac{rd\mu + d\mu Cc\mu}{r\sqrt{2}}$. Præterea eſt $VY = Sc.\mu.\sqrt{2}$, $XY = \frac{1}{\sqrt{2}}r$; ergo $VX = Sc.\mu\sqrt{2} - \frac{1}{\sqrt{2}}r$; ergo $BV = z = x - Sc.\mu.\sqrt{2} + \frac{1}{\sqrt{2}}r = \frac{\mu}{\sqrt{2}} - \frac{Sc.\mu}{\sqrt{2}} - \frac{1}{2\sqrt{2}}r = \frac{\mu - Sc.\mu}{\sqrt{2}} + \frac{r}{2\sqrt{2}}$; & differentijs ac-

$$-Sc.\mu.\sqrt{2} + \frac{1}{\sqrt{2}}r$$

ceptis $dz = \frac{d\mu - dSc.\mu}{\sqrt{2}} = \frac{rd\mu - d\mu Cc.\mu}{r\sqrt{2}}$. Quibus inventis habes coordinatas curvæ quæſitæ datas per $\mu$. Hæc autem curva, ubicumque $Sc.\mu = \frac{1}{2}r$, habet areas algebraice integrabiles; ejus autem deſcriptio a quadratura circuli dependet. Hæc duo problemata in eum finem propoſita ſunt, ut diſcas, qua methodo tractandæ ſint quæſtiones hujus generis.

22. Problema decimum. In circulo A B D E ( *Fig.* 10, 11 ) determinare omnes arcus, qui ſint in ſuorum ſinuum ratione duplicata. Ad cognoſcenda paria omnium arcuum, qui ſunt in ratione duplicata ſuorum ſinuum utor curva ſinuum circularium a f d 2 f &c. in qua ſi ſecentur a g æquales arcubus A G, ſunt f g æquales ſinibus G F. Verum non contemplor tantum partem a f d, cujus abſciſſa a d eſt æqualis ſemicircumferentiæ A B D, ſed partes omnes conſequentes, quæ ſunt infinitæ; his enim omiſſis plures arcus omittuntur, qui gaudent proprietate quæſita. Si vertice a deſcribatur quælibet parabola ſecans curvam ſinuum in punctis h, k, hæc determinabit longitudinem a m, a n duorum arcuum, qui ſunt ut quadrata ſuorum ſinuum. Quoniam curva ſinuum in puncto a facit angulum ſemirectum cum a d, & quælibet parabola, cujus vertex princeps ſit a facit angulum rectum, omnis parabola initio cadet extra curvam ſinuum. Determinetur primum parabola a f, quæ tangat curvam ſinuum in f. Vocetur radius $= r$, $ag = x$, $fg = y$; ſubtangens parabolæ in puncto f erit $= 2x$, ſubtangens curvæ ſinuum $= \frac{ry}{\sqrt{rr - yy}}$; ergo quum curvæ debeant habere communem ſubtangentem, erit $2x = \frac{ry}{\sqrt{rr - yy}}$. Abſcinde arcum $AG = ag$. Erit ut coſinus ad ſinum, ita radius ad duplicem arcum A G; ſed ut coſinus ad ſinum, ita radius ad tangentem: ergo tangens arcus A G erit æqualis duplici arcui A G, ſeu A G erit dimidium ſuæ tangentis.

23. Quoniam parabola a f non incidit in curvam ſinuum niſi in f, propte-

rea

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ſrea non poteſt aſſignari alius arcus, qui ſit ad A G in ratione ſuorum ſinuum duplicata. Si intelligatur minui parabolæ parameter, hæc ſecabit curvam in duobus punctis, quorum primum erit inter a, f, alterum inter f, e, donec parabola ſecet curvam in c. Quare in hoc caſu duo arcus habebuntur ambo minores quadrante, qui ſunt in ratione ſuorum ſinuum duplicata. Si ulterius minuatur parabolæ parameter, puncta ſectionum magis magiſque recedent, ut arcus alter minor ſit quadrante, & alter major. Duo tantum arcus habebuntur, donec parabolæ deſcriptæ ramus alter fugiat partem curvæ d 2 f 2 a; nam ubi in hanc partem parabola incidat, unus aut duo alij arcus obtinebuntur, prout curvæ aut ſeſe tangunt, aut ſeſe ſecant, qui arcus ſemicircumferentiam excedunt. Caſus autem, in quo parabola curvam tangit, exhibet arcum A 2 G, qui additus ſemicircumferentiæ eſt æqualis ſuæ tangentis dimidio. Quatuor itaque arcus invenientur, donec parabola fugiat tertium ramum 2 a 3 f 2 d. Contactus exhibet arcum A 3 G, qui additus integræ circumferentiæ eſt æqualis ſuæ tangentis dimidio. Hoc modo progrediens videbis, creſcere numerum arcuum, qui ſunt in ratione duplicata ſuorum ſinuum, quo minuitur parabolæ parameter; ita ut arcus infiniteſimus habeat infinitos arcus, ad quos ſit in ratione ſinuum duplicata.

24. Ex his omnibus apparet poſſe per interſectionem parabolæ, & curvæ ſinuum ſolvi problema: dato arcu alios invenire, ad quos habeat rationem ſinuum duplicatam. Quod problema generatim ſumptum eſſe tranſcendens, neque ſolvi poteſt, niſi aut per curvas traſcendentes, aut per ſeries infinitas. Novi aliquando accidere, ut dato arcu alius determinetur algebraice: nimirum quum iſti inter ſeſe tenent proportionem rationalem. Quomodo hoc fiat declarandum eſt, ſoluto generaliter hoc problemate. Invenire duos arcus, qui & ſint in ratione data, & ſint ut quadrata ſuorum ſinuum. Ratio data ſit $m:n$. Unus ex arcubus quæſitis ſit $mv$, alter erit $nv$; ergo $mv:nv$, ſeu $m:n::\overline{Sc.mv}^2:\overline{Sc.nv}^2$, ergo $\sqrt{m}:\sqrt{n}::Sc.mv:Sc.nv$. Introducamus notam formulam ſinus arcus multipli, & inveniemus

$$\sqrt{m}:\sqrt{n}::\frac{\overline{Cc.v+\sqrt{-1}.Sc.v}^m-(\overline{Cc.v-\sqrt{-1}.Sc.v}^m}{2r^{m-1}\sqrt{-1}}:$$

$$\frac{\overline{Cc.v+\sqrt{-1}.Sc.v}^n-(\overline{Cc.v-\sqrt{-1}.Sc.v}^n}{2r^{n-1}\sqrt{-1}}.$$ Per hanc formulam determinabitur arcus $v$, quo determinato habemus arcus $mv$, $nv$, qui requiruntur.

25. Si ratio $m:n$ ſit rationalis, ut duo numeri $m$, $n$ ſint integri, quod ſemper fieri poteſt, elatis binomijs ad ſuas poteſtates, omnia imaginaria abibunt, & obtinebitur algebraica æquatio, per quam definietur $Sc.v$, adeoque invenietur algebraice arcus $v$. Exemplis res illuſtranda. Sit $m=1$; $n=2$; erit

$1:\sqrt{2}::Sc.v:\frac{2Cc.v.Sc.v}{r}::r:2Cc.v$, ſeu $Cc.v=\frac{r}{\sqrt{2}}$, quem ſcimus eſſe coſinum ſemiquadrantis: igitur unus arcus erit quadrans, alter ſemiquadrans, quod notiſſimum eſt. Quapropter ſi parabola ſecet curvam ſinuum in c, ordinata ex altero ſectionis puncto ducta biſecabit a b. Verum infiniti ſunt arcus,

quorum coſinus $= \frac{r}{2}$, exiſtente ſinu aut poſitivo, aut negativo: Hi omnes ſi comparentur cum eorum duplis ſunt in ratione ſuorum ſinuum duplicata. Sit $m=1$, $n=3$ fiet $1:\sqrt{3}::Sc.r:\frac{3\overline{Cc.r}^{2}.Sc.r-\overline{Sc.r}^{3}}{r^{2}}::r^{3}:3\overline{Cc.r}^{2}-\overline{Sc.r}^{2}::r^{2}:4\overline{Cc.r}^{2}-r^{2}$: igitur $Cc.r=\pm\frac{r\sqrt{\sqrt{3}+1}}{2}$. Arcus itaque, cujus hic eſt coſinus ad ſuum triplum erit in ratione directa duplicata ſinuum. Quo majores erunt numeri $m$, $n$, eo altior proveniet æquatio, ſed ſemper algebraica, ſi ipſi ſint numeri rationales.

26. Quod ſi fuerint ſurdi, aut etiam tranſcendentes, inveniemus ſeriem, per quam $Cc.r$ poterit determinari. Ea autem ſic erit, poſito $m=1$, & $n$ quocumque numero $1:\sqrt{n}::r^{n-1}Sc.r:n\overline{Cc.r}^{n-1}Sc.r-\frac{n.\overline{n-1}.\overline{n-2}}{2.3}\overline{Cc.r}^{n-3}\overline{Sc.r}^{3}+\frac{n.\overline{n-1}.\overline{n-2}.\overline{n-3}.\overline{n-4}}{2.3.4.5}\overline{Cc.r}^{n-5}\overline{Sc.r}^{5}-$ &c. quæ ſi dividatur per $Sc.r$, tum pro $\overline{Sc.r}^{2}$, ſcribatur $r^{2}-\overline{Cc.r}^{2}$, ejecto ſinu $r$ nihil continebit incogniti præter $Cc.r$. Hoc problema propoſui, quia ab eo dependet problema quod ſequitur.

27. Problema undecimum. Invenire infinitas lunulas circulares quadraturam algebraicam admittentes. Hypocrates Chius in circulo, (*Fig.* 12) cujus centrum C, ſumpſit quadrantem CAB, ejuſque chordam AB biſecavit radio CE in K. Tum facto centro in K intervallo KA$=$KC deſcripſit circulum ADBC. His effectis quoniam circuli ſunt in duplicata radiorum ratione conſtat circulum AEB eſſe ad circulum ADBC, ut quadratum AC ad quadratum AK; ſed quadratum AC eſt duplum quadrati KA; Ergo circulus AEB eſt duplus circuli ADBC; ergo quadrans CAB æqualis ſemicirculo ADB; igitur ablato communi ſpatio AEBK remanebit lunula ADBE æqualis triangulo ABC. Prima hæc lunula algebraice quadrabilis obſervata fuit a Græco Hypocrate.

28. Quærantur aliæ infinitæ lunulæ algebraice quadrabiles. Sit ſector CAB (*Fig.* 13) diviſus bifariam a radio CE, qui dividit chordam AB ad angulos rectos in H. Centro K poſito in radio CE, & radio KA intelligatur deſcriptus alius ſector KADB. Si iſti duo ſectores æquales fuerint, tunc dempto communi ſpatio KAEB remanebit lunula ADBE æqualis rectilineo ACBK, atque adeo algebraice quadrabilis. Quærendæ itaque conditiones, ſub quibus æquales ſunt ſectores. Produc CE in D, ut habeantur ſectorum dimidia BCE, BKD pariter æqualia. Age CF parallelam KB, & duc FG normalem radio CE. Quando arcus ſectorum æqualium ſunt reciproce ut radij, erit EB:DB::KB:CB; ſed ex ſectorum KBD, FCE ſimilitudine eſt DB:EF::KB:CF$=$CB: ergo componendo rationes

EB:EF::$KB^{2}:CB^{2}$; atqui KB:CF::BH:FG; igitur

EB:EF::$HB^{2}:GF^{2}$; ſcilicet arcus EB, EF ſunt in ſuorum ſinuum ratione duplicata.

29. Itaque in dato circulo A B F per problema præcedens determinentur duo arcus E B, E F, qui ſint in ſuorum ſinuum ratione duplicata, & juncta C F, ex puncto B agatur B K parallela C F; centro K intervallo K B deſeribatur ſector K B D A, & jungautur K A, C A: ajo lunulam A D B E eſſe algebraice quadrabilem, & æqualem ſpatio rectilineo K A C B. Demonſtratio peragitur inverſo ordine. Nam ex conſtructione eſt

$EB:EF::HB^2:GF^2::KB^2:CF^2=CB^2$; atqui $EF:DB::CF=CB:KB$; ergo $EB:DB::KB:CB$; ſunt igitur arcus E B, D B in ratione ſuorum radiorum reciproca, ergo æquales ſectores C B E, K B D, adeoque etiam dupli C B A, K B A; ergo dempto communi K A E B remanet lunula A D B E æqualis rectilineo C A K B. Antequam aliquot determiuationes peragam, duo hæc jubeo te diligenter diſtinguere. Aliud eſt lunulas eſſe algebraice quadrabiles, aliud eſt lnuulas algebraice quadrabiles poſſe algebraice determinari. Primum ſemper verum eſt, quoties arcus E B, E F ſiut, ut ſuorum ſinuum quadrata. Secundum obtineri non poteſt, niſi qnum arcus E B, E F ſunt in ratione effabili.

30. Si punctum F eadat in quadrante E I, punctnm K cadet inter C, H, atque adeo ſector K A D B erit minor ſemieireulo. Si F cadat in I, punctum K coineidet cum H, & ſector K A D B erit ſemicircnlus, & orietur lunnla Hypocratis. Si punctum F cadat in ſecuudo quadrante I L; punctum K cadet ultra puncta C, H, & ſector ſit major ſemieirculo. Hæ ſolutiones ſemper obiinentur excepto caſu, ubi arcus dati A B dimidium E B foret æquale ſuæ taugentis dimidio, ut conſtat ex ſuperiore problemaie.

31. Verum decreſeente arcu E B unus, aut duo arcus oriuntur, qui deſinunt in ſemicirculum L O E, ad quos E B eſt in duplicata ratione ſuorum ſinuum. Videamus quid accidat in lnnulis. Punctum ſit 2 F. Tunc ducta B K parallela C 2 F invenies ſectorem C A B eſſe æqualem toti eireulo radi K B addito ſectore K A D B; ergo dempto communi A E B K remanebit rectilineum A K B C æquale toti circulo radi K B addita lunula A E B D, quæ quantitas proinde eſſet algebraice quadrabilis. Quod idem verificaretur, ſi punctum 2 F tranſiret in ultimum quadrautem O E, eum hoc tantum diſcrimine quod K colloearetur poſt puncta C, H. Si comparatio fiat inter arcum E B deſinentem in ſecundum quadrantem tum arcu E B F deſinente in quadrante tertio (*Fig.* 14) ducatur B K parallela C F & centro K deſeribatur circulus A B M. Inveuies ſpatium circulare C A E B = integro circulo A B M addito ſectora K A B. Adde utrique ſpatium C A D B, & fiet lunula A E B D æqualis eireulo A B M ſimnl cum quadrilatero K A C B; igitur differentia inter lunulam A E B D, & eireulum A B M, æquabit quadrilaterum K A C B, adeoque erit algebraice quadrabilis. Idem proveniet, ſi punctum F tranſeat in tertium quadrantem, cum hoc tauium diſcrimine, quod K ſiium erit inter puncta C, H. Si comparentnr duo arcus E A B, E A F (*Fig.* 15) qui deſinant in tertium ſectorem, ducta B K parallela C F, invenies totum eirculum E A B, addito ſectore C A L B æqualem toti circulo A D B M, addito ſectore K B D A. Auferatur ſpatium K B L A, & fiet totus eirculus E A B, addito quadrilatero C B K A, æqualem circulo A D B M, addita lunula A D B L: ergo quadrilaternm C B K A æquabit differentiam circuli A D B M addita lunula A D B L, & cirenli E A B, quæ differentia propterea erit algebraice quadrabilis. Hæc exempla ſint ſatis; ex bis enim potes cognoſcere, quid accidat in infinitis caſibus, ubi comparantur arcus ſervantes duplicatam rationem ſinuum, qui arcus excedunt ſemel, bis, ter totam circumſerentiam.

32 Problema duodecimum. In ſuperficie quartæ partis hemiſphærii ABDG (*Fig.* 16) ſic ducere lineam AQD, ut lunula AQDA ſit algebraice quadrabilis. Per axem AC ducantur circuli maximi infinite proximi AQM, Aqm ſecantes curvam AQD in Q, q, & circulum BMD in M, m. Ex punctis Q, q in axem AC demittantur normales QP, qp, tum in BC ex punctis M, m normales MN, mn, demum ex Q agatur QO parallela Mm. Vocetur modo $BM = s$, $Mm = ds$, $BN = x$, $Nn = dx$, $AP = z$, $Pp = dz$, $PQ = y$, radius ſphæræ $= a$. Hiſce præmiſſis manifeſtum eſt, trilineum AQq eſſe elementum lunulæ; ſed trilineum QOq eſt infiniteſimum ſecundi ordinis, quum AQO ſit primi; ergo illo omiſſo lunulæ elementum erit AQO. Ut valor hujus trilinei analytice reperiatur, paralleli Mm agantur arcus Vu, Zz infinite proximi, & ex V in AC demittatur normalis VT. Vocetur $AT = t$, $VT = u$. Quoniam Vu eſt arcus deſcriptus radio VT, valet proportio $a : VT = u :: Mm = ds : Vu = \frac{uds}{a}$; & quoniam VZ eſt elementum arcus AV, cujus ſinus verſus eſt $AT = t$, ſinus rectus $VT = u$, erit $VZ = \frac{adt}{u}$; ergo rectangulum $VuzZ = \frac{uds}{a} \cdot \frac{adt}{u} = dsdt$; igitur integrando in ſuppoſitione $ds$ conſtantis habebitur trilineum $AVu = tds$, & facta $AT = AP$, hoc eſt $t = z$, erit trilineum $AQO = zds$. Inventum eſt itaque lunulæ elementum.

33. Latet neminem, $ds = \frac{adx}{\sqrt{2ax - xx}}$; ergo lunulæ elementum $AQO = \frac{azdx}{\sqrt{2ax - xx}}$; ergo integrando pars lunulæ intercepta inter ſuum perimetrum AQ, & arcum circularem AQ erit $= S\frac{azdx}{\sqrt{2ax - xx}}$, & poſitis tam $x$, quam $z = a$, habebitur integra lunula. Quapropter deſcripto quolibet circulo maximo AQM, & ducta MN normali in BC, ſi ita determinetur $AP = z$ per $BN = x$, ut formula $\frac{azdx}{\sqrt{2ax - xx}}$ evadat algebraice integrabilis; demum ex puncto P agatur radio AC normalis PQ in plano circuli APM, punctum Q erit in curva quæſita. Omnia alia curvæ puncta ſimili ratione determinantur. Infinitis modis determinari poteſt $z$ per $x$, ut formula fiat algebraice integrabilis. Indicabo illum, qui maxima ſimplicitate donatum eſt. Sit $z = \sqrt{2ax - xx}$ hoc eſt $AP = MN$, erit lunula $= Sadx = ax$, & facta $x = a$, tota lunula $= aa$. Quare ſi poſito circulo maximo quolibet AQM, ducatur MN perpendicularis in BC, tum abſcindatur $AP = MN$; & in plano circuli agatur normalis PQ, punctum Q, & alia ſimiliter determinata erunt in curva AQD, cujus integra lunula AQDA æqualis erit radij quadrato. Si veiis lunulam habere ad quadratum radij rationem datam, ponenda eſt $z = m\sqrt{2ax - xx}$, & invenies integram lunulam $= maa$; hoc eſt ad $aa$ eſſe ut $m : 1$.

34. Quod ſi cupias, non lunulam AQDA, ſed ungulam AHBDQA eſſe algebraice quadrabilem, adverte, trilineum AQO inventum eſſe $= zds$; ergo poſita $z = AC = a$, erit trilineum $AMm = ads$; ergo $MQOm = \overline{a - z} \cdot ds = \frac{\overline{a - z} \cdot adx}{\sqrt{2ax - xx}}$, ſed MQOm eſt elementum ungulæ; ergo integran-

do

do erit pars ungulæ interceptæ a quadrante $\overline{AHB}$, ab arcubus circularibus BM, MQ, & ab ejus perimetro $QA = S\frac{\overline{a-z}.adx}{\sqrt{2ax-xx}}$, in qua, facta integratione, ſi ponatur tam $z$, quam $x = a$, habebitur totius ungulæ ſuperficies. Igitur, ut hæc algebraice exprimatur, neceſſe eſt, ita dari $z$ per $x$, ut formula evadat algebraice integrabilis. Sit $z = x$, ſeu $AP = BN$. Ungula fiet

$S\frac{a.\overline{a-x}.dx}{\sqrt{2ax-xx}} = a\sqrt{2ax-xx}$, & facta $x = a$, tota ungula $= aa$. Conſtructio curvæ eodem modo peragitur. Nam accepto quolibet circulo maximo AQM, ductoque arcus BM ſinu MN, abſcinde $AP = BN$, & in circuli maximi AQMC plano duc PQ normalem AC; punctum Q, & alia ſimiliter determinata erunt in curva requiſita. Quoniam arcuum BM, AQ ſinus verſi BN, AP æquales ſunt, æquales erunt arcus BM, AQ. Itaque abſciſſo ubique $AQ = BM$ obtinebitur curva, quæ ita per motum deſcribi poteſt ad modum ſpyralis. Interea dum circuli maximi quadrans AHB data cum velocitate rotatur circa axem AC, punctum A moveatur æquabiliter per circumferentiam quadrantis, ita in punctorum A, B eadem ſit velocitas, punctum A ſemper reperietur in curva paulo ante deſcripta, quæ ungulam habet abſolute quadrabilem.

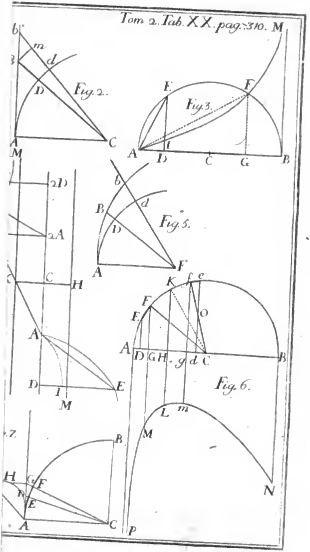


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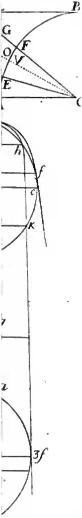
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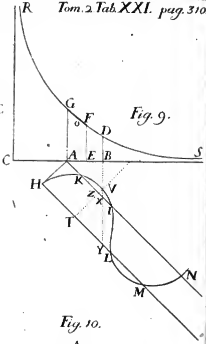
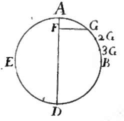
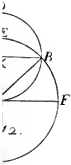




*Fig. 9.*

*Fig. 10.*






12.

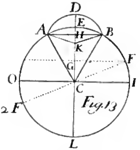
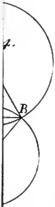



Fig. 13




4.

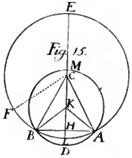



Fig. 15.

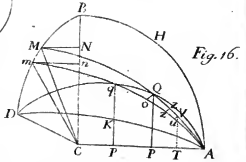



Fig. 16.

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